Ricci soliton on Sasakian manifolds admitting Zamkovoy connection

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Abstract. Object of this paper is to study Ricci soliton on concircularly flat, W_2-flat, W_3-flat, W_4-flat Sasakian manifolds with respect to Zamkovoy connection. Besides these, we discuss Ricci soliton on a Sasakian manifold satisfying $W_2^*(\xi, Y).R^* = 0$, where R^* denotes Riemannian curvature tensor with respect to Zamkovoy connection and W_2^*-denotes the W_2-curvature tensor with respect to Zamkovoy connection.
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1. Introduction

The notion of Sasakian structure [16] was introduced by Japanese mathematician S. Sasaki in the year 1960. If a contact metric structure is normal then the structure is said to have a normal contact metric structure or a Sasakian structure. Thus a manifold with Sasakian structure is a normal contact metric manifold. In some respect Sasakian manifold may be viewed as an odd dimensional analogous of Kâhler manifold. Sasakian manifold was further studied by many authors. For details, we refer ([4], [9], [11], [5]) and the references therein.

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The concircular curvature tensor on Riemannian manifold was introduced and defined by K. Yano [20]. The concircular curvature tensor of rank 3 is given by

$$W(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{1}$$

The W_2-curvature tensor [12], W_3-curvature tensor [13] and W_4-curvature tensor [13] are, respectively, given by

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY], \tag{2}$$

$$W_3(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(X,Z)Y - g(Y,Z)QX], \tag{3}$$

$$W_4(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ], \tag{4}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is set of all vector fields of the manifold M and $R(X,Y)Z$ is Riemannian curvature tensor and r, S, Q are the scalar curvature, Ricci tensor, Ricci operator, respectively.

The notion of Ricci flow was first introduced by R. S. Hamilton in the early 1980s. R. S. Hamilton [6] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S, \tag{5}$$

where g is Riemannian metric, S is Ricci tensor and t is time. Solitons for the Ricci flow is the solutions of the above equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ), where g is Riemannian metric, V is a vector field and λ is a scalar, which satisfies the equation:

$$L_V g + 2S + 2\lambda g = 0, \tag{6}$$

where S is Ricci curvature tensor, $L_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$ respectively. If the vector field V is gradient of a smooth function, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton and the associated function is called the potential function. Ricci soliton, was further studied by many researcher. For instance, we see ([10], [14], [17], [18]) and their references.

In 2008, S. Zamkovoy [21] introduced the notion of a new canonical connection named as Zamkovoy connection for para-contact manifolds. And this connection was defined as a canonical paracontact connection whose torsion is the obstruction of paracontact manifold to be a para Sasakian manifold. Zamkovoy connection was further studied by many authors et al. ([7], [8], [1], [2], [3]). The Zamkovoy connection ∇^* for an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined as

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y, \tag{7}$$

for all $X, Y \in \chi(M)$.

Definition 1.1. *An n-dimensional Sasakian manifold M is said to be concircularly flat with respect to Zamkovoy connection if $W^*(X,Y)Z = 0$, for all X, Y, $Z \in \chi(M)$.*

Definition 1.2. *An n-dimensional Sasakian manifold M is said to be W_2- flat with respect to Zamkovoy connection if $W_2^*(X,Y)Z = 0$, for all X, Y, $Z \in \chi(M)$.*

Definition 1.3. *An n-dimensional Sasakian manifold M is said to be W_3- flat with respect to Zamkovoy connection if $W_3^*(X,Y)Z = 0$, for all X, Y, $Z \in \chi(M)$.*

Definition 1.4. *An n-dimensional Sasakian manifold M is said to be W_4- flat with respect to Zamkovoy connection if $W_4^*(X,Y)Z = 0$, for all X, Y, $Z \in \chi(M)$.*

Definition 1.5. *An n-dimensional Sasakian manifold M is said to be η-Einstein manifold if its Ricci tensor is of the form*

$$S(X,Y) = k_1 g(X,Y) + k_2 \eta(X)\eta(Y),$$

for all X, $Y \in \chi(M)$, where k_1 and k_2 are scalars.

