Coincidence and fixed point results for (Ψ, L)-M-weak contraction mapping on Mb-metric spaces

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Abstract. In this paper, we introduce the $(\Psi, L) - M - weak$ contraction and we prove some common fixed point results for self-mappings T and S and some fixed point results for a single mapping T by using a $(c) - comparison$ function and a comparison function in the sense of $Mb - metric$ space.

Keywords: D-metric space, b-metric space, common fixed point.

1. Introduction

In 2020, Malkawi et al [1], defined $Mb - metric$ space, the idea of this definition has weaker than the tetrahedral inequality axiom for $D - metric$ space [2] and generalization of b-metric [3,4].

Also, many authors gives many fixed point theorems in a $b - metric$ space and Quasi Metric Spaces (see, [6-22]).

We begin by presenting the definition of $Mb - metric$ space.

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Definition 1.1 ([1]). *Let X be a non empty set and $R>1$ be a real number. A function $M: X\times X\times X\to[0,\infty)$ is called $Mb-$metric space, if it satisfies the following properties for each $\zeta,\eta,\gamma\in X$.*

$(M1): M(\zeta,\eta,\gamma)\geq 0.$

$(M2): M(\zeta,\eta,\gamma)=0$ *iff* $\zeta=\eta=\gamma.$

$(M3): M(\zeta,\eta,\gamma)=M(p(\zeta,\eta,\gamma))$; *for any permutation* $p(\zeta,\eta,\gamma)$ *of* $\zeta,\eta,\gamma.$

$(M4): M(\zeta,\eta,\gamma)\leq R\left[M(\zeta,\eta,\nu)+M(\zeta,\nu,\gamma)+M(\nu,\eta,\gamma)\right].$

and a pair (X,M) is called $Mb-$metric space.

The following $Mb-Cauchy$, $Mb-convergent$ and $Mb-complete$ spaces are defined by Malkawi et al. [1]

Definition 1.2 ([1]). *Let (X,M) be an $Mb-$metric space. A sequence $\{\zeta_n\}$ on X is said to be*

(I) $Mb-Cauchy$ iff for given $\epsilon>0$, there exists a positive integer N such that $M(\zeta_n,\zeta_m,\zeta_p)<\epsilon$ for all $m,n,p\geq N$.

(II) $Mb-convergent$ iff there exists an element ζ in X such that for given $\epsilon>0$, there exists a positive integer N such that $M(\zeta_n,\zeta_m,\zeta)<\epsilon$ for all m,n $\geq N$. So we called $\{\zeta_n\}$ $Mb-converges$ to ζ and ζ is a limit of $\{\zeta_n\}$.

(III) The $Mb-$metric space (X,M) is $Mb-$complete if every $Mb-Cauchy$ sequence is $Mb-convergent$.

In the following definition, Shatanawi [5] is defined (c)-comparison function with base R.

Definition 1.3 ([5]). *Let R be a constant $R\geq 1$. A map $\Psi:[0,+\infty)\to[0,+\infty)$ is called a $(c)-comparison$ function with base R if Ψ satisfies the following:*

(i) Ψ is monotone increasing,

(ii) $\sum_{n=0}^{\infty}R^n\Psi^n(Rt)$ converges for all $t\geq 0$.

Now, we present some important definitions and theories.

Definition 1.4. *A single-valued mapping $f:X\to X$ is called a strong almost contraction if there a constant $\delta\in[0,1]$ some $L\geq 0$ and for $R\geq 1$ such that*

$$\begin{aligned}M(f\zeta,f\eta,f\gamma)\leq\frac{\delta}{R}\max\{&RM(\zeta,\eta,\gamma),RM(\zeta,\zeta,f\zeta),RM(\eta,\eta,f\eta),RM(\gamma,\gamma,f\gamma),\\&\frac{1}{2}\left[M(\zeta,\zeta,f\eta)+M(\eta,\eta,f\gamma)+M(\gamma,\gamma,f\zeta)\right]\}+LM(\eta,\eta,f\gamma),\end{aligned}$$

for all $\zeta,\eta,\gamma\in X$.

Definition 1.5. *Let (X,M) be an $Mb-$metric space. Let $T:X\to X$ be mapping, a map T is called a $(\delta,L)-weak$ contraction if there exist a constant $\delta\in[0,1]$ some $L\geq 0$ and for $R\geq 1$ such that*

$$M(T\zeta,T\eta,T\gamma)\leq\frac{\delta}{R}M(\zeta,\eta,\gamma)+LM(\eta,\eta,T\gamma). \tag{1.1}$$

Because of the symmetry of the distance, the $(\delta, L)-$*weak contraction condition implicitly includes the following dual one:*

$$M(T\zeta, T\eta, T\gamma) \leq \frac{\delta}{R} M(\zeta, \eta, \gamma) + LM(\zeta, \zeta, T\eta). \tag{1.2}$$

Thus by (1.1) *and* (1.2)*, the of* $(\delta, L)-$*weak contraction condition can be replaced by the following condition:*

$$M(T\zeta, T\eta, T\gamma) \leq \frac{\delta}{R} M(\zeta, \eta, \gamma) + Lmin\{M(\eta, \eta, T\gamma), M(\zeta, \zeta, T\eta)\}.$$

Definition 1.6. *Let* (X, M) *be an* $Mb-$ *metric space. A map* T *is called* $(\Psi, L)-$ *weak contraction if there exist a comparison function* Ψ*, some* $L \geq 0$ *and for all* $R \geq 1$ *such that*

$$M(T\zeta, T\eta, T\gamma) \leq \frac{1}{R} \Psi(RM(\zeta, \eta, \gamma)) + LM(\eta, \eta, T\gamma). \tag{1.3}$$

Because of the symmetry of the distance, the $(\Psi, L)-$*weak contraction condition implicitly includes the following dual one:*

$$M(T\zeta, T\eta, T\gamma) \leq \frac{1}{R} \Psi(RM(\zeta, \eta, \gamma)) + LM(\zeta, \zeta, T\eta). \tag{1.4}$$

Thus by (1.3) *and* (1.4)*, the* $(\Psi, L)-$*weak contraction condition can be replaced by the following condition:*

$$M(T\zeta, T\eta, T\eta) \leq \frac{1}{R} \Psi(RM(\zeta, \eta, \gamma)) + Lmin\{M(\eta, \eta, T\gamma), M(\zeta, \zeta, T\eta)\}.$$

