Comparative numerical study of time fractional coupled Korteweg-de Vries equation

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Abstract. In this paper, we construct an analytical solution for a system of the Fractional coupled Korteweg-de Vries differential equations in the sense of Caputo definition. We give the basic properties of the fractional differential equation and a coupled Korteweg-de Vries equation. We applied the homotopy analysis transform method to obtain the analytic solution for this equation, and we compared the result with other obtained solutions by different numerical methods. A comparison study observed the efficacy and accuracy of the present algorithm.
Keywords: fractional coupled Korteweg de Vries, Caputo fractional derivative, homotopy analysis transform method.

1. Introduction

Fractional differential equations are one of the important branches in the field differential equations, because of various phenomena in physics and chemistry, electronic and electrical, medicine and epidemiology spread, engineering, can be formulated into a fractional differential equation such as the coupled Korteweg-de Vries equation [1]:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \eta \frac{\partial^3 u(x,t)}{\partial x^3} + \gamma u(x,t)\frac{\partial u(x,t)}{\partial x} + \mu v(x,t)\frac{\partial v(x,t)}{\partial x}, \\ \frac{\partial v(x,t)}{\partial t} &= \lambda \frac{\partial^3 v(x,t)}{\partial x^3} - \nu v(x,t)\frac{\partial v(x,t)}{\partial x}. \end{aligned} \tag{1}$$

The solution of coupled Korteweg-de Vries is introduced using different numerical methods, to obtain the accurate approximations and their properties, such as

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the Spectral collocation method used by Khader [4], and Hussain [5]. Bashan [6] used the Finite difference method and differential quadrature method. Bakodah applied the decomposition method [7], homotopy perturbation method investigated by Bothayna S. Kashkari [8], homotopy analysis method (HAM) in [11] used by Abbsbandy and homotopy analysis transform method [12] by Saad.

The homotopy analysis method is one of the most important approximate methods for solving the linear and nonlinear differential equations, as partial differential equations [13], delay differential equtions [14], fuzzy differential equations [15], including the fractional differential equations [16]. HAM proposed in Ph.D. dissertation in 1992 of Shijun Liao [17],[18], and [19]. The method has many applications in various classes of famous differential equations. For fractional differential equations, the scientists used the coupled Laplace transform with the homotopy analysis method (HATM), to write a simple algorithm corresponding to this type of equations, easily solved by Mathematica and Maple. The method introduced by Sunil Kumar [21], [22], Devendra Kumar [23], Majid Khan [24].

In this paper, we investigate the HATM for the following fractional differential equation [1]

$$\begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \eta\frac{\partial^3 u(x,t)}{\partial x^3} + \gamma u(x,t)\frac{\partial u(x,t)}{\partial x} + \mu v(x,t)\frac{\partial v(x,t)}{\partial x}, \\ &\frac{\partial^\beta v(x,t)}{\partial t^\beta} = \lambda\frac{\partial^3 v(x,t)}{\partial x^3} - \nu u(x,t)\frac{\partial v(x,t)}{\partial x}, \end{aligned} \tag{2}$$

where ${}^C D^\alpha$ is the fractional derivative operator of order α in the sense of Caputo. We compare the numerical solutions with the result obtained by Mohamed Elbadri [1], who used the Naturel decomposition method (NDM), and spectral collection method [2] .

2. Basic definitions of fractional calculus

In this section, we give some basic definitions of fractional calculus theory used in this paper:

Definition 2.1. *A real function $h(t)$ is said C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$ where $h_1(t) \in C(0,\infty)$ and it is said to be in space C_n if and only if $h^{(n)} \in C_\mu, n \in \mathbb{N}$.*

Definition 2.2. *The Riemann-Liouville fractional operator of order $\alpha \geq 0$, of a function $h \in C_\alpha, \alpha \geq -1$ is defined as [1]*

$$\begin{aligned} &I^\alpha h(t) == \frac{1}{\Gamma(\alpha)}\int_0^t (t-\alpha)^{(\alpha-1)} h(\tau) d\tau, \mu > 0, t > 0, \\ &I^0 h(t) = h(t), \end{aligned}$$

where $\Gamma(.)$ is well-known Gamma function.

Definition 2.3. *The Caputo fractional derivative of h, $h \in C_{-1}^{m}$ is defined as*

$$D_C^{\alpha} h(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\varsigma)^{(m-\alpha-1)} h^{(m)}(\varsigma) d\varsigma,$$

where $m-1 < \alpha < m, m \in \mathbb{N}$

Definition 2.4. *The Laplace transform for the function $h(t)$ of the Caputo fractional derivative given by [22]:*

$$L[D_t^{\alpha} h(t)] = s^{\alpha} L[h(t)] - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} h^{(k)}(0), \quad n-1 < \alpha \leq n. \tag{3}$$

3. The homotopy analysis transform method

Firstly, we give some basic idea of the homotopy analysis transform method, so we consider the following fractional differential equation to illustrate the application of the HATM for the fractional differential equation:

$$^{C}D^{\alpha} h(x,t) + \mathfrak{R} h(x,t) + \mathfrak{N} h(x,t) = f(x,t), \quad 0 < \alpha \leq 1, \tag{4}$$

where $^{C}D_t^{\alpha} h(x,t)$ is the Caputo derivative of $h(x,t)$, $\mathfrak{R}$ and $\mathfrak{N}$ are the linear and nonlinear operator respectively, $f(x,t)$ is the source term.

