The basis of knowledge space via Boolean matrix

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Abstract. Knowledge space theory as a mathematical theory in learning evaluation is a crucial component in mathematical psychology. In real world, basis of knowledge space can be significantly much smaller than the knowledge space, which may be essential for some purposes. In this paper, a new mechanism drawn lessons from Boolean matrix is formulated to search for the basis. In this process, according to the link between knowledge space and its projections, the judgement theorems of atoms (or irreducible elements) are put forward to suit for different situations. Furthermore, an algorithm for basis is presented. And numerical experiments are conducted to evaluate the effectiveness of the proposed approach.

Keywords: algorithm, basis, boolean matrix, knowledge space.

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1. Introduction

Knowledge space theory furnishes a mathematical cognitive science for knowledge assessment with existing software. It was introduced by Falmagne, Koppen, Villano, Doignon and Johannesen [11]. It has been developed into the most effective language of knowledge representation in the self-adaptive learning and testing system [1, 3, 7, 13, 16]. On the other hand, knowledge space has become an important role of mathematical psychology.

From Doignon and Falmagne [4], the core concept of knowledge space theory is knowledge state, which gathers particular elements of knowledge and denotes by K the subset of the whole considered problems Q. A knowledge state delineates the items answered correctly in ideal condition by an individual. In other words, elements in a knowledge state are interpreted as items mastered by an individual without lucky guess and careless. Thereby the standard answers for each item are clearly classified as "correct" and "wrong". In this case, knowledge space theory takes careless about answer process. Usually, the construction of knowledge spaces has tree main three main approaches: the query to experts [6, 9, 15], skill map construction [2, 14] and data-driven [10, 19].

A knowledge structure is a pair $(Q, \mathcal{K})$ in which Q is a nonempty set of examined problems and $\mathcal{K}$ is a collection of subsets of Q, including at least Q and $\emptyset$. Subsets in $\mathcal{K}$ are referred to as knowledge states. It is conceivable that $\cup\mathcal{K} = Q$, $\mathcal{K}$ is occasionally identified with a knowledge structure on Q. Furthermore, a knowledge structure $(Q, \mathcal{K})$ is said to be knowledge space when $\mathcal{K}$ is closed under union. Another terminology is quasi ordinal space, a knowledge space closed under intersection. As well known, knowledge spaces are structures for surmise relations. A compressed way of storing surmise relations is to store the basis of a knowledge space. A basis of knowledge space is the smallest subset of states from which the raw space can be reconstructed by closure under union.

It should be point out that the knowledge space often too large for an efficient adaptive assessment even though the domain of problems is small. The extreme case is the power set of a domain with n items. In this case, the cardinal number of knowledge states is 2^n. The basis of a knowledge space is usually much smaller than the complete space which results in a less storage in computer memory. On the other hand, as we know, it is easy to determine a quasi ordinal space corresponding to a surmise relation as well as a granular knowledge space corresponding to a surmise function. In conclusion, there is correspondence between the basis and surmise relation or surmise function. Thereby basis play a crucial role in searching surmise relation and surmise function. Wild [20] proposed a wildcard-based compression for knowledge space and learning space. Dowling [6] applied the basis of a knowledge space for controlling the questioning of an expert. Subsequently, Dowling et al. [8] suggested a procedure for computing the intersection of knowledge spaces using only their basis. Matayoshi [17] has extended several results on the properties of well-graded, from which the author has used the base of knowledge space to present theorems of well-grade. In addition, Dowling [5] proposed a simple algorithm for building the base. Rusch and Wille [18] searched for the basis by transfer a knowledge space into a knowledge context according to formal concept analysis. In this paper, our results provide a better understanding of method for searching base from Boolean matrix point of view and will allow for designing better, specialized algorithm.

The paper proceeds as follows: In Section 2, we briefly review basic notions of knowledge space. In Section 3, we first investigate the properties of atoms with the perspective of covering rough sets. Then incorporating with matrix viewpoint, the connection of bases between the complete space and its projection is investigated. In this case, judgement theorems of irreducible state are developed, which becomes a deriving force for formulating a fast approach for the basis. Furthermore, experimental analysis to further illustrate the effectiveness of our method is presented in Section 4. The paper is concluded with a summary at the final part.

2. Preliminaries

Knowledge spaces can be summarized by a subfamily of their states faithfully. In other terms, any state of the knowledge space can be interpreted as forming the union of some states in the subfamily. Therefore, from reference [12], we have the following basic notions.

Definition 2.1. *The span of a family of sets $\mathcal{G}$ is the family $S(\mathcal{G})$ containing any set which is the union of some subfamily of $\mathcal{G}$.*

In this fashion, $S(\mathcal{G})$ is closed under union. Basis of a knowledge space is the collection of small sets of knowledge states which determine knowledge space by means of a span. Consequently, a knowledge space admits at leat one base. And it has a unique base especially in case $\mathcal{K}$ is finite.

By definition, a minimal knowledge state is called an atom in the knowledge space. Obviously, the base of a knowledge space is formed by the collection of all the atoms. From reference [12], another characterization of the atom in a knowledge space is given as follows.

Theorem 2.1. *A state K is a knowledge space $(Q, \mathcal{K})$ is an atom if and only if $K \in \mathcal{F}$ for any states $\mathcal{F}$ such that $K = \cup\mathcal{F}$.*

Substructure and subspace in the setting have been investigated in [12]. It defined a restriction of knowledge state of a nonempty subset of domain.

Definition 2.2. *Wit a knowledge space $(Q, \mathcal{K})$, take any nonempty proper subset Q' of Q, the family*

$$\mathcal{K}_{|Q'} = \{W \subseteq Q' | W = K \cap Q' \text{ for some } K \in \mathcal{K}\} \tag{1}$$

is called the projection of $\mathcal{K}$ on Q'.