Theorem 1.1. *Assume that* $\zeta_n \to \gamma$ *as* $n \to +\infty$ *in an* $Mb-$ *metric space* (X, M) *such that* $M(\gamma, \gamma, \gamma) = 0$*. Then* $\lim_{n\to+\infty} M(\zeta_n, \zeta_n, \eta) = M(\gamma, \gamma, \eta)$ *for every* $\eta \in X$*.*

Proof. The proof is clear. □

Theorem 1.2. *Let* (X, M) *be an* $Mb-$*complete metric space and* $T : X \to X$ *be a* $(\Psi, L)-$*weak contraction mapping with a* $(c)-$*comparison function, with* M *are satisfies*

$$M(\zeta, \eta, \eta) \leq R\,[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)] \tag{1.5}$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta), \tag{1.6}$$

then T *has a fixed point.*

Proof. Let $\zeta_0 \in X$ and $\zeta_n = T\zeta_{n+1}$ for all $n \in \mathbb{N}$. Now T is a $(\Psi, L) - weak$ contraction, we get

$$\begin{aligned} M(T\zeta_{n-1}, T\zeta_{n-1}, T\zeta_n) &\leq \frac{1}{R}\Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + LM(\zeta_n, \zeta_n, T\zeta_{n-1}) \\ &= \frac{1}{R}\Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) \end{aligned}$$

thus

$$M(\zeta_n, \zeta_n, \zeta_n) \leq \frac{1}{R}\Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n))$$

by induction, we attain

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R}\Psi^n(RM(\zeta_0, \zeta_0, \zeta_1))$$

for all $n \in \mathbb{N}$. By (1.5), (1.6), for $m > n$, we have

$$\begin{aligned} M(\zeta_n, \zeta_n, \zeta_m) &\leq R\left[M(\zeta_n, \zeta_n, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_m, \zeta_m)\right] \\ &= R\left[M(\zeta_n, \zeta_n, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_{n+1}, \zeta_m)\right] \\ &\leq RM(\zeta_n, \zeta_n, \zeta_{n+1}) + R^2\left[M(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+2}) + M(\zeta_{n+2}, \zeta_{n+2}, \zeta_m)\right] \\ &\leq ... \leq \sum_{k=n}^{m-1} R^k M(\zeta_k, \zeta_k, \zeta_{k+1}) \\ &\leq \sum_{k=n}^{\infty} R^k M(\zeta_k, \zeta_k, \zeta_{k+1}) \\ &\leq \sum_{k=n}^{\infty} \frac{1}{R}\Psi^k(RM(\zeta_0, \zeta_0, \zeta_1)). \end{aligned}$$

Then $\sum_{k=n}^{\infty} R^k \Psi^k(RM(\zeta_0, \zeta_0, \zeta_1))$ is $Mb - convergent$ because Ψ is a $(c) - comparison$ function, and so $\{\zeta_n\}$ is an $Mb - Cauchy$ sequence in X. Since X is $Mb - complete$ and then $\{\zeta_n\}$ is $Mb - converges$ with respect to τ_M. to a point $\gamma \in X$ such that $\lim_{n\to\infty} M(\zeta_n, \zeta_n, \gamma) = M(\gamma, \gamma, \gamma) = 0$.

We claim that $M(\gamma, \gamma, T\gamma) = 0$. By contradiction suppose that $M(\gamma, \gamma, T\gamma) > 0$. Since Ψ is $(c) - comparison$ function, then $\Psi(t) < t$ for $t > 0$. Also since $\lim_{n\to\infty} M(\zeta_n, \zeta_n, \gamma) = M(\gamma, \gamma, \gamma) = 0$, then there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(\zeta_n, \zeta_n, \gamma) < \frac{M(\gamma,\gamma,T\gamma)}{2R}$.

Consequently, we get

$$\begin{aligned} M(\gamma, \gamma, T\gamma) &\leq R\left[M(\gamma, \gamma, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_{n+1}, T\gamma)\right] \text{ by (1.5) and (1.6)} \\ &= R\left[M(\gamma, \gamma, \zeta_{n+1}) + M(T\zeta_n, T\zeta_n, T\gamma)\right] \\ &\leq R\left[M(\gamma, \gamma, \zeta_{n+1}) + \frac{1}{R}\Psi(RM(\zeta_n, \zeta_n, \gamma)) + LM(\gamma, \gamma, \zeta)\right] \\ &< R\left[M(\gamma, \gamma, \zeta_{n+1}) + \frac{M(\gamma, \gamma, T\gamma)}{2R} + LM(\gamma, \gamma, \zeta_{n+1})\right]. \end{aligned}$$

By (1.6) and letting $n \to \infty$, we attain

$$0 < M(\gamma, \gamma, T\gamma) < \frac{M(\gamma, \gamma, T\gamma)}{2}$$

which is a contradiction. Therefore, $M(T\gamma, \gamma, \gamma) = 0$ and $\gamma = T\gamma$. □

Theorem 1.3. *Let (X, M) be an $Mb-$complete metric space and $T : X \to X$ be a $(\Psi, L)-$weak contraction mapping, suppose T satisfies the following condition: There exist a comparison function Ψ_1 and some $L_1 > 0$, such that*

$$M(T\zeta, T\zeta, T\eta) \leq \frac{1}{R}\Psi_1(RM(\zeta, \zeta, \eta)) + L_1 M(\zeta, \zeta, T\eta), \text{ for } R > 1 \tag{1.7}$$

for all $\zeta, \eta, \gamma \in X$. Then T has a unique fixed point.

Proof. Suppose that, there are two fixed point γ and ν of T. If $M(\gamma, \gamma, \nu) = 0$, it is obvious that $\gamma = \nu$. Assume that $M(\gamma, \gamma, \nu) > 0$. For (1.7) with $\zeta = \gamma$ and $\eta = \nu$, we have

$$\begin{aligned} 0 < M(\gamma, \gamma, \nu) &= M(T\gamma, T\gamma, T\nu) \\ &\leq \frac{1}{R}\Psi_1(RM(\gamma, \gamma, \nu)) + L_1 M(\gamma, \gamma, T\gamma) \\ &= \frac{1}{R}\Psi_1(RM(\gamma, \gamma, \nu)) \\ &< M(\gamma, \gamma, \nu) \end{aligned}$$

which is a contradiction. Therefore T has a unique fixed point. □

2. The main result

In this section, we present some of the main findings related to the fixed point theories and $(\Psi, L) - M - weak$ contraction mapping, and we start with the following two definitions.