Firstly, by applying the Laplace transform L on the (4), we obtain,

$$s^{\alpha} L[h(x,t)] - s^{\alpha-1} h(x,0) + L[\mathfrak{R} h(x,t)] + L[\mathfrak{N} h(x,t)] = L[f(x,t)], 0 < \alpha \leq 1 \tag{5}$$

after simplification,

$$L[h(x,t)] = \frac{1}{s} h(x,0) + \frac{1}{s^{\alpha}} (L[f(x,t)] - L[\mathfrak{R} h(x,t)] - L[\mathfrak{N} h(x,t)]), 0 < \alpha \leq 1. \tag{6}$$

By applying the homotopy analysis method shown in [10],[11] and [12], we define the non-linear operator

$$\begin{aligned} N[\phi(x,t,q)] = L[\phi(x,t,q)] - \frac{1}{s} h(x,0) - \frac{1}{s^{\alpha}} (L[f(x,t)] \\ + L[\mathfrak{R} \phi(x,t,q)] + L[\mathfrak{N} \phi(x,t,q)]). \end{aligned} \tag{7}$$

where ϕ is a real-valued function of x, t, and $q \in [0,1]$. The zeroth-order deformation constructed by Liao [10],[11] is

$$(1-q) \mathcal{L}[\phi(x,t,q) - h_0(x,t)] = \hbar q N[\phi(x,t,q)], \tag{8}$$

where $\hbar \neq 0$ is nonzero convergent control parameter, $h_0(x,t)$ is the initial guess, N is the nonlinear operator and $\mathcal{L}$ is an injective linear operator. Here, we choose the linear operator as a Laplace operator $\mathcal{L} = L$. Obviously $\phi(x,t,0) = u_0(x,t)$ and $\phi(x,t,1) = h(x,t)$. Expanding $\phi(x,t,q)$ in Taylor series with respect q,

$$\phi(x,t,q) = \sum_{i=0}^{n} h_i(x,t)q^i,$$

where

$$h_i(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}|_{q=0}, \tag{9}$$

by differentiating (8) m times with respect q, and taking $q = 0$, we obtain the mth order deformation equation

$$L[h_m(x,t) - \ell_m h_{m-1}(x,t)] = \hbar H(x,t) R_m(h_{m-1}(x,t)). \tag{10}$$

Applying L^{-1} in (10) we get:

$$h_m(x,t) = \ell_m h_{m-1}(x,t) + \hbar L^{-1}[R_m(h_{m-1}(x,t))], \tag{11}$$

where

$$\ell_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Now, for the system (2), we define a system of nonlinear operator as:

$$\begin{aligned} N_1[\phi_(x,t,q), \psi_(x,t,q)] &= L[\phi] - \frac{1}{s}u(x,0) - \frac{1}{s^\alpha}(\eta\frac{\partial^3\phi}{\partial x^3} + \gamma\phi\frac{\partial\phi}{\partial x} + \mu\psi\frac{\partial\psi}{\partial x}), \\ N_2[\phi_(x,t,q), \psi_(x,t,q)] &= L[\psi] - \frac{1}{s}v(x,0) - \frac{1}{s^\beta}(\lambda\frac{\partial^3\psi}{\partial x^3} - \nu\phi\frac{\partial\psi}{\partial x}) \end{aligned} \tag{12}$$

the mth order deformation equation is

$$\begin{aligned} L[u_m(x,t) - l_m u_{m-1}(x,t)] &= \hbar H(x,t) K_m[\overrightarrow{u}_{m-1}(x,t), \overrightarrow{v}_{m-1}(x,t)], \\ L[v_m(x,t) - l_m v_{m-1}(x,t)] &= \hbar H(x,t) R_m[\overrightarrow{u}_{m-1}(x,t), \overrightarrow{v}_{m-1}(x,t)], \end{aligned} \tag{13}$$

where

$$\begin{aligned} K_n &= u_{n-1}(x,t) - \frac{(1-l_n)}{s}u_0(x,t) \\ &\quad + \frac{1}{s^\alpha}(L[\eta\frac{\partial^3 u_{n-1}}{\partial x^3} + \gamma\sum_{i=0}^{n-1} u_i(x,t)\frac{\partial u_{n-1-i}}{\partial x} + \mu\sum_{i=0}^{n-1-i} v_i\frac{\partial v_{n-1-i}}{\partial x}]), \\ R_n &= v_{n-1}(x,t) - \frac{(1-l_n)}{s}v_0(x,t) \\ &\quad + \frac{1}{s^\beta}(L[\lambda\frac{\partial^3 v_{n-1}}{\partial x^3} - \nu\sum_{i=0}^{n-1} u_i(x,t)\frac{\partial v_{n-1-i}}{\partial x}]). \end{aligned}$$