Depending on the context, it can also refer to $\mathcal{K}_{|Q'}$ as a substructure of $\mathcal{K}$. Every set $W = K \cap Q'$ is said to be the trace of the state K on Q'.

Definition 2.3. *Suppose that $(Q, \mathcal{K})$ is a knowledge structure with $\cup\mathcal{K} = Q$ and $|Q| \geq 2$, and take that Q' be any proper nonempty set of Q. Define a relation $\sim_{Q'}$ on $\mathcal{K}$ by*

$$K \sim_{Q'} L \Leftrightarrow K \cap Q' = L \cap Q'. \tag{2}$$

Thus, $\sim_{Q'}$ is an equivalence relation on $\mathcal{K}$. We denote by $[K]$ the equivalence class of $\sim_{Q'}$ containing K.

For any state K in $\mathcal{K}$ and with $[K]$ as in Definition 2.3, we define the family

$$\mathcal{K}_{[K]} = \{M \subseteq Q | M = L - \cap[K] \text{ for some } L \sim_{Q'} K\}. \tag{3}$$

3. Basis of knowledge space

3.1 Perspective of covering rough sets

From the perspective of topological theory, we know that any knowledge space $(Q, \mathcal{K})$ is in fact a cover approximate space and $\mathcal{K}$ is a cover on Q. According to covering rough sets theory [21], and the base is actually the reduction of cover. Then we shall have the following definition from reference.

Definition 3.1. *Let $(Q, \mathcal{K})$ be a knowledge space and $K \in \mathcal{K}$. If K is represented as the join of several states of $\mathcal{K} - \{K\}$, then K is called an reducible state of $\mathcal{K}$. Otherwise, K is referred to as irreducible sate of $\mathcal{K}$.*

Thus, all the atoms of $\mathcal{K}$ are the irreducible states. Then we will present the judgement approach of reducible state to trim the proof process.

Theorem 3.1. *Take that K is a reducible state in knowledge space $\mathcal{K}$ and $L \in \mathcal{K}$. L is reducible in $\mathcal{K}$ if and only if it is a reducible state in $\mathcal{K} - \{K\}$.*

Proof. If L is reducible in $\mathcal{K} - \{K\}$, then it is the joint of some knowledge sates in $\mathcal{K} - \{K, L\}$. Namely, L may be interpreted as the join of several states of $\mathcal{K} - \{L\}$. That is to say, L is a reducible state of $\mathcal{K}$.

Conversely, take that L is reducible in $\mathcal{K}$, then there is a sequence of states $K_1, K_2, \cdots, K_t$ such that $L = \bigcup_{i=1}^{t} K_i$. For any $0 < i \leq t$, it is easy verified that $K_i \subset L$. In this case, if $K_i \neq K$, then we have proved L is a reducible state of $\mathcal{K} - \{K\}$. Otherwise, suppose that $K_1 = K$ without losing generality. As K is a reducible state of $\mathcal{K}$, there exists a sequence of states $L_1, L_2, \cdots, L_p$ such that $K_1 = \bigcup_{i=1}^{p} L_i$. Since $K_1 \subset L$, we have any L_i, $0 < i \leq p$, must be unequal to L. In other terms, $L = (\bigcup_{i=1}^{p} L_i) \cup (\bigcup_{i=2}^{t} K_i)$. In such case, we conclude that L is a reducible state of $\mathcal{K} - \{K\}$. □

This theorem ensures that a new reducible state will not be produced when deleting a reducible state on $\mathcal{K}$. With perspective of this, we can reduct knowledge space by removing all reducible states or one by one at every time. Ultimately, we will obtain a family of all irreducible states, which is called the reduct of $\mathcal{K}$ [21]. Namely, the base of knowledge space.

Lemma 3.1. *Let $(Q, \mathcal{K})$ is a knowledge space. Then $\mathcal{B}$ is the base of $\mathcal{K}$ if and only if for any $B \in \mathcal{B}$ and $\mathcal{F} \subset \mathcal{B}$, $\mathcal{B} \in \mathcal{F}$ implies $B \neq \bigcup \mathcal{F}$.*

Proof. Suppose existing $B \in \mathcal{B}$ and $\mathcal{F} \subset \mathcal{F}$ such that $B \notin \mathcal{F}$ and $B = \bigcup \mathcal{F}$. Then the span $S(\mathcal{B} - \{B\}) = \mathcal{K}$, which is contradicted with the definition of base. Conversely, if $\mathcal{B}$ is not the base of $\mathcal{K}$, there is $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the base. Take $B \in \mathcal{B} - \mathcal{B}_1$, there is $\mathcal{F} \subseteq \mathcal{B}_1$ satisfying $B = \bigcup \mathcal{F}$ but $B \notin \mathcal{F}$, contradiction. □

In addition, the following results will be easily catched. Firstly, for any $q \in Q$, we denote

$$\sigma(q) = \{K \in \mathcal{K} | q \in K \ \wedge \ (\forall L \in \mathcal{K} \ \wedge \ q \in L \ \wedge \ L \subseteq K \Rightarrow K = L)\},$$

which is said to be the minimal description sate set with respect to item q in $\mathcal{K}$. It is easily checked that any $K \in \sigma(q)$ is an atom at q. For convenience, for $\mathcal{K}' \subseteq \mathcal{K}$, the minimal description state set with respect to item q in $\mathcal{K}'$ is denoted $\sigma_{\mathcal{K}'}(q)$.