Definition 2.1. *Let (X, M) be an $Mb - metric$ space and $T, S : X \to X$ be two mappings. The pair (T, S) is called a $(R, L) - M - weak$ contraction if for $R \geq 1$ and some $L \geq 0$. such that*

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) \leq \frac{1}{R} &\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, T\eta), \\ &\frac{1}{2}\left(M(\zeta, \zeta, S\eta) + M(T\zeta, \eta, \eta)\right)\} \\ &+ L\min\{M(\zeta, \zeta, S\eta), M(T\zeta, \eta, \eta)\}. \end{aligned} \tag{2.1}$$

for all $\zeta, \eta \in X$.

Definition 2.2. *Let (X, M) be an $Mb-$metric space and $T, S : X \to X$ be two mappings. The pair (T, S) is called a $(\Psi, L) - M - weak$ contraction if there exist a control function, some $L \geq 0$ and for $R \geq 1$. such that*

$$M(T\zeta, T\zeta, S\eta) \leq \frac{1}{R}\Psi(\max\{RM(\zeta,\zeta,\eta), RM(\zeta,\zeta,T\zeta), RM(\eta,\eta,T\eta), \frac{1}{2}M(\zeta,\zeta,S\eta) + M(T\zeta,\zeta,\zeta)\}) + L\min\{M(\zeta,\zeta,S\eta), M(T\gamma,\eta,\eta)\}. \tag{2.2}$$

for all $\zeta, \eta \in X$.

Now, we present the first result using an $Mb - complete$ metric space and $(\Psi, L) - M - weak$ contraction to give a common fixed point for two maps T and S.

Theorem 2.1. *Let (X, M) be an $Mb-$complete metric space and $T, S : X \to X$ be two mappings such that the pair (T, S) is a $(\Psi, L) - M - weak$ contraction with*

$$M(\zeta,\eta,\eta) \leq R\,[M(\zeta,\gamma,\gamma) + M(\gamma,\eta,\eta)] \tag{2.3}$$

and

$$M(\zeta,\zeta,\eta) = M(\zeta,\eta,\eta). \tag{2.4}$$

If Ψ is a $(c) - comparison$ function, then T and S have a common fixed point.

Proof. Pick $\zeta_0 \in X$. Let $\zeta_1 = T\zeta_0$. Also, put $\zeta_2 = S\zeta_1$. Keep going this process, (ζ_n) is constructed sequence in X such that $\zeta_{2n+1} = T\zeta_{2n}$ and $\zeta_{2n+2} = S\zeta_{2n+1}$. Suppose $M(\zeta_n, \zeta_n, \zeta_{n+1}) = 0$ for some $n \in \mathbb{N}$. Without loss of generality, assume $n = 2k$ for some $k \in \mathbb{N}$. Thus $M(\zeta_{2k}, \zeta_{2k}, \zeta_{2k+1}) = 0$. Now, by (2.2), we have

$$\begin{aligned}
&M(\zeta_{2k+1},\zeta_{2k+1},\zeta_{2k+2}) = M(T\zeta_{2k},T\zeta_{2k},S\zeta_{2k+1})\\
&\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{2k},\zeta_{2k},\zeta_{2k+1}), RM(\zeta_{2k},\zeta_{2k},T\zeta_{2k}), RM(\zeta_{2k+1},\zeta_{2k+1},S\zeta_{2k+1}),\\
&\frac{1}{2}\,[M(\zeta_{2k},\zeta_{2k},S\zeta_{2k+1}) + M(T\zeta_{2k},T\zeta_{2k},\zeta_{2k+1})]\}\\
&+ L\min\{M(T\zeta_{2k},T\zeta_{2k},\zeta_{2k+1}), M(\zeta_{2k},\zeta_{2k},S\zeta_{2k+1})\}\\
&= \frac{1}{R}\Psi(\max\{RM(\zeta_{2k+1},\zeta_{2k+1},\zeta_{2k+2}),\\
&\frac{1}{2}[M(\zeta_{2k},\zeta_{2k},\zeta_{2k+2}) + M(\zeta_{2k+1},\zeta_{2k+1},\zeta_{2k+2})]\}\\
&+ L\min\{M(\zeta_{2k+1},\zeta_{2k+1},\zeta_{2k+1}), M(\zeta_{2k},\zeta_{2k},\zeta_{2k+2})\}\\
&\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{2k+1},\zeta_{2k+1},\zeta_{2k+2}),\\
&\frac{1}{2}\,[M(\zeta_{2k},\zeta_{2k},\zeta_{2k+1}) + M(\zeta_{2k+1},\zeta_{2k+1},\zeta_{2k+2})]\}\\
&\leq \frac{1}{R}\Psi(RM(\zeta_{2k+1},\zeta_{2k+1},\zeta_{2k+2})).
\end{aligned}$$

Since $\Psi(t) < t$ for all $t > 0$, we conclude that $M(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2}) = 0$. By $(M2)$ and $(M3)$ of the definition of $Mb-metric$ spaces, we have $\zeta_{2k+1} = \zeta_{2k+2}$. So, $\zeta_{2k} = \zeta_{2k+1} = \zeta_{2k+2}$. Therefore, $\zeta_{2k} = T\zeta_{2k} = S\zeta_{2k}$ and hence ζ_k is a fixed point of T and S. Thus, we may assume that $M(\zeta_n, \zeta_n, \zeta_{n+1}) \neq 0$, for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$. If n is even $n = 2t$ for some $t \in \mathbb{N}$. By (2.2), we have

$$\begin{aligned}
M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}) &= M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t}) \\
&= M(T\zeta_{2t}, T\zeta_{2t}, S\zeta_{2t+1}) \\
&\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, T\zeta_{2t}), \\
&RM(\zeta_{2t-1}, \zeta_{2t-1}, S\zeta_{2t-1}), \\
&\frac{1}{2}\left[M(\zeta_{2t}, \zeta_{2t}, S\zeta_{2t-1}) + M(T\zeta_{2t}, T\zeta_{2t}, \zeta_{2t-1})\right]\}) \\
&+ L\min\{M(\zeta_{2t}, \zeta_{2t}, S\zeta_{2t-1}), M(T\zeta_{2t}, T\zeta_{2t}, \zeta_{2t-1})\} \\
&= \frac{1}{R}\Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}), \\
&\frac{1}{2}\left[M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t}) + M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t-1})\right]\}) \\
&+ L\min\{M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t}), M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t-1})\}.
\end{aligned}$$

From (2.3) and (2.4), we conclude

$$\begin{aligned}
M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}) &\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}), \\
&\frac{1}{2}\left[M(\zeta_{2t-1}, \zeta_{2t-1}, \zeta_{2t}) + M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})\right]\}) \\
&\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})\}. \qquad (2.5)
\end{aligned}$$