By applying the inverse Laplace operator for (13), we obtain:

$$
\begin{aligned}
u_m(x,t) &= \ell_m u_{m-1}(x,t) + \hbar L^{-1} K_m[\overrightarrow{u}_m(x,t), \overrightarrow{v}_m(x,t)],\\
v_m(x,t) &= \ell_m v_{m-1}(x,t) + \hbar L^{-1} R_m[\overrightarrow{u}_m(x,t), \overrightarrow{v}_m(x,t)],
\end{aligned}
\tag{14}
$$

where L^{-1} is the inverse Laplace transform operator, and in this case, we mention that $u(x,t)$, and $v(x,t)$ is written as a series

$$
\begin{aligned}
u(x,t) &= u_0(x,t) + \sum_{i=1}^{\infty} u_i(x,t),\\
v(x,t) &= v_0(x,t) + \sum_{i=1}^{\infty} v_i(x,t),
\end{aligned}
\tag{15}
$$

with the initial conditions,

$$
u(x,0) = u_0(x,t), \qquad v(x,0) = v_0(x,t).
$$

4. Application

In this section, we evaluate the coupled Korteweg-de Vries (2) using HATM with two different initial conditions.

Example 4.1. We consider (2) with $\eta = -a, \gamma = -6a, \mu = 2b, \lambda = -r$ and $\nu = 3r$:

$$
u(x,0) = \tfrac{\zeta}{a}\left(sech(\tfrac{1}{2}\sqrt{\tfrac{\zeta}{a}}x)\right)^2, \qquad v(x,0) = \tfrac{\zeta}{\sqrt{2a}}\left(sech(\tfrac{1}{2}\sqrt{\tfrac{\zeta}{a}}x)\right)^2. \tag{16}
$$

for $\eta = -1, \gamma = -6, \mu = 6, \lambda = -1$ and $\nu = 3$, we obtained the first-order approximation :

$$
\begin{aligned}
u_0(x,t) &= (sech(\frac{x}{2}))^2,\\
u_1(x,t) &= -\frac{\hbar t^\alpha (sech[\frac{x}{2}]^4 \tanh[\frac{x}{2}]^2 + sech[\frac{x}{2}]^2 \tanh[\frac{x}{2}]^3)}{\Gamma[\alpha+1]},\\
u_2(x,t) &= -\frac{\hbar t^\alpha (sech[\frac{x}{2}]^4 \tanh[\frac{x}{2}]^2 + sech[\frac{x}{2}]^2 \tanh[\frac{x}{2}]^3)}{\Gamma[\alpha+1]}\\
&\quad + \hbar(\frac{\hbar t^\alpha \operatorname{sech}^6(\frac{x}{2})(t^\alpha(22\cosh(x) + \cosh(2x) - 39)\Gamma(\alpha+\beta+1) - 12\Gamma(2\alpha+1)t^\beta(2\cosh(x)-3))}{8\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)}\\
&\quad - \frac{\hbar t^\alpha \left(\tanh\left(\frac{x}{2}\right)\operatorname{sech}^4\left(\frac{x}{2}\right) + \tanh^3\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right)}{\Gamma(\alpha+1)}),
\end{aligned}
$$

for $v(x,t)$, we have:

$$
\begin{aligned}
v_0(x,t) &= \frac{\operatorname{sech}^2\left(\frac{x}{2}\right)}{\sqrt{2}},\\
v_1(x,t) &= \frac{\sqrt{2}ht^\beta \tanh\left(\frac{x}{2}\right)\operatorname{sech}^4\left(\frac{x}{2}\right)}{\Gamma(\beta+1)} - \frac{3ht^\beta \tanh\left(\frac{x}{2}\right)\operatorname{sech}^4\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)}\\
&\quad - \frac{ht^\beta \tanh^3\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)},\\
v_2(x,t) &= -\frac{3h^2\operatorname{sech}^6\left(\frac{x}{2}\right)t^{\alpha+\beta}}{2\sqrt{2}\Gamma(\alpha+\beta+1)} + \frac{3h^2\cosh(x)\operatorname{sech}^6\left(\frac{x}{2}\right)t^{\alpha+\beta}}{2\sqrt{2}\Gamma(\alpha+\beta+1)}\\
&\quad + \frac{9h^2t^{2\beta}\operatorname{sech}^6\left(\frac{x}{2}\right)}{8\sqrt{2}\Gamma(2\beta+1)} - \frac{7h^2t^{2\beta}\cosh(x)\operatorname{sech}^6\left(\frac{x}{2}\right)}{4\sqrt{2}\Gamma(2\beta+1)}\\
&\quad + \frac{h^2t^{2\beta}\cosh(2x)\operatorname{sech}^6\left(\frac{x}{2}\right)}{8\sqrt{2}\Gamma(2\beta+1)} - \frac{3ht^\beta \tanh\left(\frac{x}{2}\right)\operatorname{sech}^4\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)}\\
&\quad + \frac{\sqrt{2}ht^\beta \tanh\left(\frac{x}{2}\right)\operatorname{sech}^4\left(\frac{x}{2}\right)}{\Gamma(\beta+1)} + \ldots
\end{aligned}
$$