This paper is organized as follows:

After introduction a short description of Sasakian manifold is given in section (2). In section (3), we have discussed some properties of Sasakian manifold with respect to Zamkovoy connection ∇^* and obtained Riemannian curvature tensor R^*, Ricci tensor S^*, scalar curvature tensor r^*, Ricci operator Q^* with respect to ∇^*. Section (4), section (5), section (6) and section (7) contain Ricci soliton on concircularly flat, W_2-flat, W_3-flat and W_4- flat Sasakian manifolds, respectively, with respect to the Zamkovoy connection. In section (8), we have discussed Ricci soliton on Sasakian manifold satisfying $W_2^*(\xi,Y).R^* = 0$.

2. Preliminaries

Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ , a 1-form η and a Riemannian metric g satisfying

$$\phi^2 Y = -Y + \eta(Y)\xi, \eta(\xi) = 1, \eta(\phi X) = 0, \ \phi\xi = 0, \tag{8}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{9}$$
$$g(X, \phi Y) = -g(\phi X, Y), \eta(Y) = g(Y, \xi), \tag{10}$$

for all X, $Y \in \chi(M)$.

An almost contact metric manifold M is said to be Sasakian manifold if the following relations hold in M

$$g(X, \phi Y) = \nabla\eta(X, Y), \tag{11}$$
$$\nabla_X \xi = -\phi X, \tag{12}$$
$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{13}$$
$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{14}$$

for all X, $Y \in \chi(M)$, where ∇_X denotes the covariant differentiation with respect to X.

In a Sasakian manifold equipped with the structure (ϕ, ξ, η, g) the following relations also hold ([19], [15]):

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{15}$$
$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{16}$$
$$S(X, \xi) = (n-1)\eta(X), \tag{17}$$
$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{18}$$
$$Q\xi = (n-1)\xi, \tag{19}$$
$$S(X, Y) = g(QX, Y), S^2(X, Y) = S(QX, Y). \tag{20}$$

In Sasakian manifold, using (12) and (15) equation (7) reduces to

$$\nabla^*_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y, \tag{21}$$

with torsion tensor $T^*(X, Y) = 2g(X, \phi Y)\xi$.

3. Some properties of Sasakian manifold with respect to Zamkovoy connection

By the the help of (21), (12) and (13) we get the following results:

$$\nabla^*_X \eta(Y) = \eta(\nabla_X Y) + g(X, \phi Y), \tag{22}$$

$$\begin{aligned} \nabla^*_X(\phi Y) &= \nabla_X(\phi Y) - g(\phi X, \phi Y)\xi \\ &\quad -\eta(X)Y + \eta(X)\eta(Y)\xi, \end{aligned} \tag{23}$$

$$\begin{aligned} \nabla^*_X g(Y, \phi Z) &= g(\nabla_X Y, \phi Z) + \eta(X)g(\phi Y, \phi Z) + g(Y, \nabla_X(\phi Z)) \\ &\quad -\eta(X)g(Y, Z) + \eta(X)\eta(Y)\eta(Z) \end{aligned} \tag{24}$$

Let R^* be the Riemannian curvature tensor with respect to Zamkovoy connection which is defined as

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z. \tag{25}$$

By the help of (21), (22), (23) and (24) we obtain the expressions for $\nabla^*_X \nabla^*_Y Z$, $\nabla^*_Y \nabla^*_X Z$ and $\nabla^*_{[X,Y]} Z$ to get the Riemannian curvature tensor with respect to Zamkovoy connection as

$$\begin{aligned} R^*(X,Y)Z &= R(X,Y)Z - g(Z,\phi X)\phi Y - g(Y,\phi Z)\phi X \\ &\quad -2g(Y,\phi X)\phi Z + g(X,Z)\eta(Y)\xi - \eta(X)g(Y,Z)\xi \\ &\quad +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X, \end{aligned} \tag{26}$$