If $\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})\} = RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})$, then (2.5) yields a contradiction. Thus, $\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})\} = RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1})$ and hence

$$M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}) \leq \frac{1}{R}\Psi(RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1})). \qquad (2.6)$$

If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$. By similar arguments as above, we will show that

$$M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t+2}) \leq \frac{1}{R}\Psi(RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})). \qquad (2.7)$$

From (2.6) and (2.7), we get

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R}\Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)). \qquad (2.8)$$

Repeat (2.8) $n-times$, we have $M(\zeta_n,\zeta_n,\zeta_{n+1}) \leq \frac{1}{R}\Psi^n(RM(\zeta_0,\zeta_0,\zeta_1))$. For $n,m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} M(\zeta_n,\zeta_n,\zeta_m) &\leq \sum_{i=n}^{m-1} R^i M(\zeta_i,\zeta_i,\zeta_{i+1}) \\ &\leq \sum_{i=n}^{m-1} R^i \Psi^i(RM(\zeta_0,\zeta_0,\zeta_1)) \\ &\leq \sum_{i=n}^{\infty} R^i \Psi^i(RM(\zeta_0,\zeta_0,\zeta_1)). \end{aligned}$$

We have $\sum_{i=n}^{\infty} R^i\Psi^i(M(\zeta_0,\zeta_0,\zeta_1))$ $Mb-converges$ because Ψ is (c)-comparison, and so $\lim_{n\to+\infty}\sum_{i=n}^{\infty} R^i\Psi^i(M(\zeta_0,\zeta_0,\zeta_1)) = 0$.

Thus, $\lim_{n,m\to+\infty} M(\zeta_n,\zeta_n,\zeta_m) = 0$. Thus (ζ_n) is an $Mb-Cauchy$ sequence in X. Now X is $Mb-complete$, so there exists $\gamma \in X$ such that $\zeta_n \to \gamma$ with $M(\gamma,\gamma,\gamma) = 0$. So,

$$\lim_{n,m\to+\infty} M(\zeta_n,\zeta_n,\zeta_m) = \lim_{n\to\infty} M(\zeta_n,\zeta_n,\gamma) = M(\gamma,\gamma,\gamma) = 0. \tag{2.9}$$

Now, we prove that $S\gamma = \gamma$ and $T\gamma = \gamma$. Since $M(\zeta_{2n+1},\zeta_{2n+1},\gamma) \to M(\gamma,\gamma,\gamma) = 0$ and $M(\zeta_{2n+2},\zeta_{2n+2},\gamma) \to M(\gamma,\gamma,\gamma) = 0$, By using $Theorem$ 1.1, we get

$$\lim_{n\to+\infty} M(\zeta_{2n+1},\zeta_{2n+1},S\gamma) = M(\gamma,\gamma,S\gamma) \tag{2.10}$$

and

$$\lim_{n\to+\infty} M(\zeta_{2n+2},\zeta_{2n+2},S\gamma) = M(\gamma,\gamma,T\gamma). \tag{2.11}$$

By using (2.2), we have

$$\begin{aligned} &M(\zeta_{2n+1},\zeta_{2n+1},S\gamma) = M(T\zeta_{2n},T\zeta_{2n},S\gamma) \\ &\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{2n},\zeta_{2n},\gamma), RM(\zeta_{2n},\zeta_{2n},T\zeta_{2n}), RM(\gamma,\gamma,S\gamma), \\ &\frac{1}{2}\left[M(T\zeta_{2n},T\zeta_{2n},\gamma) + M(\zeta_{2n},\zeta_{2n},S\gamma)\right]\}) \\ &+ L\min\left\{M(T\zeta_{2n},T\zeta_{2n},\gamma), M(\zeta_{2n},\zeta_{2n},S\gamma)\right\} \\ &\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{2n},\zeta_{2n},\gamma), RM(\zeta_{2n},\zeta_{2n},\zeta_{2n+1}), RM(\gamma,\gamma,S\gamma), \\ &\frac{1}{2}\left[M(\zeta_{2n+1},\zeta_{2n+1},\gamma) + M(\zeta_{2n},\zeta_{2n},S\gamma)\right]\}) \\ &+ L\min\left\{M(\zeta_{2n+1},\zeta_{2n+1},\gamma), M(\zeta_{2n},\zeta_{2n},S\gamma)\right\}. \end{aligned}$$

In the above inequality, let $n \to +\infty$ and using (2.9),(2.10), we get that $M(\gamma,\gamma,S\gamma) \leq \frac{1}{R}\Psi(RM(\gamma,\gamma,S\gamma))$. Since $\Psi(t) < t$ for all $t > 0$, we conclude that $M(\gamma,\gamma,S\gamma) = 0$. By using $(M2)$ and $(M3)$ of the definition of $Mb-metric$ space, we get that $S\gamma = \gamma$. Similarity, we may show that $T\gamma = \gamma$. □

In the following theorem, we give the 2^{nd} result.

Theorem 2.2. *Let (X, M) be an $Mb-$complete metric space and $T, S : X \to X$ be two mapping such that*

$$M(T\zeta, T\zeta, S\eta) \le \frac{1}{R}\Psi\left(\max\left\{RM(\zeta,\zeta,\eta), RM(\zeta,\zeta,T\zeta), RM(\eta,\eta,S\eta), \frac{1}{2}\left[M(T\zeta,T\zeta,\eta)+M(\zeta,\zeta,S\eta)\right]\right\}\right) + L\min\left\{M(\zeta,\zeta,T\zeta), M(\zeta,\zeta,S\eta), M(T\zeta,T\zeta,\eta)\right\} \tag{2.12}$$

with

$$M(\zeta,\eta,\eta) \le R\left[M(\zeta,\gamma,\gamma)+M(\gamma,\eta,\eta)\right]$$

and

$$M(\zeta,\zeta,\eta) = M(\zeta,\eta,\eta),$$

for all $\zeta, \eta \in X$. If Ψ is a $(c)-$comparison function, then the common fixed point of T and S is unique.