Example 1. Let $Q = \{a, b, c, d\}$ and $\mathcal{K} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. Then the minimal description with respect to every item is $\sigma(a) = \{\{a\}\}$, $\sigma(b) = \{\{b\}\}$, $\sigma(c) = \{\{b, c\}\}$, $\sigma(d) = \{\{a, d\}, \{b, c, d\}\}$.

Theorem 3.2. *Let $(Q, \mathcal{K})$ be a knowledge space and $K \in \mathcal{K}$ be reducible. Then $\sigma(q) = \sigma_{\mathcal{K}-\{K\}}(q)$ for each $q \in Q$.*

Proof. Actually, if K is reducible in $\mathcal{K}$, then $K \notin \sigma(q)$ for any $q \in Q$. However, for any $L \in \mathcal{K}$, there is K' belonging to $\mathcal{K} - \{K\}$ such that $K' \subseteq L$ when $K \subseteq L$. Thus $\sigma(q) = \sigma_{\mathcal{K}-\{K\}}(q)$ for each $q \in Q$. □

For instance, $\{a, b, d\}$ is reducible in $\mathcal{K}$. It is easy to see that the minimal descriptions of all items stay the same even when removing such sate from $\mathcal{K}$.

Corollary 3.1. *For a knowledge space $\mathcal{K}$ on Q and any $q \in Q$, $\sigma_{\mathcal{B}}(q) = \sigma(q)$, where $\mathcal{B}$ is the base of $\mathcal{K}$.*

3.2 Boolean matrix viewpoint

Rusch and Wille [18] transferred a knowledge space into knowledge context and used double arrows to reduce the knowledge context without changing the basis of knowledge space. In this fashion, the dual of the reduced knowledge context is actually the base of knowledge space. This section we shall use a novel method to search for the base efficiently.

Described verbally, a subset A in Q can be regarded as a mapping $A : Q \to \{0, 1\}$, $A(q)$ being interpreted as "the degree which q is contained to A". That is, the characteristic mapping from Q to $\{0, 1\}$ with

$$A(q) = \begin{cases} 1, & q \in A, \\ 0, & q \notin A \end{cases}.$$

Let $Q = \{q_1, q_2, \cdots, q_n\}$, then A can be denoted by a $1 \times n$ row vector, of which each coordinate is either 0 or 1. For instance, suppose that $Q = \{q_1, q_2, q_3, q_4\}$, then $A = (1, 0, 0, 1)$ means a subset $\{q_1, q_4\}$ of Q.

As inspired from Rusch and Wille [18], a knowledge space may be converted to no matter a knowledge context but also a Boolean matrix called knowledge matrix and denoted by $M_{\mathcal{K}}$. From the perspective of calculation, we adopt Boolean matrix to find the basis of knowledge space. In addition, it should be noted that a knowledge matrix is consisted of the collection of all knowledge states with row vectors forming with a presented ordering.

Example 2. Let $\mathcal{K} = \{\emptyset, \{q_1\}, \{q_1, q_2\}, \{q_2, q_3\}, \{q_1, q_2, q_3\}\}$ is a knowledge space with $Q = \{q_1, q_2, q_3\}$. Then the corresponding knowledge matrix is

$$M_{\mathcal{K}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Actually, in the search of base, it is not needed to consider the two states $\emptyset$ and Q if Q is not an irreducible state. Thus the above knowledge matrix can briefly denoted by

$$M_{\mathcal{K}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

However, if a knowledge space is trivial, then its base $\mathcal{B} = \{Q\}$. Excluding this fashion, if no otherwise specified, the knowledge matrix corresponded to a non-trivial knowledge space neglects the states $\emptyset$ as well as Q when Q is reducible. That is to say, all zero vector and all one vector are excluded by the knowledge matrix. Furthermore, for a knowledge matrix $M_{\mathcal{K}}$, let $M_{\mathcal{K}}(i, :)$ and $M_{\mathcal{K}}(:, j)$ denote the i-th row and the j-th column vectors of $M_{\mathcal{K}}$, respectively. And take $M_{\mathcal{K}}(i, j)$ be the crossed member of i-th row and the j-th column vectors.

Corollary 3.2. *Let $\mathcal{K} = \{\emptyset, K_1, \cdots, K_m, Q\}$ be a knowledge space with $Q = \{q_1, q_2, \cdots, q_n\}$ and its basis $\mathcal{B}$. Assume that $M_{\mathcal{K}}$ is the corresponding knowledge matrix with a preseted ordering. If $|M_{\mathcal{K}}(i, :)| = 1$, $0 < i \leq m$, then the corresponding state K_i belongs to $\mathcal{B}$.*

Proof. Straightforward. □

Corollary 3.3. *If $|M_{\mathcal{K}}(:,j)| = 1$, $0 < j \leq n$, then the corresponding state K belongs to the basis $\mathcal{B}$.*

Proof. $|M_{\mathcal{K}}(:,j)| = 1$ indicates the corresponding state K is not the joint of some other sates. That is to say, K is irreducible in $\mathcal{K}$, then $K \in \mathcal{B}$. □

Summing up above corollaries, a thing they have in common is that such knowledge state belonging to the base is an atom at only one item. A bit more technically, if the state K is an atom at item q, then $K - \{q\}$ is not atom any more. Furthermore, if K and q are chopped away from $(Q, \mathcal{K})$, it is conceivable that the following conclusion can be formulated.