Consequently one can easily bring out the followings:

$$S^*(Y,Z) = S(Y,Z) + 2g(Y,Z) - (1+n)\eta(Y)\eta(Z), \tag{27}$$
$$S^*(Y,\xi) = 0 = S^*(\xi,Z), \tag{28}$$
$$Q^*Y = QY + 2Y - (1+n)\eta(Y)\xi, \tag{29}$$
$$Q^*\xi = 0 \tag{30}$$
$$r^* = r + n - 1, \tag{31}$$
$$R^*(X,Y)\xi = 0, \tag{32}$$
$$R^*(\xi,Y)Z = 0, \tag{33}$$
$$R^*(X,\xi)Z = 0. \tag{34}$$

Let M be an n-dimensional Sasakian manifold admitting Zamkovoy connection ∇^*, then (i) The curvature tensor R^* of ∇^* is given by (26), (ii) The Ricci tensor S^* of ∇^* is given by (27), (iii) The scalar curvature r^* of ∇^* is given by (31), (iv) The Ricci tensor S^* of ∇^* is symmetric.

Consider a Ricci soliton (g,ξ,λ) on M, then from (6) we have

$$\begin{aligned} 0 &= \left(L_\xi g\right)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) \\ &= -g(\phi Y,Z) + g(\phi Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) \\ &= S(Y,Z) + \lambda g(Y,Z). \end{aligned} \tag{35}$$

Setting $Z=\xi$ in (35), we get

$$S(Y,\xi) = -\lambda\eta(Y). \tag{36}$$

Replacing Y by QY in (36), we get

$$S^2(Y,\xi) = \lambda^2\eta(Y). \tag{37}$$

4. Ricci soliton on concircularly flat Sasakian manifold with respect to Zamkovoy connection

Theorem 4.1. *If a concircularly flat Sasakian manifold M admits a Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection, then the Ricci soliton is steady, shrinking or expanding according as $r = (n^2+1), r > (n^2+1),$ or $< (n^2+1)$.*

Proof 4.1. Let M be a concircularly flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W^*(X,Y)Z = 0$, where W^* is the concircular curvature tensor with respect to Zamkovoy connection and X, Y, $Z \in \chi(M)$.

For a concircularly flat Sasakian manifold, it follows from (1) that

$$R^*(X,Y)Z = \frac{r^*}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{38}$$

where R^* is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product of (38) with a vector field V we get

$$g(R^*(X,Y)Z,V) = \frac{r+n-1}{n(n-1)}[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)]. \tag{39}$$

Taking an orthonormal frame field and contracting (39) over X and V we get

$$S^*(Y,Z) = \frac{r+n-1}{n}g(Y,Z). \tag{40}$$

Using (27) in (40) and setting $Z = \xi$, we get

$$S(Y,\xi) = \frac{r-n-1}{n}\eta(Y) + (1+n)\eta(Y). \tag{41}$$

By the help of (36) and (41), we obtain $\lambda = -\frac{r-(n^2+1)}{n}$.

Therefore, the Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection is steady, shrinking or expanding according as $r = (n^2+1)$, $r > (n^2+1)$, *or* $r < (n^2+1)$.

5. Ricci soliton on W_2-flat Sasakian manifold with respect to Zamkovoy connection

Theorem 5.1. *If a W_2-flat Sasakian manifold M admits a Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection, then the Ricci soliton is steady, shrinking or expanding according as $r = (n^2+1)$, $r > (n^2+1)$, or $r < (n^2+1)$.*

Proof 5.1. Let M be a W_2-flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W_2^*(X,Y)Z = 0$, where W_2^* is the W_2-curvature tensor with respect to Zamkovoy connection and $X, Y, Z \in \chi(M)$.