Proof. From *Theorem* 2.1, the existence of the common fixed point of T and S followed. To prove the uniqueness of the common fixed point of T and S, assume there exist two fixed points ν, κ of T and S. So $T\nu = S\nu = \nu$ and $T\kappa = S\kappa = \kappa$. Now, we have to show that $\nu = \kappa$. By (2.12), we have

$$\begin{aligned} M(\nu,\nu,\kappa) &= M(T\nu,T\nu,S\kappa) \\ &\le \frac{1}{R}\Psi\left(\max\left\{RM(\nu,\nu,\kappa), RM(\nu,\nu,T\nu), RM(\kappa,\kappa,S\kappa), \right.\right. \\ &\quad \left.\left. \frac{1}{2}\left[M(T\nu,T\nu,\kappa)+M(\kappa,\kappa,T\nu)\right]\right\}\right) \\ &\quad + L\min\left\{M(\nu,\nu,T\nu), M(T\nu,T\nu,\kappa), M(\kappa,\kappa,T\nu)\right\} \\ &\le \frac{1}{R}\Psi\left(\max\left\{RM(\nu,\nu,\kappa), RM(\nu,\nu,T\nu), RM(\kappa,\kappa,\kappa), \right.\right. \\ &\quad \left.\left. \frac{1}{2}\left[M(T\nu,T\nu,\kappa)+M(\kappa,\kappa,\nu)\right]\right\}\right) \\ &\quad + L\min\left\{M(\nu,\nu,\nu), M(\nu,\nu,\kappa), M(\kappa,\kappa,\nu)\right\} \\ &= \frac{1}{R}\Psi(RM(\nu,\nu,\kappa)). \end{aligned}$$

Since $\Phi(t) < t$ for all $t > 0$, we conclude that $M(\nu,\nu,\kappa) = 0$. By $(M2)$ and $(M3)$ of the definition of $Mb-metric$ spaces, we get that $\nu = \kappa$. □

Taking $T = S$ in Theorems 2.1 and 2.2, we have the following results

Corollary 2.1. *Let (X,M) be an $Mb-$complete metric space and $T:X\to X$ be a mapping such that*

$$\begin{aligned}M(T\zeta,T\zeta,T\eta)\leq \frac{1}{R}\Psi(\max\{&RM(\zeta,\zeta,\eta),RM(\zeta,\zeta,T\zeta),RM(\eta,\eta,T\eta),\\ &\frac{1}{2}[M(T\zeta,T\zeta,\eta)+M(\zeta,\zeta,S\eta)]\})\\ &+L\min\{M(\zeta,\zeta,T\eta),M(T\zeta,T\zeta,\eta)\}\end{aligned}$$

with

$$M(\zeta,\eta,\eta)\leq R\,[M(\zeta,\gamma,\gamma)+M(\gamma,\eta,\eta)]$$

and

$$M(\zeta,\zeta,\eta)=M(\zeta,\eta,\eta),$$

for all $\zeta,\eta\in X$. If Ψ is a $(c)-$comparison function, then T has a fixed point.

Corollary 2.2. *Let (X,M) be an $Mb-$complete metric space and $T:X\to X$ be a mapping such that*

$$\begin{aligned}M(T\zeta,T\zeta,T\eta)\leq \frac{1}{R}\Psi(\max\{&RM(\zeta,\zeta,\eta),RM(\zeta,\zeta,T\zeta),RM(\eta,\eta,T\eta),\\ &\frac{1}{2}[M(T\zeta,T\zeta,\eta)+M(\zeta,\zeta,S\eta)]\})\\ &+L\min\{M(\zeta,\zeta,T\zeta),M(\zeta,\zeta,T\eta),M(T\zeta,T\zeta,\eta)\}\end{aligned}$$

with

$$M(\zeta,\eta,\eta)\leq R\,[M(\zeta,\gamma,\gamma)+M(\gamma,\eta,\eta)]$$

and

$$M(\zeta,\zeta,\eta)=M(\zeta,\eta,\eta),$$

for all $\zeta,\eta\in X$. If Ψ is a $(c)-$comparison function, then T has a unique fixed point.

Corollary 2.3. *Let (X,M) be an $Mb-$metric space and $T,S:X\to X$ be two mappings such that*

$$\begin{aligned}M(T\zeta,T\zeta,T\eta)\leq \frac{1}{R}\Psi(\max\{&RM(S\zeta,S\zeta,S\eta),RM(S\zeta,S\zeta,T\zeta),RM(S\eta,S\eta,T\eta),\\ &\frac{1}{2}\,[M(T\zeta,T\zeta,S\eta)+M(S\zeta,S\zeta,T\eta)\}])\\ &+L\min\{M(S\zeta,S\zeta,T\eta),M(S\eta,S\eta,T\zeta)\}\end{aligned}$$

with

$$M(\zeta,\eta,\eta)\leq R\,[M(\zeta,\gamma,\gamma)+M(\gamma,\eta,\eta)]$$

and

$$M(\zeta,\zeta,\eta)=M(\zeta,\eta,\eta),$$

for all $\zeta, \eta \in X$*. Also, suppose that*

(1) $TX \subseteq SX$

(2) SX *is an* $Mb-$ *complete subspace of the* $Mb-$ *metric space* X*.*

If Ψ *a* $(c)-$ *comparison function, then* T *and* S *have a coincidence point. Moreover, the point of coincidence of* T *and* S *is unique.*

Corollary 2.4. *Let* (X, M) *be an* $Mb-$*metric space and* $T, S : X \to X$ *be two mappings such that*

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) \leq \frac{1}{R}\Psi(\max\{&RM(S\zeta, S\zeta, S\eta), RM(S\zeta, S\zeta, T\zeta), RM(S\eta, S\eta, T\eta), \\ &\frac{1}{2}\left[M(T\zeta, T\zeta, S\eta) + M(S\zeta, S\zeta, T\eta)\right]\}) \\ &+ L\min\{M(T\zeta, T\zeta, S\zeta), M(S\zeta, S\zeta, T\eta), M(S\eta, S\eta, T\zeta)\} \end{aligned}$$

with

$$M(\zeta, \eta, \eta) \leq R\left[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)\right]$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

for all $\zeta, \eta \in X$*. Also, suppose that*

(1) $TX \subseteq SX$

(2) SX *is an* $Mb-$ *complete subspace of the* $Mb-$ *metric space* X*.*

If Ψ *a* $(c)-$ *comparison function, then the point of coincidence of* T *and* S *is unique; that is, if* $T\nu = S\nu$ *and* $T\kappa = S\kappa$*, then* $T\nu = T\kappa = S\kappa = S\nu$*.*

In Theorems 2.1 and 2.2, the $(c)-$ *comparison* function can be replaced by a comparison function if we formulated the counteractive condition to a suitable form. For example, we have the following result.