Theorem 3.3. *Take K ba an atom at item $q \in Q$ uniquely from knowledge space $(Q, \mathcal{K})$. Then $(\mathcal{K} - \{K\})_{|Q-\{q\}}$ is a projection of $\mathcal{K} - \{K\}$ as well as a knowledge space on $Q - \{q\}$.*

Proof. With hypothesis, we have $\sigma(q) = \{K\}$. Assume that $M_{\mathcal{K}}$ is the knowledge matrix, $K = K_i \in \mathcal{K}$ and $q = q_j \in Q$. If $|M_{\mathcal{K}}(:,j)| = 1$, then any states in $\mathcal{K}$ do not contain q_j except for K_i. In other terms, any state $L \in \mathcal{K} - \{K_i\}$ can be interpreted as a join of some states in $\mathcal{B} - \{K_i\}$, where $\mathcal{B}$ is the base of $\mathcal{K}$. Namely, $(\mathcal{K} - \{K_i\})_{|Q-\{q_j\}}$ equaling to $\mathcal{K} - \{K_i\}$ is a knowledge space on $Q - \{q_j\}$. For another case with $|M_{\mathcal{K}}(i,:)| = 1$, it is easy to verify that $(\mathcal{K} - \{K_i\})_{|Q-\{q_j\}} = \mathcal{K}_{|Q-\{q_j\}}$. And thus $(\mathcal{K} - \{K_i\})_{|Q-\{q_j\}}$ is a knowledge space on $Q - \{q_j\}$ obviously. Stated thus, $(\mathcal{K} - \{K\})_{|Q-\{q\}}$ is a projection of $\mathcal{K} - \{K\}$ as well as a knowledge space on $Q - \{q\}$. □

Then the following corollary will be easily got.

Corollary 3.4. *Take that $Q' = \{q \in Q \,||\sigma(q)| = 1\}$, where $|\sigma(q)|$ is the cardinality of $\sigma(q)$. Then $(\mathcal{K} - \bigcup_{q \in Q'} \sigma(q))_{|Q-Q'}$ is a projection of $\mathcal{K} - \bigcup_{q \in Q'} \sigma(q)$ and a knowledge space on $Q - Q'$ as well.*

We examine an example of these conclusions.

Example 3. Consider the knowledge space $\mathcal{K} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,e\}, \{a,b,c,e\}, \{a,b,c,d\}, \{a,b,c,d,e\}\}$. By definition, it is easy to obtain the knowledge matrix

$$M_{\mathcal{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

At the same time, we find that $Q' = \{a, b, d\}$ and $(\mathcal{K} - \bigcup_{q \in Q'} \sigma(q))_{|\{c,e\}} = \{\emptyset, \{c\}, \{e\}, \{c, e\}\}$ is obviously a knowledge space on $\{c, e\}$. Then the new knowledge matrix is

$$M_{(\mathcal{K} - \bigcup_{q \in Q'} \sigma(q))_{|\{c,e\}}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Nevertheless, expecting a learning space, many knowledge spaces may exist no knowledge space with only one item. On the other hand, atom at more than one item can be found everywhere. In such cases, we are interested in searching for the irreducible states. As a matter of fact, it is easy to formulate the following corollary explicitly incorporating with knowledge matrix.

Theorem 3.4. *Let $(Q, \mathcal{K})$ be a knowledge space without atoms at only one item and $M_{\mathcal{K}}$ is the corresponding knowledge matrix. Take that $\mathcal{G} = \{K_j \in \mathcal{K} || A(K_j)| = min_{t=1}^{|\mathcal{K}|}\{|M_{\mathcal{K}}(t, :)|\}\}$. Then any $K \in \mathcal{G}$ is an irreducible state of $\mathcal{K}$.*

Proof. By hypothesis, if K is a reducible state of $\mathcal{K}$, then K is a join of some other knowledge states, which is in contradiction with $|A(K)|$ is the minimum. □

Another interesting thing coming into our eyes is that the link between the base of knowledge space and the base of its projection. Firstly, we define an equivalence relation on $\mathcal{K}$.

Definition 3.2. *Suppose that $(Q, \mathcal{K})$ is a knowledge structure with $\cup\mathcal{K} = Q$ and $|Q| \geq 2$, and take that Q' be any proper nonempty set of Q. Define a relation $\approx_{Q'}$ on $\mathcal{K}$ by*

$$K \approx_{Q'} L \Leftrightarrow K - Q' = L - Q'. \tag{4}$$

Thus, $\approx_{Q'}$ is an equivalence relation on $\mathcal{K}$. We denote by $[K]_{\approx}$ the equivalence class of $\approx_{Q'}$ containing K.

Lemma 3.2. *Let $\mathcal{K}$ be any $\cup$-closed (close under union) family. Take any $Q' \subset Q$. Then the following statements are true.*

1. *If $\mathcal{K}$ is closed under intersection, then $K \approx_{Q'} \cap[K]_{\approx}$.*
2. *$[K]_{\approx}$ is $\cup$-closed.*
3. *The function $h : [K]_{\approx} \mapsto K - Q'$ is a well defined bijection of $\mathcal{K}_{\approx}$ onto $\mathcal{K}_{|Q-Q'}$, where $\mathcal{K}_{\approx} = \{[K]_{\approx} | K \in \mathcal{K}\}$.*
4. *$\approx_{Q'} = \sim_{Q-Q'}$.*

Proof. (1) By hypothesis, $\cap[K]_{\approx} \in \mathcal{K}$, and $\cap[K]_{\approx} - Q' = \bigcap_{L\in[K]_{\approx}}(L - Q') = K - Q'$. Thus, $K \approx_{Q'} \cap[K]_{\approx}$.

(2) For any $L, H \in [K]_{\approx}$, we have $L - Q' = H - Q'$ and $L \cup H \in \mathcal{K}$, then $(L \cup H) - Q' = (L - Q') \cup (H - Q') = L - Q'$, which means that $L \cup H \approx_{Q'} K$. Hence, $L \cup H \in [K]_{\approx}$.