Example 4.2. For the second test example, we consider (2), with the new initial conditions

$$
u(x,0) = \frac{4c^2e^{cx}}{(1+e^{cx})^2}, \quad v(x,0) = \frac{4c^2e^{cx}}{(1+e^{cx})^2}. \tag{17}
$$

Applying the HATM, with $\eta=-1, \gamma=-6, \mu=3, \lambda=-1$, and $\nu=3$, we obtain

$$
\begin{aligned}
u_0(x,t) &= \frac{4c^2e^{cx}}{(e^{cx}+1)^2},\\
u_1(x,t) &= -\frac{1}{\Gamma(\alpha+1)}\Bigg(- \frac{12c^2e^{cx}\left(\frac{4c^3e^{cx}}{(e^{cx}+1)^2} - \frac{8c^3e^{2cx}}{(e^{cx}+1)^3}\right)}{(e^{cx}+1)^2}\\
&\quad - 4c^2\Bigg(\frac{c^3e^{cx}}{(e^{cx}+1)^2} - \frac{6c^3e^{2cx}}{(e^{cx}+1)^3} + 3ce^{cx}\left(\frac{6c^2e^{2cx}}{(e^{cx}+1)^4} - \frac{2c^2e^{cx}}{(e^{cx}+1)^3}\right)\\
&\quad + \hbar t^\alpha e^{cx}\left(- \frac{2c^3e^{cx}}{(e^{cx}+1)^3} + \frac{18c^3e^{2cx}}{(e^{cx}+1)^4} - \frac{24c^3e^{3cx}}{(e^{cx}+1)^5}\right)\Bigg)\Bigg),\\
u_2(x,t) &= -\frac{ht^\alpha}{\Gamma(\alpha+1)}\Bigg(- \frac{12c^2e^{cx}\left(\frac{4c^3e^{cx}}{(e^{cx}+1)^2} - \frac{8c^3e^{2cx}}{(e^{cx}+1)^3}\right)}{(e^{cx}+1)^2}\\
&\quad - 4c^2\Bigg(\frac{c^3e^{cx}}{(e^{cx}+1)^2} - \frac{6c^3e^{2cx}}{(e^{cx}+1)^3}\\
&\quad + e^{cx}\left(- \frac{2c^3e^{cx}}{(e^{cx}+1)^3} + \frac{18c^3e^{2cx}}{(e^{cx}+1)^4} - \frac{24c^3e^{3cx}}{(e^{cx}+1)^5}\right)\\
&\quad + 3ce^{cx}\left(\frac{6c^2e^{2cx}}{(e^{cx}+1)^4} - \frac{2c^2e^{cx}}{(e^{cx}+1)^3}\right)\Bigg)\Bigg) + \ldots
\end{aligned}
$$

and for $v(x,t)$, we obtain

$$
\begin{aligned}
v_0(x,t) &= \frac{4c^2e^{cx}}{(e^{cx}+1)^2},\\
v_1(x,t) &= \frac{4c^5\hbar e^{cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^2} - \frac{56c^5\hbar e^{2cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^3}\\
&+ \frac{48c^5\hbar e^{2cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^4} + \frac{144c^5\hbar e^{3cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^4}\\
&- \frac{96c^5\hbar e^{3cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^5} - \frac{96c^5\hbar e^{4cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^5},\\
v_2(x,t) &= \frac{48c^5\hbar e^{2cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^4} + \frac{144c^5\hbar e^{3cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^4}\\
&- \frac{96c^5\hbar e^{3cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^5} - \frac{96c^5\hbar e^{4cx}t^\beta}{\Gamma(\beta+1)(e^{cx}+1)^5}\\
&+ \frac{48c^8\hbar^2 e^{4cx}t^{\alpha+\beta}}{(e^{cx}+1)^6\Gamma(\alpha+\beta+1)} + \frac{4c^8\hbar^2 e^{cx}t^{2\beta}}{\Gamma(2\beta+1)(e^{cx}+1)^6}\\
&- \frac{56c^8\hbar^2 e^{2cx}t^{2\beta}}{\Gamma(2\beta+1)(e^{cx}+1)^6} + \frac{72c^8\hbar^2 e^{3cx}t^{2\beta}}{\Gamma(2\beta+1)(e^{cx}+1)^6} + \cdots .
\end{aligned}
$$

