Study on Kenmotsu manifolds admitting generalized Tanaka-Webster connection

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Abstract. In this study, we use the generalized Tanaka-Webster connection to observe curvature properties and Ricci soliton in Kenmotsu manifold. Here we look at projective curvature tensor, conharmonic curvature tensor, Weyl projective curvature tensor and recurrent conditions of Kenmotsu manifold with generalized Tanaka-Webster connection. Likewise, we demonstrate identical conditions for a Ricci soliton in a Kenmotsu manifold with generalized Tanaka-Webster connection.

Keywords: Kenmotsu manifolds, generalized Tanaka-Webster connection, Ricci solitons, shrinking, expanding, steady.

1. Introduction

The Tanaka-Webster connection is canonical affine connection defined on a non-degenerate pseudo-Hermition CR-manifold [19, 22]. The generalized Tanaka-

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Webster connection(GTWC) for contact metric manifolds by using the canonical connection turned into first studied by Tanno [20]. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. Using the GTWC, few geometers have studied some characterizations of real hypersurfaces in complex space forms [18]. Kenmotsu manifolds introduced by Kenmotsu in 1971[9]. Recently many authors[15, 7, 16] have been studied GTWC connection in Kenmotsu manifolds.
Ricci soliton, introduced by Hamilton [8] are natural generalizations of the Einstein metrics and is defined on a Riemannian manifold (M, g). A Ricci soliton (g, V, λ) defined on (M, g) as

$$(L_{\mathrm{X}}g)(U, V) + 2S(U, V) + 2\lambda g(U, V) = 0, \tag{1.1}$$

where L_{X} denotes the Lie-derivative of Riemannian metric g along a vector field X, λ is a consant and U, V are arbitrary vector fields on M. A Ricci soliton is said to shrinking or steady or expanding to the extent that λ is negative, zero or positive respectively. Ricci solitons have been studied extensively in the context of contact geometry; we may refer to [17, 4, 14, 11, 10, 12, 13]) and references therein.

2. Preliminaries

A smooth manifold M of $(2n+1)$-dimension is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1, 1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 U = -U + \eta(U)\xi,\ \phi\xi = 0,\ g(U, \xi) = \eta(U),\ \eta(\xi) = 1, \eta \circ \phi = 0, \tag{2.1}$$

and

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V). \tag{2.2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [1] if

$$(\nabla_U \phi)V = -g(U, \phi V)\xi - \eta(V)\phi U, \tag{2.3}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold the following relations hold [6].

$$\nabla_U \xi = U - \eta(U)\xi, \tag{2.4}$$

$$(\nabla_U \eta)V = g(\nabla_U \xi, V), \tag{2.5}$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{2.6}$$

$$R(\xi, U)V = \eta(V)U - g(U, V)\xi, \tag{2.7}$$

$$\eta(R(U, V)W) = g(U, W)\eta(V) - g(V, W)\eta(U), \tag{2.8}$$

$$S(U, \xi) = -2n\eta(U), \tag{2.9}$$

$$Q\xi = -2n\xi, \tag{2.10}$$

$$S(\phi U, \phi V) = S(U, V) + 2n\eta(U)\eta(V), \tag{2.11}$$

for any vector fields U, V, W on M, where R denote the curvature tensor of type $(1, 3)$ on M.

3. Curvature properties of Kenmotsu manifolds admitting generalized Tanaka-Webster connection

Througout this paper we associate $*$ with the quantities with respect to generalized Tanaka-Webster connection. The generalized Tanaka-Webster connection ∇^* associated to the Levi-Civita connection ∇ is given by [21, 7]

$$\nabla^*_U V = \nabla_U V - \eta(V)\nabla_U \xi + (\nabla_U \eta)(V)\xi - \eta(U)\phi V, \tag{3.1}$$

for any vector fields U, V on M.