(3) That h is a well define function is due to Eq.(4). Clearly, $h(\mathcal{K}_{\approx}) = \mathcal{K}_{|Q-Q'}$ by the definitions of h and $\mathcal{K}_{|Q-Q'}$. Assume that $h([K]_{\approx}) = K - Q' = h([L]_{\approx}) = L - Q' = X$ for some $[K]_{\approx}$ and $[L]_{\approx}$ in $\mathcal{K}_{\approx}$. Whether or not $X = \emptyset$, this entails $K \approx_{Q'} L$ and thus $[K]_{\approx} = [L]_{\approx}$.

(4) Straightforward. □

Example 4. Consider the knowledge space $\mathcal{K} = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c, e\}, \{b, d, f\}, \{a, b, c, d\}, \{a, b, c, e\}, \{b, c, d, e\}, \{b, c, d, f\}, \{b, c, e, f\}, \{a, b, d, f\}, \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{a, b, c, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}, \{a, b, d, c, e, f, g\}\}$. Take $Q' = \{b, c, e, g\}$. Then

1. $[\{a, b, d, f\}]_{\approx} = \{\{a, b, c, d, e, f, g\}, \{a, b, c, d, e, f\}, \{a, b, c, d, f\}, \{a, b, d, f\}\}$,
2. $[\{b, d, f\}]_{\approx} = \{\{b, d, e, f\}, \{b, c, d, f\}, \{b, d, f\}\}$,
3. $[\{a, b, d\}]_{\approx} = \{\{a, b, c, d, e\}, \{a, b, c, d\}, \{a, b, d\}\}$,
4. $[\{b, d\}]_{\approx} = \{\{b, c, d, e\}, \{b, c, d\}, \{b, d\}\}$,
5. $[\{a, c\}]_{\approx} = \{\{a, b, c, e\}, \{a, b, c\}, \{a, b\}, \{a, c\}\}$,
6. $[\{c\}]_{\approx} = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{b, c, e\}\}$,
7. $[\{a, b, c, e, f\}]_{\approx} = \{\{a, b, c, e, f\}\}$,
8. $[\{b, c, e, f\}]_{\approx} = \{\{b, c, e, f\}\}$.

As illustrated before, a quasi ordinal space is a knowledge space closed under intersection. In the first place, we consider the base of quasi ordinal space in the following.

Theorem 3.5. *Let* $(Q, \mathcal{K})$ *be a quasi ordinal space and* $\mathcal{B}$ *is the base of* $\mathcal{K}$. *Denote* $\mathcal{L} = \{L \in \mathcal{K} : |L| = min\{|K||K \in \mathcal{K} - \{\emptyset\}\}\}$ *and* $Q' = \bigcup \mathcal{L}$. *If* $Q - Q' \neq \emptyset$, *then* $\mathcal{L} \subset \mathcal{B}$ *and* $\mathcal{B}_{|Q-Q'} = \{B - Q'|B \in \mathcal{B},\ B - Q' \neq \emptyset\}$ *is the base of* $\mathcal{K}_{|Q-Q'}$.

Proof. For any $L \in \mathcal{L}$, by definition, L is obviously an irreducible state in $\mathcal{K}$. Then $\mathcal{L} \subseteq \mathcal{B}$. Next, if $\mathcal{B}_{|Q-Q'}$ is not the base of $\mathcal{K}_{|Q-Q'}$, from Lemma 1, there exists $W \in \mathcal{B}_{|Q-Q'}$ and $\mathcal{F} \subset \mathcal{B}_{|Q-Q'}$ such that $W = \bigcup \mathcal{F}$ while $W \notin \mathcal{F}$. By definition of $\mathcal{B}_{|Q-Q'}$, a state B in $\mathcal{B}$ satisfying $W = B - Q'$ as well as $K \in \mathcal{B}$ satisfying $F = K - Q'$ for $F \in \mathcal{F}$ can be found. Take $H_F = B \cap K$, then $H_F \in \mathcal{K}$ as $\mathcal{K}$ is a quasi ordinal space. Thus there is $\mathcal{H}_F \subset \mathcal{B}$ such that $H_F = \bigcup \mathcal{H}_F$ for any $F \in \mathcal{F}$. Let $A = B - \bigcup\{H_F|F \in \mathcal{F}\}$, then $A \subset Q - Q'$. Since $\mathcal{K}$

is a quasi ordinal space, we have $A \in \mathcal{K}$. Thus there exists $\mathcal{G} \subset \mathcal{B}$ such that $A = \bigcup \mathcal{G}$. In this fashion, it is easy to see that $\bigcup(\{\mathcal{H}_F | F \in \mathcal{F}\} \cup \{\mathcal{G}\}) \subset \mathcal{B}$ and $B = \bigcup(\{\mathcal{H}_F | F \in \mathcal{F}\} \cup \{G\})$. This contradicts to the definition of $\mathcal{B}$. □

More normally, if Q' is any nonempty proper subset of Q, then we shall have the following verdict.

Corollary 3.5. *Let $(Q, \mathcal{K})$ be a quasi ordinal space and $\emptyset \neq Q' \subset Q$. Then $\mathcal{B}_{|Q-Q'} = \{B - Q' | B \in \mathcal{B},\ B - Q' \neq \emptyset\}$ is the base of $\mathcal{K}_{|Q-Q'}$.*

Proof. Straightforward. □

Example 5. Let $\mathcal{K} = \{\emptyset, \{a,b\}, \{d,e\}, \{a,b,f\}, \{c,d,e\}, \{a,b,d,e\}, \{a,b,c,d,e\}, \{a,b,d,e,f\}, Q\}$ be a quasi ordinal space with $Q = \{a,b,c,d,e,f\}$. From Theorem 3.5, $\mathcal{L} = \{\{a,b\}, \{d,e\}\}$. Take $Q' = \bigcup \mathcal{L} = \{a,b,d,e\}$. Obviously, $\{a,b\}, \{d,e\} \in \mathcal{B}$ and $\mathcal{B}_{|Q-Q'} = \{\{c\}, \{f\}\}$ is the basis of $\mathcal{K}_{|Q-Q'}$. From knowledge matrix, the process as follows:

$$M_{\mathcal{K}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow M_{\mathcal{K}_{|Q-Q'}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