For a W_2^*-flat Sasakian manifold, it follows from (2) that

$$R^*(X,Y)Z = \frac{1}{n-1}\left[g(Y,Z)Q^*X - g(X,Z)Q^*Y\right], \tag{42}$$

where R^* is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product of (42) with a vector field V we get

$$g(R^*(X,Y)Z,V) = \frac{1}{n-1}\left[g(Y,Z)g(Q^*X,V) - g(X,Z)g(Q^*Y,V)\right]. \tag{43}$$

Taking a frame field and contracting (43) over X and V we get

$$S^*(Y,Z) = \frac{1}{n-1}\left[r^*g(Y,Z) - S^*(Y,Z)\right]. \tag{44}$$

Using (27), (31) in (44), we get

$$nS(Y,Z) = (r-n-1)g(Y,Z) + n(n+1)\eta(Y)\eta(Z). \tag{45}$$

Setting $Z=\xi$ in (45), we have

$$nS(Y,\xi) = (r-n-1)\eta(Y) + n(n+1)\eta(Y) \tag{46}$$

By the help of (36) and (46), we obtain

$$\lambda = -\frac{r-(n^2+1)}{n}.$$

This gives the theorem.

6. Ricci soliton on W_3-flat Sasakian manifold with respect to Zamkovoy connection

Theorem 6.1. *If a W_3-flat Sasakian manifold M of dimension n (>3) admits a Ricci soliton (g,ξ,λ) with respect to Zamkovoy connection, then the Ricci soliton is steady, shrinking or expanding according as* $r = -(n-1)(n-3)$, $r < (n-1)(n-3)$ *or* $r > (n-1)(n-3)$.

Proof 6.1. Let M be a W_3-flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W_3^*(X,Y)Z = 0$, where W_3^* is the W_3-curvature tensor with respect to Zamkovoy connection and $X,Y,Z \in \chi(M)$.

For a W_3^*-flat Sasakian manifold, it follows from (3) that

$$R^*(X,Y)Z = \frac{1}{n-1}\left[S^*(X,Z)Y - g(Y,Z)Q^*X\right], \tag{47}$$

where R^* is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product of (47) with a vector field V we get

$$g\left(R^{*}\left(X,Y\right)Z,V\right)=\frac{1}{n-1}\left[S^{*}\left(X,Z\right)g\left(Y,V\right)-g\left(Y,Z\right)g\left(Q^{*}X,V\right)\right]. \tag{48}$$

Taking an orthonormal frame field and contracting (48) over X and V we get

$$S^{*}\left(Y,Z\right)=\frac{1}{n-1}\left[S^{*}\left(Y,Z\right)-r^{*}g\left(Y,Z\right)\right]. \tag{49}$$

Using (27), (31) in (49), we get

$$S\left(Y,Z\right)=\frac{(-r-3n+5)}{(n-2)}g\left(Y,Z\right)+(n+1)\,\eta\left(Y\right)\eta\left(Z\right). \tag{50}$$

Setting $Z=\xi$ in (50), we have

$$S\left(Y,\xi\right)=\frac{(-r-3n+5)}{(n-2)}\eta\left(Y\right)+(n+1)\,\eta\left(Y\right). \tag{51}$$

By the help of (36) and (51), we obtain

$$\lambda=\frac{r-(n-1)(n-3)}{n-2}.$$

Therefore, the Ricci soliton (g,ξ,λ) *with respect to Zamkovoy connection is steady, shrinking or expanding according as* $r=-(n-1)(n-3)$, $r<(n-1)(n-3)$ *or* $r>(n-1)(n-3)$.

Corollary 6.1. *If* (g,ξ,λ) *be a Ricci soliton withrespect to Zamkovoy connection on a 3-dimensional Sasakian manifold, then the Ricci soliton is steady, shrinking or expanding according as* $r=0$, $r<0$ *or* $r>0$.