Using (2.4) and (2.5), the above equation yields,

$$\nabla^*_U V = \nabla_U V + g(U, V)\xi - \eta(V)U - \eta(U)\phi V. \tag{3.2}$$

By taking $V = \xi$ in (3.2) and using (2.4) we obtain

$$\nabla^*_U \xi = 0. \tag{3.3}$$

We now calculate the Riemann curvature tensor R^* using (3.2) as follows:

$$R^*(U, V)W = R(U, V)W + g(V, W)U - g(U, W)V. \tag{3.4}$$

Using (2.6) and taking $W = \xi$ in (3.4) we get

$$R^*(U, V)\xi = 0. \tag{3.5}$$

On contracting (3.4), we obtain the Ricci tensor S^* of a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* as

$$S^*(V, W) = S(V, W) + 2ng(V, W). \tag{3.6}$$

This gives

$$Q^*V = QV + 2nV. \tag{3.7}$$

Contracting with respect to V and W in (3.6), we get

$$r^* = r + 2n(2n + 1), \tag{3.8}$$

where r^* and r are the scalar curvatures with respect to the generalized Tanaka-Webster connection ∇^* and the Levi-Civita connection ∇ respectively.

Now, the projective curvature tensor [23] P^* with respect to the generalized Tanaka-Webster connection ∇^* is defined by

$$P^*(U,V)W = R^*(U,V)W - \frac{1}{2n}\{S^*(V,W)U - S^*(U,W)V\}. \tag{3.9}$$

If projective curvature tensor P^* with respect to the generalized Tanaka-Webster connection ∇^* vanishes, then from (3.9), we have

$$R^*(U,V)W = \frac{1}{2n}\{S^*(V,W)U - S^*(U,W)V\}. \tag{3.10}$$

Now, in view of (3.4) and (3.6), (3.10) takes the form

$$\begin{aligned} &g(R(U,V)W,Y) + g(V,W)g(U,Y) - g(U,W)g(V,Y) \\ &= \frac{1}{2n}[\{S(V,W) + 2ng(V,W)\}g(U,Y) \\ &- \{S(U,W) + 2ng(U,W)\}g(V,Y)]. \end{aligned} \tag{3.11}$$

Now, taking $Y = \xi$ in (3.11), we obtain

$$S(V,W)\eta(U) - S(U,W)\eta(V) = 2n\{g(U,W)\eta(U) - g(V,W)\eta(U)\}. \tag{3.12}$$

Again setting $U = \xi$ in (3.12), we get

$$S(V,W) = -2ng(V,W). \tag{3.13}$$

Contracting the above equation (3.13), we get

$$r = -2n(2n+1). \tag{3.14}$$

Using (3.13) in (3.10), we have $R^* = 0$.

Thus, we state the following:

Theorem I. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection. In M, vanishing of projective curvature tensor with respect to generalized Tanaka-Webster connection leads to vanishing of curvature tensor with respect to generalized Tanaka-Webster connection.*

By interchanging U and V in (3.9), we have

$$P^*(V,U)W = R^*(V,U)W - \frac{1}{2n}\{S^*(U,W)V - S^*(V,W)U\}. \tag{3.15}$$

On adding (3.9) and (3.15) and using the fact that $R(U,V)W + R(V,U)W = 0$, we get

$$P^*(U,V)W + P^*(V,U)W = 0. \tag{3.16}$$

From (3.4), (3.9) and first Bianchi identity $R(U,V)W+R(V,W)U+R(W,U)V = 0$ with respect to ∇, we obtain

$$P^*(U,V)W + P^*(V,W)U + P^*(W,U)V = 0. \tag{3.17}$$

Hence, from (3.16) and (3.17), shows that projective curvature tensor with respect to generalized Tanaka-Webster connection in a Kenmotsu manifold is skew-symmetric and cyclic.

Now, by taking $W = \xi$ in (3.9), using (3.5) and (3.6), we get

$$P^*(U,V)\xi = 0. \tag{3.18}$$

From (3.18), we state the following.

Theorem II. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection is ξ-projectively flat.*

Definition 3.1. *A Kenmotsu manifold is said to be ϕ-projectively semisymmetric with respect to the generalized Tanaka-Webster connection ∇^* if*

$$P^*(U,V).\phi = 0. \tag{3.19}$$

for any vector fields U,V on M.