However, if $\mathcal{K}$ is a knowledge space without closing under intersection, it is worth pointing out that $B - Q'$ is not necessarily in $\mathcal{B}_{|Q-Q'}$ for some $B \in \mathcal{B}$, where $\mathcal{B}$ and $\mathcal{B}_{|Q-Q'}$ are the bases of $\mathcal{K}$ and $\mathcal{K}_{|Q-Q'}$, respectively. For instance, from Example 4, we know $\mathcal{K}_{|\mathcal{Q}-\mathcal{Q}'} = \{\emptyset, \{a\}, \{d\}, \{f\}, \{a,d\}, \{a,f\}\{d,f\}, \{a,d,f\}\}$ and the state $\{a,b,c,d,e,f,g\}$ is an irreducible state of $\mathcal{K}$.

Nevertheless, $\{a,b,c,d,e,f,g\} - Q' = \{a,d,f\}$ is a reducible state of $\mathcal{K}_{|Q-Q'}$.

Fortunately, considering Q' a knowledge state in the given knowledge space, excellent results will be came out as follows.

Theorem 3.6. *Let $(Q, \mathcal{K})$ be a knowledge space with knowledge matrix $M_{\mathcal{K}}$. Take Q' a nonempty state in $\mathcal{K}$ such that $|Q'| < max_{i=1}^{|\mathcal{K}|}\{|M_{\mathcal{K}}(i,:)|\}$. Assume that $\mathcal{B}$ and $\mathcal{B}_{|Q-Q'}$ are the bases of $\mathcal{K}$ and $\mathcal{K}_{|Q-Q'}$, respectively. Then the following conclusions hold.*

1. *$\mathcal{B}_{|Q-Q'} = \{B - Q' | B \in \mathcal{B} \wedge (B - Q') \neq \emptyset\}$.*
2. *If $L \in [K]_{\approx}$ is a reducible state such that $L - Q' \in \mathcal{B}_{|Q-Q'}$, then $\min\{|B| | B \in [K]_{\approx}\} < |L|$.*
3. *If $L - Q' \in \mathcal{B}_{|Q-Q'}$ and $\min\{|B| | B \in [K]_{\approx}\} = |L|$ for $L \in [K]_{\approx}$, then $L \in \mathcal{B}$.*

Proof. (1) Suppose $B \in \mathcal{K}$ with $B - Q' \neq \emptyset$. So B is not included in Q'. If $B \cap Q' = \emptyset$, then $B - Q'$ is obviously in $\mathcal{B}_{|Q-Q'}$. Conversely, if $B \cap Q' \neq \emptyset$, assume that $B - Q'$ be a reducible sate in $\mathcal{K}_{|Q-Q'}$. Then there exists $\mathcal{W} \subseteq \mathcal{B}_{|Q-Q'}$ such that $B - Q' = \bigcup \mathcal{W}$. Actually, there are some $L \in \mathcal{K}$ satisfying $W = L - Q'$ for each $W \in \mathcal{W}$. Without loss of generality, take $\mathcal{L} \subseteq \mathcal{K}$ such that any sate in it is one-to-one correspondence with that in $\mathcal{W}$. So it is easy to see that $B - Q' = \bigcup \mathcal{L} - Q'$. Furthermore, there is $\mathcal{H} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{L} = \mathcal{H}$ as $\bigcup \mathcal{L}$ is a state in $\mathcal{K}$. That is to say, $B - Q' = \bigcup \mathcal{H} - Q'$. Moreover, $B = B \cup (B - Q') = (\bigcup \mathcal{H} \cup B) - B \cap Q'$. In other terms, $B \cap Q' \in \bigcup \mathcal{H}$. Since $B - Q' = \bigcup \mathcal{H} - Q'$, it must be $B \subseteq \bigcup \mathcal{H}$. Therefore, $B = (\bigcup \mathcal{H} \cup B) - B \cap Q' = \bigcup \mathcal{H}$. If $|\mathcal{H}| > 1$, then it is contradicted to B is an irreducible state. Otherwise, $\mathcal{H} = \{B\}$. This implies $B - Q'$ is an irreducible state in $\mathcal{B}_{|Q-Q'}$.

Next we prove that any state in $\mathcal{B}_{|Q-Q'}$ corresponds to a state in $\mathcal{B}$. Conversely, suppose this condition is not fulfilled by $W \in \mathcal{B}_{|Q-Q'}$. By definition, there is $\mathcal{L} \subseteq \mathcal{K}$ such that $W = L - Q'$ for any $L \in \mathcal{L}$. Then such state L is not in $\mathcal{B}$. Furthermore, we can find $\mathcal{F} \subseteq \mathcal{B}$ with $L = \bigcup \mathcal{F}$. In other word, $W = \bigcup_{F \in \mathcal{F}} (F - Q')$ and $F - Q' \neq \emptyset$. This implies W is a reducible state since $F - Q' \in \mathcal{B}_{|Q-Q'}$ for any $F \in \mathcal{F}$, a contradiction.