5. Convergence analysis

Theorem 5.1. *If the series solutions $\sum_{i=0}^n u_i(x,t)$, and $\sum_{i=0}^n v_i(x,t)$ are convergent, where u_m and v_m are governed by equation* (9)*, then must be solutions of* (11)*.*

Proof of Theorem 5.1. *Suppose that $\sum_{i=0}^n u_i(x,t)$ and $\sum_{i=0}^n v_i(x,t)$ are convergent, i.e $\lim_{n\longrightarrow\infty} u_n(x,t) = \lim_{n\longrightarrow\infty} v_n(x,t) = 0$. From the equation* (13)*, we get:*

$$
\begin{aligned}
\hbar H \sum_{m=1}^\infty K_m &= \lim_{k\longrightarrow\infty}\sum_{m=0}^k L[u_m - \ell_m u_{m-1}]\\
&= L[\lim_{k\longrightarrow\infty}\sum_{m=0}^k[u_m - \ell_m u_{m-1}]] = L[\lim_{k\longrightarrow\infty} u_k],
\end{aligned}
$$

where L is a Laplace operator. Since $\lim_{k\longrightarrow\infty} u_k = 0$, $H \neq 0$ and $\hbar \neq 0$, this implies $\sum_{m=1}^\infty K_m = 0$. Now, expand $N_1[\Phi(x,t,q), \psi(x,t,q)]$ about $q = 0$, then set $q = 1$,

$$N_1[\Phi(x,t,1), \psi(x,t,q)] = 0,$$

we can see that $u(x,t) = \Phi(x,t,1) = \sum_{i=0}^\infty u_i(x,t)$, and $v(x,t) = \psi(x,t,1) = \sum_{i=0}^\infty v_i(x,t)$ solve equation (2)*.*

Theorem 5.2 ([20])**.** *Let the solution component $h_0(x,t), h_1(x,t), h_2(x,t), \ldots$ be defined as* (11)*. The series solution $\sum_{m=0}^\infty h_m(x,t)$ defined in* (15) *converge if there exist $0 < \gamma < 1$ such that $\|h_{m+1}(x,t)\| \leq \gamma\|h_m(x,t)\|, \forall m > m_0$, for some $m_0 \in \mathbb{N}$.*

Theorem 5.3 ([20]). *Suppose that the series solution $\sum_{m=0}^{\infty} h_m(x,t)$ is convergent to the solution $h(x,t)$. If the truncated series $\sum_{m=0}^{k} h_m(x,t)$ is used as approximation to the solution $h(x,t)$, then the truncated error satisfies*

$$\| h(x,t) - \sum_{m=0}^{k} h_m(x,t) \| \leq \frac{1}{1-\gamma}\gamma^{k+1} \| h_0(x,t) \| .$$

6. Numerical result and discussion

In Fig. 2 and 3, we plot the 5-th order approximation solutions and the exact solution of (2), we can see that the graphs coincide with the graphs of the exact solution.

Now, to control the convergence of the series in the frame of HATM, by the auxiliary convergence parameter $\hbar$, we plot the so-called $\hbar$-curve for $u_{tt}(0,0)(\hbar)$ and $v_{tt}(0,0)(\hbar)$ In Fig1,4,6 and 8. we can see from those figures that the best value of $\hbar$ for the first example is in the convergence region $-1.2 \leq \hbar \leq -0.6$, and for the second example is $-1.3 \leq \hbar \leq -0.7$ so we have a freedom to choice $\hbar$ in this region to get a convergence series. In table 6.1, and 6.3 we compare our results with other result obtained using the NDM solutions [1], and the spectral collection method [2]. In Table 6.2, the approximation solutions for different value of α, and β are given.

Table 6.1. *Comparison of our results of $u(x,t)$, and $v(x,t)$ with other result obtained by other numerical methods for different values of $\alpha = \beta = 1$.*