Now, (3.19) turns into

$$(P^*(U,V).\phi)W = P^*(U,V)\phi W - \phi P^*(U,V)W = 0. \tag{3.20}$$

Making use of (3.9), (3.4) and (3.6) in (3.20), we get

$$\begin{aligned} &R(U,V)\phi W - \phi R(U,V)W - \frac{1}{2n}\{S(V,\phi W)U - S(U,\phi W)V \\ &+ S(U,W)\phi V - S(V,W)\phi U\} = 0. \end{aligned} \tag{3.21}$$

Taking V by ξ in (3.21), using (2.6) and (2.9), we get

$$S(U,\phi W)\xi = -2ng(U,\phi W)\xi. \tag{3.22}$$

Taking an inner product with ξ and replacing U by ϕU, using (2.2) and (2.11) in (3.22), we get

$$S(U,W) = -2ng(U,W). \tag{3.23}$$

and

$$r = -2n(2n+1). \tag{3.24}$$

Again by substituting (3.24) in (3.9), we obtain

$$P^*(U,V)W = R(U,V)W + \{g(V,W)U - g(U,W)V\}. \tag{3.25}$$

Thus, we can state the following:

Theorem III. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection be ϕ-projectively semisymmetric if and only if $S(U,W) = -2ng(U,W)$.*

Further if $P^* = 0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

Suppose $(P^*(U,V).S^*)(W,Y) = 0$ holds in a Kenmotsu manifold M. Then, we have

$$S^*(P^*(U,V)W,Y) + S^*(W,P^*(U,V)Y) = 0. \tag{3.26}$$

Taking $U = \xi$ in the equation (3.26), we get

$$S^*(P^*(\xi,V)W,Y) + S^*(W,P^*(\xi,V)Y) = 0. \tag{3.27}$$

By using (3.9), equation (3.27) turns into

$$S^*(V,W)\eta(Y) + S^*(V,Y)\eta(W) = 0. \tag{3.28}$$

In view of equation (3.6), (3.28) becomes

$$S(V,W)\eta(Y) + S(V,Y)\eta(W) + 2n\{g(V,W)\eta(Y) + g(V,Y)\eta(W)\} = 0. \tag{3.29}$$

In (3.29), taking $Y = \xi$ and contracting with respect to V and W, we get

$$S(V,W) = -2ng(V,W).$$

and

$$r = -2n(2n+1). \tag{3.30}$$

Again by substituting (3.30) in (3.9), we obtain

$$P^*(U,V)W = R(U,V)W + \{g(V,W)U - g(U,W)V\}. \tag{3.31}$$

Thus, we can state that

Theorem IV. *In a Kenmotsu manifold M with generalized Tanaka-Webster connection, $P^*.S^* = 0$ if and only if $S(V,W) = -2ng(V,W)$.*

Further if $P^* = 0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

The conharmonic curvature tensor [5] K^* with respect to the generalized Tanaka-Webster connection ∇^* is defined by

$$\begin{aligned} K^*(U,V)W = R^*(U,V)W - \frac{1}{2n-1}\{&S^*(V,W)U - S^*(U,W)V \\ &+ g(V,W)Q^*U - g(U,W)Q^*V\}. \end{aligned} \tag{3.32}$$

If conharmonic curvature tensor K^* with respect to the generalized Tanaka-Webster connection ∇^* vanishes, then from (3.32), we have

$$\begin{aligned} R^*(U,V)W = \frac{1}{2n-1}\{&S^*(V,W)U - S^*(U,W)V \\ &+ g(V,W)Q^*U - g(U,W)Q^*V\}. \end{aligned} \tag{3.33}$$

By using (3.4), (3.6) and (3.7) in (3.33), we get

$$\begin{aligned} & g(R(U,V)W,Y) + g(V,W)g(U,Y) - g(U,W)g(V,Y) \\ & = \frac{1}{2n-1}[\{S(V,W) + 4ng(V,W)\}g(U,Y) - \{S(U,W) + 4ng(U,W)\} \\ & g(V,Y) + S(U,Y)g(V,W) - S(V,Y)g(U,W)]. \end{aligned} \tag{3.34}$$