(2) Since $L - Q' \in \mathcal{B}_{|Q-Q'}$, there exists $B \in \mathcal{B}$ satisfying $L - Q' = B - Q'$. Now we only need to present $|B| < |L|$. If not, suppose that there is an index set T of $\mathcal{B}$ such that $L = \bigcup_{i \in T} B_i$ as L is a reducible state, then $B \notin \{B_i | i \in T\}$. However, $B - Q' = (\bigcup_{i \in T} B_i) - Q' = \bigcup_{i \in T} (B_i - Q')$, which contradicts to $B - Q'$ belonging to $\mathcal{B}_{|Q-Q'}$ in the light of item (1). In conclusion, $min\{|B| | B \in [K]_{\approx}\} < |L|$.

(3) It follows form item (2). □

Corollary 3.6. *Let $\mathcal{K}_{|Q-Q'}$ be a projection of knowledge space $\mathcal{K}$ on a nonempty proper subset Q' of Q. Assume that $\mathcal{B}$ and $\mathcal{B}_{|Q-Q'}$ are the base of $\mathcal{K}$ and $\mathcal{K}_{|Q-Q'}$, respectively. For any $K \in \mathcal{K}$, K is an irreducible state if the following conditions hold*

1. *if $K - Q' \neq \emptyset$, then $K - Q' \in \mathcal{B}_{|Q-Q'}$;*
2. *$|K - Q'| \leq |W|$ for any $W \in [K]_{\approx}$.*

Proof. It follows from Theorem 3.6. □

However, there is no evidence to see that the opposite is true.

Example 6. Consider the knowledge space $\mathcal{K}$ from Example 4, take $Q' = \{a, b, d, f\}$. By definition of projection, $\mathcal{K}_{|Q-Q'} = \{\emptyset, \{c\}, \{c, e\}, \{c, e, g\}\}$. It is easily checked that $\{\{b, c, e\}, \{a, b, c, e\}, \{b, c, e, f\}, \{a, b, c, d, e\}, \{a, b, c, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}\}$ is a equivalence class corresponding the state $\{c, e\}$ in $\mathcal{K}_{|Q-Q'}$. Since $\{c, e\}$ is an irreducible state and $|\{b, c, e\}| = 3$ is the minimum in its class, then $\{b, c, e\}$ is an irreducible state evidently in $\mathcal{K}$. In addition, $\{b, c, e, f\}$ is an irreducible state in $\mathcal{K}$ while it does not meet the condition (2) of above corollary.

Similar to Theorem 3.5, we shall have the following corollary

Corollary 3.7. *Take that* $\mathcal{L} = \{L \in \mathcal{K} || L| = min\{|K| | K \in \mathcal{K} - \{\emptyset\}\}\}$ *from knowledge space* $(Q, \mathcal{K})$. *Let* $Q' = \bigcup \mathcal{L}$. *If* $Q - Q' \neq \emptyset$, *then* $\mathcal{L} \subset \mathcal{B}$ *and* $\mathcal{B}_{|Q-Q'} = \{B - Q' | B \in \mathcal{B},\ B - Q' \neq \emptyset\}$.

Proof. It follows immediately from the item (1) of Theorem 3.6. □

Conversely, we shall have the following result to search for irreducible states from projection to the raw knowledge space. We first present a fact to trim some proofs. Actually, for any $Q' \in \mathcal{K} - \{\emptyset, Q\}$, there is a projection function $\delta : 2^{Q-Q'} \to 2^Q$ defined by

$$\delta(L) = \{K \in \mathcal{K} | K - Q' = L\},\ L \in \mathcal{K}_{Q-Q'}. \tag{5}$$

Obviously, δ is a surjection. In addition, it is easily verified that $\delta(L) = [K]_{\approx}$.

Theorem 3.7. *Let* $(Q, \mathcal{K})$ *be a knowledge space and* $Q' \in \mathcal{K} - \{\emptyset, Q\}$. *Take that*

$$\mathcal{L} = \{L \in \mathcal{K}_{Q-Q'} || L| = \min\{|K| | K \in \mathcal{K}_{Q-Q'} - \{\emptyset\}\}\}. \tag{6}$$

For $L \in \mathcal{L}$, *if* $K \in \delta(L)$ *such that* $|K| = \min\{|H| | H \in \delta(L)\}$, *then* $K \in \mathcal{B}$.

Proof. By hypothesis, any $L \in \mathcal{L}$ is irreducible. Otherwise, it will be a reducible state. Then L can be generated by forming the union of some knowledge states in $\mathcal{K}_{Q-Q'}$, which contradicts to $|L|$ is the minimum. Then according to the item (3) of Theorem 3.6, we have $K \in \mathcal{B}$. □

The next chief problem attaches our attention is the connection of bases between $\mathcal{K}$ and its sub-subspace, that is, the projection of the projection of $\mathcal{K}$.

Theorem 3.8. *Let* $(Q, \mathcal{K})$ *be a knowledge space and* $Q' \in \mathcal{K} - \{\emptyset, Q\}$. *Denote by* $\mathcal{K}' = \mathcal{K}_{|Q-Q'}$ *and take* $Q'' \in \mathcal{K}' - \{\emptyset, Q'\}$. *Suppose that*

$$\mathcal{H} = \{H \in \mathcal{K}'_{Q'-Q''} || H| = \min\{|K| | K \in \mathcal{K}'_{Q'-Q''} - \{\emptyset\}\}\}. \tag{7}$$

For any $H \in \mathcal{H}$, *there exists* $K \in \mathcal{K}$ *with* $H = K - Q' \cup Q''$ *such that* $K \in \mathcal{B}$.

Proof. It follows from Theorem 3.7. □

It is worth mentioning that the inspiration of these results is came from Boolean matrix. That is to say, all of them can be realised by computer. As discussed by these results, we shall develop a fast algorithm to search the base of knowledge space in the following.