7. Ricci soliton on W_4-flat Sasakian manifold with respect to the Zamkovoy connection

Theorem 7.1. *If a* W_4*-flat Sasakian manifold* M *admits a Ricci soliton* (g,ξ,λ) *with respect to Zamkovoy connection, then the Ricci soliton is always shrinking.*

Proof 7.1. Let M be a W_4-flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W_4^{*}\left(X,Y\right)Z=0$, where W_4^{*} is the W_4-curvature tensor with respect to Zamkovoy connection and $X,Y,Z\in\chi\left(M\right)$. For a W_4^{*}-flat Sasakian manifold, it follows from equation (4) that

$$R^{*}\left(X,Y\right)Z=-\frac{1}{n-1}\left[g\left(X,Z\right)Q^{*}Y-g\left(X,Y\right)Q^{*}Z\right], \tag{52}$$

where R^{*} is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product with a vector field V in (52), we get

$$g\left(R^{*}\left(X,Y\right)Z,V\right)=-\frac{1}{n-1}\left[g\left(X,Z\right)g\left(Q^{*}Y,V\right)-g\left(X,Y\right)g\left(Q^{*}Z,V\right)\right]. \tag{53}$$

Let $\{e_i\}\,(1\leq i\leq n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X=V=e_i$ in the equation (53) and taking summation over i $(1\leq i\leq n)$ and using (27), we get

$$S\left(Y,Z\right)=-2g\left(Y,Z\right)+\left(1+n\right)\eta\left(Y\right)\eta\left(Z\right). \tag{54}$$

Setting $Y=\xi$ in (54), we get

$$S\left(\xi,Z\right)=-2\eta\left(Z\right)+\left(1+n\right)\eta\left(Z\right). \tag{55}$$

Using (36) in (55), we get

$$\lambda=-\left(n-1\right)<0.$$

Therefore, the Ricci soliton (g,ξ,λ) with respect to Zamkovoy connection is always shrinking.

8. Ricci soliton on Sasakian manifold satisfying $W_2^{*}\left(\xi,Y\right).R^{*}=0$

Theorem 8.1. *If a Sasakian manifold M satisfying $W_2^{*}\left(\xi,Y\right).R^{*}=0$ admits a Ricci soliton (g,ξ,λ) with respect to Zamkovoy connection, then the Ricci soliton cannot be steady.*

Proof 8.1. Let us consider a Sasakian manifold admitting Zamkovoy connection and satisfying

$$\left(W_2^{*}\left(\xi,Y\right).R^{*}\right)\left(X,U\right)Z=0,$$

where $X,Y,Z,U\in\chi\left(M\right).$

Then we have

$$\begin{aligned}&\left(W_2^{*}\left(\xi,Y\right)R^{*}\right)\left(X,U\right)Z\\=\;&R^{*}\left(W_2^{*}\left(\xi,Y\right)X,U\right)Z+R^{*}\left(X,W_2^{*}\left(\xi,Y\right)U\right)Z\\&+R^{*}\left(X,U\right)W_2^{*}\left(\xi,Y\right)Z.\end{aligned} \tag{56}$$

Setting $U=\xi$ in (56), we have

$$\begin{aligned}&\left(W_2^{*}\left(\xi,Y\right)R^{*}\right)\left(X,\xi\right)Z\\=\;&R^{*}\left(W_2^{*}\left(\xi,Y\right)X,\xi\right)Z+R^{*}\left(X,W_2^{*}\left(\xi,Y\right)\xi\right)Z\\&+R^{*}\left(X,\xi\right)W_2^{*}\left(\xi,Y\right)Z.\end{aligned} \tag{57}$$

By the help of (34) and (57), we get

$$0=R^{*}\left(X,Q^{*}Y\right)Z. \tag{58}$$

Taking inner product of (58) with a vector field V, we get

$$g\left(R^{*}\left(X,Q^{*}Y\right)Z,V\right)=0. \tag{59}$$

Taking a frame field and contracting (59) over X and V and using (20), we get

$$0=S^{2}\left(Y,Z\right)+4S\left(Y,Z\right)+4g\left(Y,Z\right)-n\left(1+n\right)\eta\left(Y\right)\eta\left(Z\right) \tag{60}$$

Setting $Z=\xi$ in (60), we have

$$0=S^{2}\left(Y,\xi\right)+4S\left(Y,\xi\right)+4\eta\left(Y\right)-n\left(1+n\right)\eta\left(Y\right) \tag{61}$$

By the help of (37) and (61), we obtain

$$\lambda=2\pm\sqrt{n\left(n+1\right)}\neq 0.$$

Therefore, the Ricci soliton (g,ξ,λ) with respect to Zamkovoy connection cannot be steady.