Taking $Y = \xi$ in (3.34), we obtain

$$S(V,W)\eta(U) - S(U,W)\eta(V) - 2n\{g(U,W)\eta(V) - g(V,W)\eta(U)\} = 0. \tag{3.35}$$

Taking $U = \xi$ in (3.35), we get

$$S(V,W) = -2ng(V,W). \tag{3.36}$$

Contracting the equation (3.36), we get

$$r = -2n(2n+1). \tag{3.37}$$

Using (3.36) in (3.33), we have $R^* = 0$.

Theorem V. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection. In M, vanishing of conharmonic curvature tensor with respect to generalized Tanaka-Webster connection leads to vanishing of curvature tensor with respect to generalized Tanaka-Webster connection.*

In a Riemannian manifold Weyl conformal curvature tensor C^* with respect to the generalized Tanaka-Webster connection $\nabla^*$is defined as

$$\begin{aligned} C^*(U,V)W = {} & R^*(U,V)W \\ & - \frac{1}{2n-1}\{S^*(V,W)U - S^*(U,W)V + g(V,W)Q^*U \\ & - g(U,W)Q^*V\} + \frac{r^*}{2n(2n-1)}\{g(V,W)U - g(U,W)V\}. \end{aligned} \tag{3.38}$$

By making use of (3.4), (3.6), (3.7), (3.8) in (3.38) yields

$$C^*(U,V)W = C(U,V)W. \tag{3.39}$$

for all U, V, W. Thus, we state the following:

Theorem VI. *The Weyl conformal curvature tensor of Kenmotsu manifold with respect to the Levi-Civita connection and the generalized Tanaka-Webster connection are equivalent.*

Definition 3.2. *A Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* is called recurrent, if its curvature tensor R^* satisfies the condition*

$$(\nabla^*_Y R^*)(U,V)W = A(Y)R^*(U,V)W, \tag{3.40}$$

where R^* is the curvature tensor with respect to the connection ∇^* and A is 1-form.

Using (3.40), we can write

$$\nabla^*_Y R^*(U,V)W - R^*(\nabla^*_Y U,V)W - R^*(U,\nabla^*_Y V)W - R^*(U,V)\nabla^*_Y W \\ = A(Y)R^*(U,V)W. \tag{3.41}$$

Making use of (3.2), (3.4) and (3.6) in (3.41), we get

$$\begin{aligned} & g(Y,R(U,V)W)\xi - g(Y,U)R(\xi,V)W \\ & - g(Y,V)R(U,\xi)W - g(Y,W)R(U,V)\xi \\ & - \eta(R(U,V)W)Y + \eta(U)R(Y,V)W + \eta(V)R(U,Y)W + \eta(W)R(U,V)Y \\ & - \eta(Y)\{\phi R(U,V)W - R(\phi U,V)W - R(U,\phi V)W - R(U,V)\phi W\} \\ & = A(Y)\{g(V,W)U - g(U,W)V\}. \end{aligned} \tag{3.42}$$

Replacing W by ξ and using (2.1), (2.6), (2.7) and (2.8), we get

$$A(Y)\{\eta(V)U - \eta(U)V\} = g(Y,V)U - g(Y,U)V + R(U,V)Y. \tag{3.43}$$

Taking an inner product with Z in (3.43), we have

$$A(Y)\{\eta(V)g(U,Z) - \eta(U)g(V,Z)\} \\ = g(Y,V)g(U,Z) - g(Y,U)g(V,Z) + R(U,V,Y,Z). \tag{3.44}$$

Let $\{e_1, e_2, e_3, ...e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then by putting $U = Z = e_i$ in (3.44) and summing up with respect to $i, 1 \leq i \leq 2n+1$, we obtain

$$S(V,Y) = -2n\{g(V,Y) + \eta(V)A(Y)\}. \tag{3.45}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (3.45), we get

$$S(V,Y) = -2n\{g(V,Y) + \eta(V)\eta(Y)\}. \tag{3.46}$$

Thus, we state the following:

Theorem VII. *If a Kenmotsu manifold whose curvature tensor of manifold is covariant constant with respect to the generalized Tanaka-Webster connection, the manifold is recurrent and the associated* 1*-form* A *is equal to the associated* 1*-form* η*, then the manifold is an* η*-Einstein manifold.*

4. Ricci solitons in Kenmotsu manifold with generalized Tanaka-Webster connection

Suppose Kenmotsu manifold M admits a Ricci soliton with respect to the generalized Tanaka-Webster connection ∇^*. Then

$$(\overset{*}{L}_X g)(U,V) + 2S^*(U,V) + 2\lambda g(U,V) = 0. \tag{4.1}$$

If the potential vector field X is the structure vector field ξ, then since ξ is a parallel vector field with respect to the generalized Tanaka-Webster connection(from (3.3)), the first term in equation (4.1) becomes zero, hence M reduces to an Einstein manifold. In this case the results in theorem (3.4) and (3.6) hold.

If X is pointwise collinear with the structure vector field ξ, i.e. $X = b\xi$, where b is a function on M, then equation (1.1) implies that

$$\begin{aligned} bg(\nabla^*_U\xi, V) + (Ub)\eta(V) + bg(U, \nabla^*_V\xi) + (Vb)\eta(U) \\ + 2S^*(U, V) + 2\lambda g(U, V) = 0. \end{aligned} \tag{4.2}$$

Using (3.3) and (3.6) in (4.2), it follows that

$$(Ub)\eta(V) + (Vb)\eta(U) + 2S(U, V) + 2\{2n + \lambda\}g(U, V) = 0. \tag{4.3}$$

By setting $V = \xi$ in (4.3) and using (2.9), we obtain

$$(Ub) = -\{2\lambda + \xi b\}\eta(U). \tag{4.4}$$

Again, replacing U by ξ in (4.4), we get

$$(\xi b) = -\lambda. \tag{4.5}$$

Substituting this in (4.4), we have

$$(Ub) = -\lambda\eta(U). \tag{4.6}$$

By applying d on (4.6), we get

$$\lambda d\eta = 0. \tag{4.7}$$

Since $d\eta \neq 0$ from (4.7), we have

$$\lambda = 0. \tag{4.8}$$

Substituting (4.8) in (4.6), we conclude that b is a constant. Hence it is verified from (4.3) that

$$S(U, V) = -(2n + \lambda)g(U, V) + \lambda\eta(U)\eta(V). \tag{4.9}$$

This leads to the following.

Theorem VIII. *If a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection admits a Ricci soliton (g, X, λ) with X, pointwise collinear with ξ, then the manifold is an η-Einstein manifold and the Ricci soliton is steady.*

5. Example of a 5-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection

We consider the five-dimensional manifolod $M = \{(u, v, w, x, y) \in R^5\}$, where (u, v, w, x, y) are the standard coordinates in R^5. The vector fields

$$E_1 = e^{-y}\frac{\partial}{\partial u}, \quad E_2 = e^{-y}\frac{\partial}{\partial v}, \quad E_3 = e^{-y}\frac{\partial}{\partial w}, \quad E_4 = e^{-y}\frac{\partial}{\partial x}, \quad E_5 = e^{-y}\frac{\partial}{\partial y}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(W) = g(W, E_3)$ for any $W \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_3, \phi E_2 = E_4, \phi E_3 = -E_1, \phi E_4 = -E_2, \phi E_5 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_5) = 1, \quad \phi^2(W) = -W + \eta(W)E_5, \quad g(\phi W, \phi X) = g(W, X) - \eta(W)\eta(X),$$

for any $W, X \in \chi(M)$. Thus for $E_5 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the metric g. Then, we have

$$\begin{aligned}
&[E_1, E_2] = [E_1, E_3] = [E_1, E_4] = [E_2, E_3] = 0, \quad [E_1, E_5] = E_1, \\
&[E_4, E_5] = E_4, \quad [E_2, E_4] = [E_3, E_4] = 0, \quad [E_2, E_5] = E_2, \quad [E_3, E_5] = E_3.
\end{aligned}$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned}
2g(\nabla_U V, W) &= U(g(V, W)) + V(g(W, U)) - W(g(U, V)) \\
&- g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]).
\end{aligned}$$