x	t	*Exact($u(x,t)$)*	*HATM*	*Error (HATM)*	*Error(NDM)[1]*	*Error[2]*
-10	0.1	0.000164304	0.000164304	2.47239×10^{-13}	2.95039×10^{-8}	2.99039×10^{-8}
-10	0.2	0.000148670	0.000148670	1.56036×10^{-11}	2.30335×10^{-7}	2.33335×10^{-7}
-5	0.1	0.024092321	0.024092321	8.91166×10^{-12}	3.93592×10^{-6}	3.96592×10^{-6}
-5	0.2	0.021824797	0.021824797	5.83638×10^{-10}	3.08049×10^{-5}	3.38049×10^{-5}
5	0.1	0.029347625	0.029347625	8.43378×10^{-12}	2.02966×10^{-4}	3.97592×10^{-6}
5	0.2	0.032383774	0.032383773	5.22271×10^{-10}	5.15558×10^{-4}	3.78049×10^{-5}
10	0.1	0.000200678	0.000200678	2.54363×10^{-13}	2.11254×10^{-8}	2.96039×10^{-8}
10	0.2	0.000221781	0.000221781	1.65158×10^{-11}	2.28173×10^{-7}	2.37335×10^{-7}
x	t	*Exact($v(x,t)$)*	*HATM*	*Error (HATM)*	*Error(NDM)[1]*	*Error[2]*
-10	0.1	0.000116180	0.000116180	1.74825×10^{-13}	2.08624×10^{-8}	2.18624×10^{-8}
-10	0.2	0.000105125	0.000105125	1.10334×10^{-11}	1.62872×10^{-7}	1.64872×10^{-7}
-5	0.1	0.017035843	0.017035843	6.30149×10^{-12}	2.78312×10^{-6}	2.88312×10^{-6}
-5	0.2	0.015432462	0.015432462	4.12694×10^{-10}	2.21782×10^{-5}	2.87824×10^{-5}
5	0.1	0.020751905	0.020751905	5.96358×10^{-12}	2.9094×10^{-6}	2.98312×10^{-6}
5	0.2	0.022898786	0.022848786	3.69301×10^{-10}	2.238036×10^{-5}	2.47824×10^{-5}
10	0.1	0.000141901	0.000141901	1.79862×10^{-13}	2.19313×10^{-8}	2.09624×10^{-8}
10	0.2	0.000156823	0.000156823	1.16785×10^{-11}	1.79991×10^{-7}	1.72872×10^{-7}

In the above tables, we can see that the approximation solution obtained by HATM, is agreed and appropriate with the exact solution more than the NDM and spectral collection methods.

Table 6.2. *Numerical solutions of $u(x,t)$, and $v(x,t)$ for different values of α,β. Numerical solutions of $u(x,t)$, and $v(x,t)$ for different values of α,β*

x	t	*Exact* $u(x,t)$	$\alpha=\beta=0.9$	$\alpha=\beta=0.8$	$\alpha=\beta=0.6$
-10	0.1	0.000164304	0.000159449	0.000153648	0.000139462
-10	0.2	0.000148670	0.000142868	0.000136314	0.000123400
-5	0.1	0.024092321	0.023388290	0.022546135	0.020472920
-5	0.2	0.021824797	0.020954214	0.020024033	0.018074633
5	0.1	0.029347625	0.030919078	0.031537740	0.035657141
5	0.2	0.032383774	0.033919078	0.035900711	0.041940955
10	0.1	0.000200678	0.000207156	0.000215926	0.000244977
10	0.2	0.000221781	0.000232514	0.000246460	0.000290077
x	t	*Exact* $v(x,t)$	$\alpha=\beta=0.9$	$\alpha=\beta=0.8$	$\alpha=\beta=0.6$
-10	0.1	0.000116180	0.000112747	0.000108646	0.000098614
-10	0.2	0.000105125	0.000100894	0.000096389	0.000087257
-5	0.1	0.017035843	0.016538018	0.015942525	0.014476541
-5	0.2	0.015432462	0.014816867	0.014159129	0.012780695
5	0.1	0.020751905	0.021410870	0.022300550	0.025213406
5	0.2	0.022898786	0.023984410	0.025385636	0.029656733
10	0.1	0.000141901	0.000146481	0.000152682	0.000173225
10	0.2	0.000156823	0.000164412	0.000174273	0.000205115

For the second example, we compare with the exact solution, the approximate solutions for [3] in Table 5.3. In Fig. 4 the 10-th order of solution with the exact solution $u(x,t) = \frac{\zeta}{a}(\mathrm{sech}(\frac{1}{2\sqrt{\frac{\zeta}{a}}(x-\zeta t)}))^2, u(x,t) = \frac{\zeta}{\sqrt{2a}}(\mathrm{sech}(\frac{1}{2\sqrt{\frac{\zeta}{a}}(x-\zeta t)}))^2$ for example1 are presented, and $u(x,t) = v(x,t) = \frac{4c^2 e^{c(x-c^2 t)}}{(1+e^{c(x-c^2 t)})^2}$ for the second example. In Fig 6, the graphs traced for different values of α,β.