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References

[1] A. Biswas, K. K. Baishya, *A general connection on Sasakian manifolds and the case of almost pseudo symmetric Sasakian manifolds*, Scientific Studies and Research Series Mathematics and Informatics, 29 (2019), 59-72.

[2] A. Biswas, K. K. Baishya, *Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection*, Bulletin of the Transilvania University of Brasov, 12 (2020), 233-246.

[3] A.M. Blaga, *Canonical connections on para-Kenmotsu manifolds,* Novi Sad J. Math, 45 (2015), 131-142.

[4] P.C. Boyer, Galicki Krzysztof, *3-Sasakian manifolds,* Surveys in Differential Geometry, 7 (1999), 123-184.

[5] M.C. Chaki, M. Tarafdar, *On a type of Sasakian manifold,* Soochow J. Math., 16 (1990), 23-28.

[6] R.S. Hamilton, *The Ricci flow on surfaces, Math. and General Relativity (Santa Cruz, CA, 1986)*, American Math. Soc. Contemp. Math, 7 (1988), 232-262.

[7] A. Mandal, A. Das, *On M-projective curvature tensor of Sasakian manifolds admitting Zamkovoy connection,* Adv. Math. Sci. J., 9 (2020), 8929-8940.

[8] A. Mandal, A. Das, *Projective Curvature Tensor with respect to Zamkovoy connection in Lorentzian para Sasakian manifolds,* J. Indones. Math. Soc., 26 (2020), 369-379.

[9] D. Martelli, J. Sparks, Shing-Tung Yau, *Sasaki-Einstein manifolds and volume minimization,* Communications in Mathematical Physics, 280 (2008), 611-673.

[10] H.G. Nagaraja, C.R. Premalatha, *Ricci solitons in Kenmotsu manifolds,* J. of Mathematical Analysis, 3 (2012), 18-24.

[11] Z. Olszak, *Certain property of the Ricci tensor on Sasakian manifolds,* Collo. Math., 40 (1978), 235-237.

[12] G.P. Pokhariyal, R.S. Mishra, *Curvature tensors and their relativistic significance,* Yokohama Math. J., 18 (1970), 105-108.

[13] G.P. Pokhariyal, R.S. Mishra, *Curvature tensors and their relativistic significance ii,* Yokohama Math. J., 19(2) (1971), 97-103.

[14] V.V. Reddy, R. Sharma, S. Sivaramkrishan, *Space times through Hawking-Ellis construction with a back ground Riemannian metric,* Class Quant. Grav., 24 (2007), 3339-3345.

[15] S. Sasaki, *Lecture notes on almost contact manifolds part I,* Tohoku Univ., 1965.

[16] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure,* Tohoku Math. J., 2 (1960), 459-476.

[17] R. Sharma, *Certain results on K-contact and (k, μ)-contact manifolds,* J. of Geometry, 89 (2008), 138-147.

[18] M.M. Tripathi, *Ricci solitons in contact metric manifold,* ArXiv: 0801. 4222vl [math. D. G.] (2008).

[19] K. Yano, M. Kon, *Structures on manifolds,* Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 3 (1984).

[20] K. Yano, *Concircular geometry I concircular transformations,* Proc. Imp. Acad. Tokyo, 16 (1940), 195-200.

[21] S. Zamkovoy, *Canonical connections on paracontact manifolds,* Ann. Global Anal. Geom., 36 (2008), 37-60.

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