By Koszul's formula, we get

$$\begin{aligned}
&\nabla_{E_1} E_1 = -E_5, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0, \quad \nabla_{E_1} E_4 = 0, \quad \nabla_{E_1} E_5 = E_1, \\
&\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -E_5, \quad \nabla_{E_2} E_3 = 0, \quad \nabla_{E_2} E_4 = 0, \quad \nabla_{E_2} E_5 = E_2, \\
&\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_5, \quad \nabla_{E_3} E_4 = 0, \quad \nabla_{E_3} E_5 = E_3, \\
&\nabla_{E_4} E_1 = 0, \quad \nabla_{E_4} E_2 = 0, \quad \nabla_{E_4} E_3 = 0, \quad \nabla_{E_4} E_4 = -E_5, \quad \nabla_{E_4} E_5 = E_4, \\
&\nabla_{E_5} E_1 = 0, \quad \nabla_{E_5} E_2 = 0, \quad \nabla_{E_5} E_3 = 0, \quad \nabla_{E_5} E_4 = 0, \quad \nabla_{E_5} E_5 = 0.
\end{aligned}$$

Further we obtain the following:

$$\nabla^*_{E_i} E_j = 0, \qquad i, j = 1, 2, 3, 4, 5.$$

and hence

$$(\nabla^*_{E_i} \phi)E_j = 0, \qquad i, j = 1, 2, 3, 4, 5.$$

From the above expressions it follows that the manifold satisfies (2.2), (2.3) and (2.4) for $\xi = E_5$. Hence the manifold is a Kenmotsu manifold. With the help of the above results we can verify the following results.

$$\begin{aligned}
&R(E_1,E_2)E_2 = R(E_1,E_3)E_3 = R(E_1,E_4)E_4 = R(E_1,E_5)E_5 = -E_1,\\
&R(E_1,E_2)E_1 = E_2, \quad R(E_1,E_3)E_1 = R(E_5,E_3)E_5 = R(E_2,E_3)E_5 = E_3,\\
&R(E_2,E_3)E_3 = R(E_2,E_4)E_4 = R(E_2,E_5)E_5 = -E_2, \quad R(E_3,E_4)E_4 = -E_3,\\
&R(E_2,E_5)E_2 = R(E_1,E_5)E_1 = R(E_4,E_5)E_4 = R(E_3,E_5)E_3 = E_5,\\
&R(E_1,E_4)E_1 = R(E_2,E_4)E_2 = R(E_3,E_4)E_3 = R(E_5,E_4)E_5 = E_4
\end{aligned}$$

and

$$R^*(E_i,E_j)E_k = 0, \qquad i,j,k = 1,2,3,4,5.$$

From the above expressions of the curvature tensor of the Kenmotsu manifold it can be easily seen that the manifold has a constant sectional curvature -1.

Making use of the above results we obtain the Ricci tensors as follows:

$$\begin{aligned}
&S(E_1,E_1) = g(R(E_1,E_2)E_2,E_1) + g(R(E_1,E_3)E_3,E_1) + g(R(E_1,E_4)E_4,E_1)\\
&+ g(R(E_1,E_5)E_5,E_1) = -4.
\end{aligned}$$

Similarly, we have

$$S(E_2,E_2) = S(E_3,E_3) = S(E_3,E_3) = S(E_4,E_4) = S(E_5,E_5) = -4$$

and

$$S^*(E_1,E_1) = S^*(E_2,E_2) = S^*(E_3,E_3) = S^*(E_4,E_4) = S(E_5,E_5) = 0.$$

Therefore, it can be easily verified that the manifold is an Einstein manifold with respect to Levi-Civita connection.

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Accepted: November 23, 2020