It is easy to see that the algorithm is correct and has execution time in $O(|Q||\mathcal{K}|)$. However, the algorithm (SA) for constructing the base from [12] needs $O(|Q||\mathcal{K}|^2)$, and the complexity of the Rusch-Wille approach [18] from formal concept point of view takes $O(|Q||\mathcal{K}|^2 + |Q|^2|\mathcal{K}|)$.

Algorithm 1 Construction of basis based on covering rough set (CBCR)

Input: A knowledge space $\mathcal{K} = \{K_1, K_2, \cdots, K_m\}$ with $Q = \{q_1, q_2, \cdots, q_n\}$.
Output: The basis $\mathcal{B}$ of $\mathcal{K}$.
1: Generate knowledge matrix $M_{\mathcal{K}}$ and set $\mathcal{B} \leftarrow \emptyset$.
2: Select state K_i such that $|M_{\mathcal{K}}(i,:)| = 1$ or there exists $j \leq n$ satisfying $|M_{\mathcal{K}}(i,j)| = 1$ and $|M_{\mathcal{K}}(:,j)| = 1$.
3: $\mathcal{B} \leftarrow \{K_i\}$ and then update $M_{\mathcal{K}}$ to $M_{\mathcal{K}_{Q-Q'}}$ based on Theorem 3.3.
4: **While** $M_{\mathcal{K}_{Q-Q'}} \neq \emptyset$ **do**
5: Perform Step 2 and 3, and if such conditions of Step 2 do not satisfy, go to Step 6.
6: $\mathcal{B} \leftarrow \mathcal{B} \cup \{K_t\}$, where K_t such that item (3) of Theorem 3.7.
7: Update $M_{\mathcal{K}_{Q-Q'}}$
8: **End while**
9: Output $\mathcal{B}$.

Example 7. Let $(Q, \mathcal{K})$ be a knowledge space with $Q = \{q_1, \cdots, q_7\}$ and $\mathcal{K} = \{K_0, K_1, \cdots, K_{15}\}$, where $K_0 = \emptyset$ and $K_{15} = Q$. The knowledge matrix by above proper order is presented in the following:

$$M_{\mathcal{K}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Firstly, from Step 2 and 3, we have $\{K_7\} \in \mathcal{B}$ since $M_{\mathcal{K}}(7,:) = (0,0,0,0,0,0,1)$, and then

$$M_{\mathcal{K}_{Q-\{q_7\}}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

From Eq.(6), $\mathcal{L} = \{\{q_2, q_3\}, \{q_1, q_5\}, \{q_4, q_6\}\}$. By Eq.(5), We can easily check that $\delta(\{q_2, q_4\}) = \{K_1, K_9\}$, $\delta(\{q_1, q_5\}) = \{K_2, K_{10}\}$, and $\delta(\{q_4, q_6\}) = \{K_4\}$. Comparing with knowledge matrix $M_{\mathcal{K}}$, we have $|K_1| < |K_9|$, $|K_2| < |K_{10}|$.

Hence, $K_1, K_2, K_4 \in \mathcal{B}$. Update $M_{Q-\{q_7\}}$ from Step 7, the knowledge matrix of the projection becomes empty. Therefore $\mathcal{B} = \{K_1, K_2, K_4, K_7\}$.

4. Experimental analysis

In this section, to further illustrate the effectiveness of our proposed algorithm, we compare it with some existing algorithms for the bases of knowledge spaces. The experiments were performed on a personal computer with Windows 10 and Intel (R) Core (TM) i7-6700 CPU @ 3.41GHz with 8GB of memory. The algorithms are implemented by Matlab 9.3.

To achieve this task, we need some knowledge spaces. However, it is difficult to search for knowledge spaces in real life, we design a generation algorithm to construct knowledge spaces randomly. All knowledge spaces used in experimental analysis are randomly generated, which are shown in Table 1. It should be point out that every knowledge space depicted in Table 1 is closed under intersection.

Table 1: Data description of the 15 knowledge spaces

Knowledge spaces	Items	States
K1	14	5376
K2	19	207
K3	14	1036
K4	15	2576
K5	18	9216
K6	18	384
K7	17	2400
K8	17	6016
K9	22	73
K10	19	66
K11	15	4068
K12	21	928
K13	16	1388
K14	18	520
K15	14	2736

The algorithms Dowling and Wille used to numerical experiments are from References [5] and [18], respectively. And the comparison results is depicted in Table 2. From Table 2, we see that our method CBCR was the fastest on the most knowledge spaces except K5. In addition, Dowling was the most time-consuming. In K5, our approach was only 0.02 seconds longer than Wille while much faster than Dowling. To sum it up, we can conclude that the superior computational efficiency of our proposed method over the others.

Table 2: Comparison of the running time(s)

Knowledge spaces	CBCR	Dowling	Wille
K1	0.09	66.2	0.18
K2	0.01	0.11	0.18
K3	0.04	5.61	0.18
K4	0.05	15.16	0.2
K5	0.22	193.46	0.2
K6	0.01	0.36	0.2
K7	0.05	12.92	0.21
K8	0.12	75.47	0.23
K9	0.01	0.02	0.32
K10	0.08	62.41	0.32
K11	0.08	45.45	0.32
K12	0.06	1.95	0.32
K13	0.06	4.12	0.33
K14	0.02	0.65	0.33
K15	0.05	16.14	0.33

5. Conclusion and future work

Base plays an important role in knowledge space. In this article, we mainly focused on the base incorporating with the projection of knowledge space from the viewpoints of covering rough set theory and Boolean matrix. In this case, the connections of bases between raw knowledge space and its different projections have been formulated. With this mechanism, a fast approach for base has been designed subsequently. The experiment results have demonstrated that our proposed method has a superior performance on the running time by comparing with the existing algorithms. In summary, the proposed model can effectively preprocess the base of knowledge spaces.

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