Theorem 2.3. *Let* (X, M) *be an* $Mb-$ *complete metric space and* $T : X \to X$ *be a mapping, such that*

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) \leq \frac{1}{R}\Psi\left(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, T\eta)\}\right) \\ + L\min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, T\eta), M(\eta, \eta, T\zeta)\} \end{aligned} \tag{2.13}$$

for all $\zeta, \eta \in X$*,*

$$M(\zeta, \eta, \eta) \leq R\left[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)\right] \tag{2.14}$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta), \tag{2.15}$$

If Ψ *is a comparison function, then* T *has a unique fixed point.*

Proof. Choose $\zeta_0 \in X$. Put $\zeta_1 = T\zeta_0$. Also, set $\zeta_2 \in X$ such that $\zeta_2 = T\zeta_1$. Continuing the same process, we can construct a sequence (ζ_n) in X such that $\zeta_{n+1} = T\zeta_n$. If $M(\zeta_k, \zeta_k, \zeta_{k+1}) = 0$ for some $k \in \mathbb{N}$, then by the definition of $Mb-metric$ spaces, we have $\zeta_k = \zeta_{k+1} = T\zeta_k$, that is, ζ_k is a fixed point of T. Thus, we assume that $M(\zeta_n, \zeta_n, \zeta_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. By (2.13), we have

$$\begin{aligned}
&M(\zeta_n, \zeta_n, \zeta_{n+1}) = M(T\zeta_{n-1}, T\zeta_{n-1}, T\zeta_n)\\
&\leq \frac{1}{R}\Psi(\max\{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_{n-1}, \zeta_{n-1}, T\zeta_{n-1}), RM(\zeta_n, \zeta_n, T\zeta_n)\})\\
&+ L\min\{M(\zeta_{n-1}, \zeta_{n-1}, T\zeta_n), M(\zeta_{n-1}, \zeta_{n-1}, T\zeta_n), M(\zeta_n, \zeta_n, T\zeta_{n-1})\}\\
&= \frac{1}{R}\Psi(\max\{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, T\zeta_{n+1})\})\\
&+ L\min\{M(\zeta_{n-1}, \zeta_{n-1}, T\zeta_{n+1}), M(\zeta_n, \zeta_n, \zeta_n)\}\\
&= \frac{1}{R}\Psi(\max\{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, T\zeta_{n+1})\}).
\end{aligned}$$

If

$$\max\{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, \zeta_{n+1})\} = RM(\zeta_n, \zeta_n, \zeta_{n+1}),$$

then

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R}\Psi(RM(\zeta_n, \zeta_n, \zeta_{n+1})) < M(\zeta_n, \zeta_n, \zeta_{n+1}),$$

a contradiction. Thus,

$$\max\{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, \zeta_{n+1})\} = RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)$$

and hence

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R}\Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)), \quad \text{for all } n \in \mathbb{N}. \tag{2.16}$$

Repeating (2.14) $n - times$, we get that

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R}\Psi^n(RM(\zeta_0, \zeta_0, \zeta_1)).$$

Now, we will prove that (ζ_n) is an $Mb - Cauchy$ sequence in the $Mb - metric$ space (X, M). For this, given $\epsilon > 0$, since $\frac{1}{R(2+L)}(\epsilon - \Phi(\epsilon)) > 0$ and $\lim_{n\to+\infty} R^n\Psi^n(M(\zeta_0, \zeta_0, \zeta_1)) = 0$, there exists $k \in \mathbb{N}$such that $M(\zeta_n, \zeta_n, \zeta_{n+1}) < \frac{1}{R(2+L)}(\epsilon - \Phi(\epsilon))$ for all $n \geq k$. Now, given $m, n \in \mathbb{N}$ with $m > n$. Claim: $M(\zeta_n, \zeta_n, \zeta_m) < \epsilon$ for all $m > n > k$. We prove our claim by induction on m. Since $k + 1 > k$, then

$$M(\zeta_k, \zeta_k, \zeta_{k+1}) \leq \frac{1}{R\,(2 + L)}(\epsilon - \Phi(\epsilon)) < \epsilon.$$

The last inequality proves our claim for $m = k + 1$. Assume that our claim holds for $m = k$.

To prove our claim for $m = k+1$, we have

$$\begin{aligned} M(\zeta_n,\zeta_n,\zeta_{k+1}) &\le R\left[M(\zeta_n,\zeta_n,\zeta_{n+1}) + M(\zeta_{n+1},\zeta_{n+1},\zeta_{k+1}) - M(\zeta_{n+1},\zeta_{n+1},\zeta_{n+1})\right] \\ &\le R\left[M(\zeta_n,\zeta_n,\zeta_{n+1}) + M(\zeta_n,\zeta_n,\zeta_{k+1})\right] \\ &= R\left[M(\zeta_n,\zeta_n,\zeta_{n+1}) + M(T\zeta_n,T\zeta_n,T\zeta_k)\right]. \end{aligned} \tag{2.17}$$

By (2.13), we have

$$\begin{aligned} M(T\zeta_n,T\zeta_n,T\zeta_k) &\le \frac{1}{R}\Psi(\max\{RM(\zeta_n,\zeta_n,\zeta_k), RM(\zeta_n,\zeta_n,T\zeta_n), RM(\zeta_k,\zeta_k,T\zeta_k)\}) \\ &+ L\min\{M(\zeta_n,\zeta_n,T\zeta_n), M(\zeta_n,\zeta_n,T\zeta_k), M(\zeta_k,\zeta_k,T\zeta_n)\} \\ &= \frac{1}{R}\Psi(\max\{RM(\zeta_n,\zeta_n,\zeta_k), RM(\zeta_n,\zeta_n,\zeta_{n+1}), RM(\zeta_k,\zeta_k,\zeta_{k+1})\}) \\ &+ L\min\{M(\zeta_n,\zeta_n,\zeta_{n+1}), M(\zeta_n,\zeta_n,\zeta_{k+1}), M(\zeta_k,\zeta_k,\zeta_{n+1})\} \\ &\le \frac{1}{R}\Psi(\max\{RM(\zeta_n,\zeta_n,\zeta_k), RM(\zeta_n,\zeta_n,\zeta_{n+1}), RM(\zeta_k,\zeta_k,\zeta_{k+1})\}) \\ &+ LM(\zeta_n,\zeta_n,\zeta_{n+1}). \end{aligned}$$