Table 6.3. *Comparison of numerical solutions of $u(x,t)$, and $v(x,t)$ with other obtained by other numerical methods for different value of $\alpha=\beta=1$.*

x	t	*Exact*	*HATM*($\alpha=\beta=1$)	*Error (HATM)*	*Error(NDM)[1]*	*Error[3]*
-10	0.1	0.000164304	0.000164304	2.47240×10^{-13}	2.95039×10^{-8}	2.95039×10^{-8}
-10	0.2	0.000148670	0.000148670	1.56036×10^{-11}	2.30335×10^{-7}	2.30335×10^{-7}
-5	0.1	0.024092321	0.024092321	8.91164×10^{-12}	3.93592×10^{-6}	3.93592×10^{-6}
-5	0.2	0.021824797	0.021824797	5.83638×10^{-10}	3.08049×10^{-5}	3.08049×10^{-5}
5	0.1	0.029347625	0.029347625	8.43377×10^{-12}	4.11452×10^{-6}	4.11452×10^{-6}
5	0.2	0.032383774	0.032383773	5.22271×10^{-10}	3.36633×10^{-5}	3.36633×10^{-5}
10	0.1	0.000200678	0.000200678	2.54363×10^{-13}	3.10155×10^{-8}	3.10155×10^{-8}
10	0.2	0.000221781	0.000221781	1.65158×10^{-11}	2.54546×10^{-7}	2.54546×10^{-7}
x	t	*Exact*	*HATM*($\alpha=\beta=1$)	*Error (HATM)*	*Error(NDM)[1]*	*Error[3]*
-10	0.1	0.000164304	0.000164304	2.47240×10^{-13}	2.95039×10^{-8}	2.95039×10^{-8}
-10	0.2	0.000148670	0.000148670	1.56036×10^{-11}	2.30335×10^{-7}	2.30335×10^{-7}
-5	0.1	0.024092321	0.024092321	8.91164×10^{-12}	3.93592×10^{-6}	3.93592×10^{-6}
-5	0.2	0.021824797	0.021824797	5.83638×10^{-10}	3.08049×10^{-5}	3.08049×10^{-5}
5	0.1	0.029347625	0.029347625	8.43375×10^{-12}	4.11452×10^{-6}	4.11452×10^{-6}
5	0.2	0.032383774	0.032383773	5.22277×10^{-10}	3.36633×10^{-5}	3.36633×10^{-5}
10	0.1	0.000200678	0.000200678	2.54363×10^{-13}	3.10155×10^{-8}	3.10155×10^{-8}
10	0.2	0.000221781	0.000221781	1.65158×10^{-11}	2.54546×10^{-7}	2.54546×10^{-7}