If $\max\{RM(\zeta_n,\zeta_n,\zeta_k), RM(\zeta_n,\zeta_n,\zeta_{n+1}), RM(\zeta_k,\zeta_k,\zeta_{k+1})\} = RM(\zeta_n,\zeta_n,\zeta_k)$, then by (2.17) we have

$$\begin{aligned} M(\zeta_n,\zeta_n,\zeta_{k+1}) &\le R\left[M(\zeta_n,\zeta_n,\zeta_{n+1}) + \frac{1}{R}\Psi(RM(\zeta_n,\zeta_n,\zeta_k)) + LM(\zeta_n,\zeta_n,\zeta_{n+1})\right] \\ &< \left[\frac{1+L}{R\,(2+L)}(\epsilon - \Phi(\epsilon)) + \frac{1}{R}\Phi(\epsilon)\right] R \\ &< \epsilon. \end{aligned}$$

If $\max\{RM(\zeta_n,\zeta_n,\zeta_k), RM(\zeta_n,\zeta_n,\zeta_{n+1}), RM(\zeta_k,\zeta_k,\zeta_{k+1})\} = RM(\zeta_n,\zeta_n,\zeta_{n+1})$, then by (2.17) we have

$$\begin{aligned} M(\zeta_n,\zeta_n,\zeta_{k+1}) &\le R\left[M(\zeta_n,\zeta_n,\zeta_{n+1}) + \frac{1}{R}\Psi(RM(\zeta_n,\zeta_n,\zeta_{n+1})) + LM(\zeta_n,\zeta_n,\zeta_{n+1})\right] \\ &< (2+L)RM(\zeta_n,\zeta_n,\zeta_{n+1}) \\ &< \frac{\epsilon - \Phi(\epsilon)}{\epsilon} \\ &< \epsilon. \end{aligned}$$

If $\max\{RM(\zeta_n,\zeta_n,\zeta_k), RM(\zeta_n,\zeta_n,\zeta_{n+1}), RM(\zeta_k,\zeta_k,\zeta_{k+1})\} = RM(\zeta_k,\zeta_k,\zeta_{k+1})$, then by (2.17) we have

$$\begin{aligned} M(\zeta_n,\zeta_n,\zeta_{k+1}) &\le R\left[M(\zeta_n,\zeta_n,\zeta_{n+1}) + \frac{1}{R}\Psi(RM(\zeta_k,\zeta_k,\zeta_{k+1})) + LM(\zeta_n,\zeta_n,\zeta_{n+1})\right] \\ &< (R+L)M(\zeta_n,\zeta_n,\zeta_{n+1}) + RM(\zeta_k,\zeta_k,\zeta_{k+1}) \\ &< \frac{R+L}{R\,(2+L)}(\epsilon - \Phi(\epsilon)) + \frac{R}{R\,(2+L)}(\epsilon - \Phi(\epsilon)) \\ &< \epsilon. \end{aligned}$$

Thus (ζ_n) is an $Mb-Cauchy$ sequence in X. Since X is $Mb-complete$, then (ζ_n) $Mb-converges$, with respect to τ_M, to a point γ for some $\gamma \in X$ such that

$$\lim_{n,m\to+\infty} M(\zeta_n,\zeta_n,\zeta_m) = \lim_{n\to+\infty} M(\zeta_n,\zeta_n,\gamma) = M(\gamma,\gamma,\gamma) = 0. \tag{2.16}$$

Now, assume that $M(\gamma,\gamma,T\gamma) > 0$. By using (2.14) and (2.15) of the definition of $Mb-metric$ spaces and (2.13), we have

$$\begin{aligned}
M(\gamma,\gamma,T\gamma) &\leq R[M(\gamma,\gamma,\zeta_{n+1}) + M(\zeta_{n+1},\zeta_{n+1},T\gamma)] \text{ by (2.14)}\\
&= R[M(\gamma,\gamma,\zeta_{n+1}) + M(T\zeta_n,T\zeta_n,T\gamma)]\\
&\leq R[M(\gamma,\gamma,\zeta_{n+1}) + \frac{1}{R}\Psi(\max\{RM(\zeta_n,\zeta_n,\gamma),\\
&RM(\zeta_n,\zeta_n,T\zeta_n), RM(\gamma,\gamma,T\gamma)\})\\
&+ L\min\{M(\zeta_n,\zeta_n,T\zeta_n), M(\zeta_n,\zeta_n,T\gamma), M(\zeta_n,\zeta_n,T\gamma)\}]\\
&= R[M(\gamma,\gamma,\zeta_{n+1}) + \frac{1}{R}\Psi(\max\{RM(\zeta_n,\zeta_n,\gamma), \qquad (2.19)\\
&RM(\zeta_n,\zeta_n,\zeta_{n+1}), RM(\gamma,\gamma,T\gamma)\})\\
&+ L\min\{M(\zeta_n,\zeta_n,\zeta_{n+1}), M(\zeta_n,\zeta_n,T\gamma), M(\zeta_{n+1},\zeta_{n+1},S\gamma)\}].
\end{aligned}$$

Since

$$\lim_{n,m\to+\infty} M(\zeta_n,\zeta_n,\zeta_{n+1}) = \lim_{n\to+\infty} M(\zeta_n,\zeta_n,\gamma) = 0$$

and $M(\gamma,\gamma,T\gamma) > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$\max\{RM(\zeta_n,\zeta_n,\gamma), RM(\zeta_n,\zeta_n,\zeta_{n+1}), RM(\gamma,\gamma,T\gamma)\} = RM(\gamma,\gamma,T\gamma)$$

for all $n \geq n_0$. Thus (2.19) becomes

$$\begin{aligned}
M(\gamma,\gamma,T\gamma) \leq R[M(\gamma,\gamma,\zeta_{n+1}) &+ \frac{1}{R}\Psi(RM(\gamma,\gamma,T\gamma)) +\\
&L\min\{M(\zeta_n,\zeta_n,\zeta_{n+1}), M(\zeta_n,\zeta_n,T\gamma), M(\zeta_{n+1},\zeta_{n+1},T\gamma)\}],
\end{aligned}$$

for all $n \geq n_0$. On letting $n \to +\infty$ in the above inequality and using (2.18), we get that $M(\gamma,\gamma,T\gamma) \leq \frac{1}{R}\Psi(RM(\gamma,\gamma,T\gamma)) < M(\gamma,\gamma,T\gamma)$, a contradiction. Thus $M(\gamma,\gamma,T\gamma) = 0$. By using $(M2)$ and $(M3)$ of the definition of an $Mb-metric$ space, we get that $\gamma = T\gamma$; that is, γ is a fixed point of T. To prove that the fixed point of T is unique, we assume that ν and κ are fixed points of T.

Thus, we have $T\nu = \nu$ and $T\kappa = \kappa$. By (2.13), we have

Since $\Psi(t) < t$ for all $t \in \mathbb{N}$, we have $M(\nu,\nu,\kappa) = 0$. By $(M2)$ and $(M3)$, we have $\nu = \kappa$. □

Corollary 2.5. *Let (X,M) be an $Mb-metric$ space and $T,S : X \to X$ be two mapping such that for some $L \geq 0$, we have*

$$\begin{aligned}
M(T\zeta,T\zeta,T\eta) \leq \frac{1}{R}\Psi(\max\{&RM(S\zeta,S\zeta,S\eta), Rd(S\zeta,S\zeta,T\zeta), Rd(S\eta,S\eta,T\eta)\})\\
&+ L\min\{M(S\zeta,S\zeta,T\zeta), M(S\zeta,S\zeta,T\eta), M(S\eta,S\eta,T\zeta)\}
\end{aligned}$$

with

$$M(\zeta,\eta,\eta) \leq R\,[M(\zeta,\gamma,\gamma) + M(\gamma,\eta,\eta)]$$

and

$$M(\zeta,\zeta,\eta) = M(\zeta,\eta,\eta),$$

for all $\zeta,\eta \in X$. *Also, suppose that*

(1) $TX \subseteq SX$.

(2) SX *is a complete subspace of the* $Mb-metric$ *space* X.

If Ψ *is a comparison of* T *and* S *have a coincidence point. Moreover, the point of coincidence of* T *and* S *is unique.*

Example 2.1. Let $X = [0, +\infty)$. Consider the complete $Mb - metric$ space $d : X \times X \to [0, +\infty)$, $M(\zeta,\zeta,\eta) = (\zeta - \eta)^2$ with constant $R = 2$. Define the mappings $T, S : X \to X$ by $T\zeta = \frac{1}{3}\zeta$ and $S\zeta = \frac{1}{6}\zeta$. and define $\Psi : [0,+\infty) \to [0,+\infty)$ by $\Psi(t) = \frac{1}{4}\zeta$. Then:

(1) Ψ is a continuous $(c) - comparison$ function.

(2) T, S and Ψ satisfy the following inequality:

$$\begin{aligned} M(T\zeta,T\zeta,S\eta) \leq \frac{1}{R}&\Psi(\max\{RM(\zeta,\zeta,\eta), RM(\zeta,\zeta,T\zeta), RM(\eta,\eta,S\eta),\\ &\frac{1}{2}[M(T\zeta,T\zeta,\eta) + M(\zeta,\zeta,S\eta)]\})\\ &+ L\min\{M(\zeta,\zeta,T\zeta), M(\zeta,\zeta,\eta), M(T\zeta,T\zeta,\eta)\} \end{aligned}$$

Proof. Clearly, to show that Ψ is a non-decreasing continuous function and clearly M satisfies (2.14), (2.15). Now, let $t \in [0. + \infty)$. Then,

$$\Psi^n(Rt) = \Psi^n(2t) = \frac{1}{4^n}(2t).$$

Thus

$$\sum_{n=0}^{\infty} R^n \Psi^n(Rt) = \sum_{n=0}^{\infty} \frac{2^n}{4^n}(2t) = 2t \sum_{n=0}^{\infty} \frac{1}{2^n} < +\infty.$$

So, Ψ is a continuous $(c) - comparison$ function. the part (1) as required. To show (2), let $\zeta,\eta \in X$. Then

$$M(T\zeta,T\zeta,S\eta) = M\left(\frac{1}{3}\zeta, \frac{1}{3}\zeta, \frac{1}{6}\eta\right) = \left(\frac{1}{3}\zeta - \frac{1}{6}\eta\right)^2 = \frac{1}{9}\left(\zeta - \frac{1}{2}\eta\right)^2.$$

Now, we have 3 cases:

Case I. $\zeta = \frac{1}{2}\eta$. Here, we have

$$\begin{aligned} M(T\zeta,T\zeta,S\eta) = 0 \leq \frac{1}{R}&\Psi(\max\{RM(\zeta,\zeta,\eta), RM(\zeta,\zeta,T\zeta), RM(\eta,\eta,S\eta),\\ &\frac{1}{2}[M(T\zeta,T\zeta,\eta) + M(\zeta,\zeta,S\eta)]\})\\ &+ L\min\{M(\zeta,\zeta,T\zeta), M(\zeta,\zeta,S\eta), M(T\zeta,T\zeta,\eta)\} \end{aligned}$$

Case II: $\zeta > \frac{1}{2}\eta$. Here, we have

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) &= \frac{1}{9}(\zeta - \frac{1}{2}\eta)^2 \leq \frac{\zeta^2}{6} \\ &= \frac{1}{2}(2)(\frac{2}{3}\zeta)^2(\frac{1}{4}) = \frac{1}{2}\Psi(2(\zeta - \frac{1}{3}\zeta)^2) = \frac{1}{2}\Psi(2M(\zeta, \zeta, \frac{1}{3}\zeta)) \\ &= \frac{1}{R}\Psi(RM(\zeta, \zeta, T\zeta)) \\ &\leq \frac{1}{R}\Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, S\eta), \\ &\frac{1}{2}[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)]\}) \\ &+ L\min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, S\eta), M(T\zeta, T\zeta, \eta)\}. \end{aligned}$$

Case III: $\zeta < \frac{1}{2}\eta$. Here, we have

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) &= \frac{1}{9}\left(\zeta - \frac{1}{2}\eta\right)^2 \leq \frac{y^2}{36} \\ &\leq \left(\frac{25}{36}\right)\left(\frac{\eta^2}{4}\right) = \frac{1}{2}\Psi\left(2\left(\frac{25}{36}\right)\eta^2\right) = \frac{1}{2}\Psi\left(2\left(\eta - \frac{1}{6}y\right)^2\right) \\ &= \frac{1}{2}\Psi\left(2M\left(\eta, \eta, \frac{1}{6}\eta\right)\right) = \frac{1}{R}\Psi\left(RM\left(\eta, \eta, S\zeta\right)\right) \\ &= \frac{1}{R}\Psi\left(\max\left\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, S\eta),\right.\right. \\ &\left.\left.\frac{1}{2}\left[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)\right]\right\}\right) \\ &+ L\min\left\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, S\eta), M(T\zeta, T\zeta, \eta)\right\}. \end{aligned}$$

Note that, T, S and Ψ satisfy all hypotheses of Theorem 2.2. So, T and S have a unique common fixed point. □

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