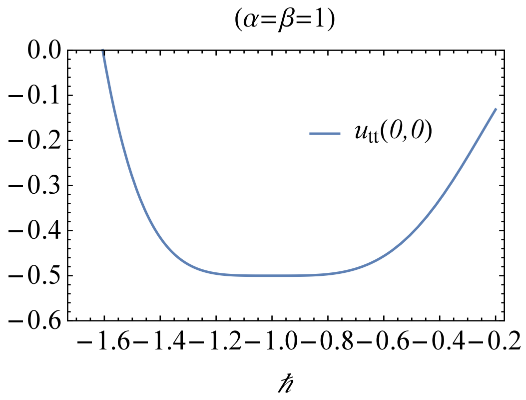

Figure 1: The $\hbar$−curve for the example 1

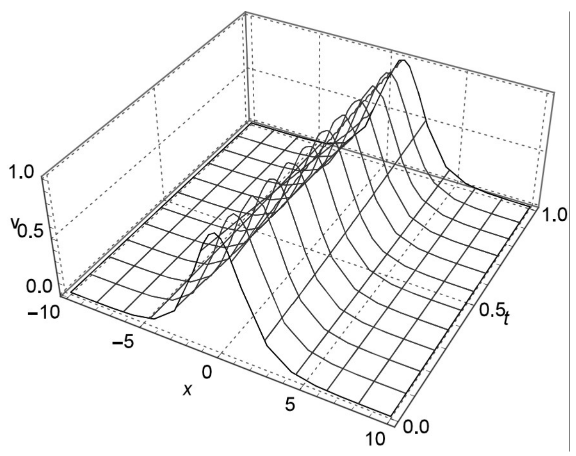

Figure 2: The approximate solution $u(x,t)$ for example 1

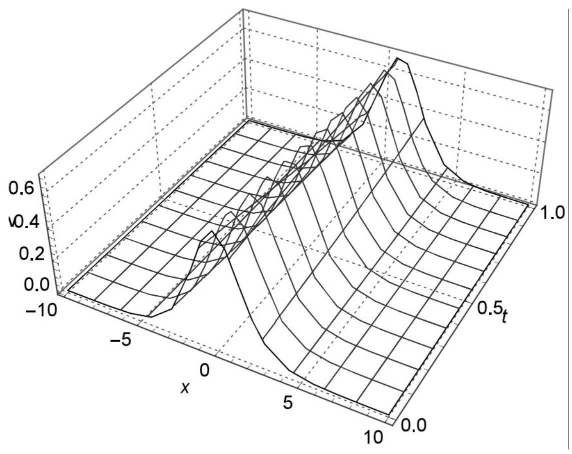

Figure 3: The approximate solution $v(x,t)$ for example 1

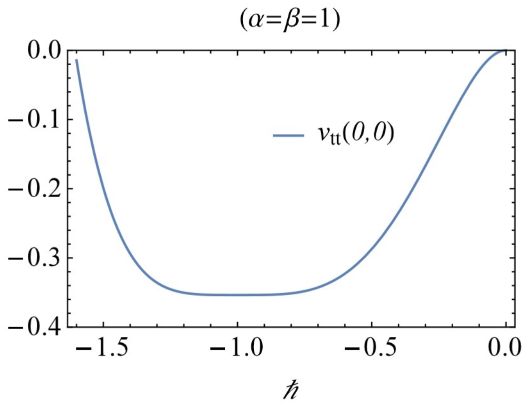

Figure 4: The $\hbar-$curve for the example 1

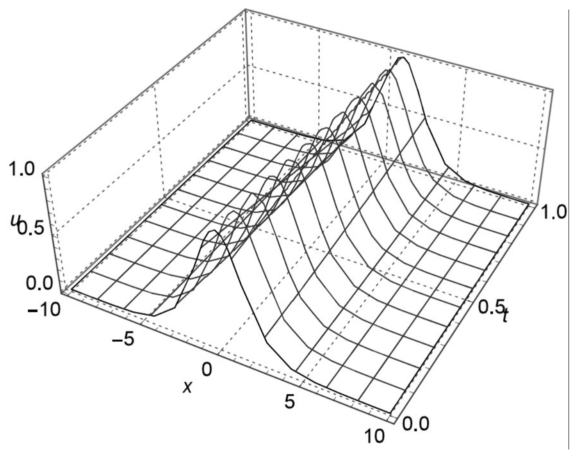

Figure 5: The approximate solution $u(x,t)$ for example 2

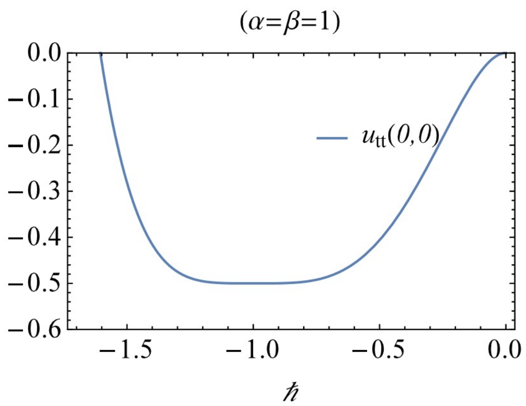

Figure 6: The $\hbar$−curve for the example 1

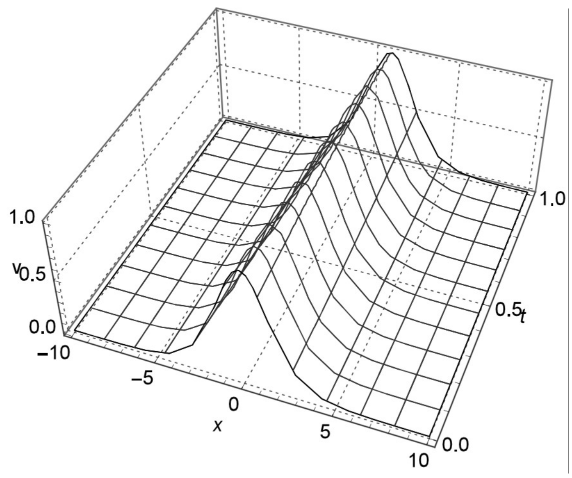

Figure 7: The approximate solution $v(x,t)$ for example 2

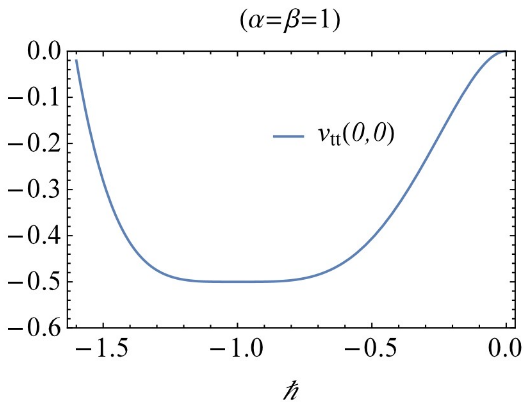

Figure 8: The $\hbar$−curve for the example 1

Concluding remarks

In this work, we successfully applied the HATM for a coupled Korteweg-de Vries equation, and we show that this method is efficient and applicable for this type of fractional differential equations, we compared our results with others obtained by different numerical method. The algorithm is powerfully for solving this kinds of equations.

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