On classes of Janowski functions associated with a conic domain

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Abstract. The purpose of this paper is to define a new class of close-to-convex function which are analytic, normalized functions in the open unit disk subordinating with a conic region and to derive initial coefficient estimates a_2 , a_3 for the function class. We also investigate interesting characteristic properties like sufficient conditions, inclusion relationship, and radius of convexity. Further by appropriate choice of parameters, we provided some new and well known results of our main result.

Keywords: Janowski starlike functions, uniformly convex, close-to-convex function, subordination, coefficient inequalities, q-calculus, differential operator.

1. Introduction

Denote by $\mathcal{A}$ the class of functions having a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \left(z \in \mathbb{U} = \{z : |z| < 1\}\right). \tag{1}$$

We let $\mathcal{S}$ to denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. We let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ to denote the well known classes of starlike, convex and close-

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to-convex (normalized) function respectively. For $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ symbolize the classes of starlike functions of order α and convex functions of order α respectively. Also let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ that are analytic in $\mathbb{U}$ and such that $Re\,(p(z)) > 0$ for all z in $\mathbb{U}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an Schwarz function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Using the concept of subordination of analytic functions, Ma and Minda [13] introduced the class $\mathcal{S}^*(\phi)$ of functions in $\mathcal{A}$ satisfying $\frac{zf^{'}(z)}{f(z)} \prec \phi$, where $\phi \in \mathcal{P}$ with $\phi^{'}(0) > 0$ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. This class specializes to several classes of univalent functions for suitable choice of ϕ. For instance, the class $\mathcal{S}^*(\frac{1+Az}{1+Bz}) =: \mathcal{S}^*[A; B]$ where $-1 \leq B < A \leq 1$, is the class of the Janowski starlike functions (see [7]).

In 1993, Goodman [4] introduced the geometrically defined class of uniformly convex functions (UCV), for which Rønning [20] and Ma and Minda [14] independently gave the following one variable characterization for the class UCV, as

$$\Re\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) > \left|\frac{zf^{''}(z)}{f^{'}(z)}\right|, \quad z \in \mathbb{U}. \tag{2}$$

Further, Bharati et al., [2] defined the subclass $k - UCV$, of order α as below: A function $f(z) \in \mathcal{A}$ is said to be k-uniformly convex of order α, if

$$\Re\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) > k\left|\frac{zf^{''}(z)}{f^{'}(z)}\right| + \alpha, \quad z \in \mathbb{U}. \tag{3}$$

Special cases of $k - UCV(\alpha)$ namely $k - UCV(0) = k - UCV$ were introduced by Kanas and Wiśniowska [8, 9]. Geometrically, a function f is in $k - UCV(\alpha)$ if and only if $1 + \frac{zf^{''}(z)}{f^{'}(z)}$ takes all values in the convex domain $\mathcal{D}_{k,\,\alpha}$ which is included in right half plane. Moreover, $\mathcal{D}_{k,\,\alpha}$ is an elliptic region for $k > 1$, a parabolic region for $k = 1$, a hyperbolic region for $0 < k < 1$, the half plane $u > \alpha$ for $k = 0$.

The function $p_{k,\alpha}(z)$ plays the role of an extremal functions those related to these conic domain $\mathcal{D}_{k,\,\alpha}$ and is given by

$$\hat{p}_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z}, & \text{if } k = 0 \\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } k = 1 \\ 1 + \frac{2(1-\alpha)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right) arc\tanh\sqrt{z}\right], & \text{if } 0 < k < 1 \\ 1 + \frac{2(1-\alpha)}{1-k^2}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{t}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1}, & \text{if } k > 1 \end{cases} \tag{4}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0, 1)$ and t is chosen such that $k = \cosh(\frac{\pi R'(t)}{4R(t)})$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, $\hat{p}_{k,\alpha}(z)$ is in $\mathcal{P}$ with the expansion of the form

$$\hat{p}_{k,\alpha}(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots, \qquad (\delta_j = p_j(k, \alpha),\, j = 1, 2, 3, \ldots), \tag{5}$$

we get

$$\delta_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)}, & \text{if } 0 \le k < 1, \\ \frac{8(1-\alpha)}{\pi^2}, & \text{if } k = 1 \\ \frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & \text{if } k > 1. \end{cases} \tag{6}$$

Now we briefly recall the q-calculus and the notations which are required for our study. Quantum calculus (q-*calculus* and h-*calculus*) is common classical calculus without the notion of limits. Here, h represents Planckís constant, while q represents quantum. Due to its application in a variety of branches such as physics, mathematics, the area of q-*calculus* has gained great importance for researchers. The first study on q-*calculus* was systematically established by Jackson [6] as q-derivative is merely a ratio which is given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (0 < q < 1).$$

Note that $\lim_{q\to 1^-} D_q f(z) = f'(z)$. Notations and symbols play an very important role in the study of q-calculus. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C})$$

and the q-shifted factorial by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\ldots\left(1-aq^{n-1}\right), & n = 1, 2, \ldots. \end{cases}$$

In [5], Ismail et. al. introduced the class $\mathcal{S}_q^*$ to be the class of functions which satisfy the condition

$$\left| \frac{zD_q f(z)}{f(z)} - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \qquad (f \in \mathcal{S}).$$

Equivalently, a function $f \in \mathcal{S}_q^*$ if and only if the following subordination condition (see [17, 24])

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1+z}{1-qz}$$

holds. Recently in [19], the authors defined the following q-differential operator

$$\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f : \mathbb{U} \to \mathbb{U}$$

given by

$$\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f = z + \sum_{n=2}^{\infty} [1-\lambda+\lambda[n]_q]^m\; \Gamma_n a_n z^n, \tag{7}$$

$$(m \in N_0 = N \cup \{0\} \text{ and } \lambda \geq 0)\,,$$

where

$$\Gamma_n = \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\ldots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\ldots(b_s;q)_{n-1}}, \quad (|q|<1)\,.$$

For detailed study and applications of the operator $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f$, refer to [19] and the references provided therein.

Throughout our present discussion, to avoid repetition, we lay down once for all that $-1 \leq B < A \leq 1$, $z \in \mathbb{U}$ and Γ_n is real.

Definition 1.1. *For $p_{k,\alpha}(z)$ defined as in (4), a function $f \in k\text{–}\mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ if and only if*

$$p(z) \prec \frac{(A+1)\hat{p}_{k,\alpha}(z)-(A-1)}{(B+1)\hat{p}_{k,\alpha}(z)-(B-1)}, \qquad (f \in \mathcal{A};\, |t| \leq 1,\, t \neq 0) \tag{8}$$

when $p(z) = \frac{tz^2[\mathcal{J}_\lambda^m(a_1,b_1;\,q,z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)g(z)][\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)g(tz)]}$ *and* $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

Analogous to the class $k-\mathcal{UCV}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$, we define the following.

Definition 1.2. *Let $\mathcal{V}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$, $|t| \leq 1$, $t \neq 0$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying the conditions*

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(tz)]} \prec \frac{(A+1)\hat{p}(z)-(A-1)}{(B+1)\hat{p}(z)-(B-1)}, \tag{9}$$

where $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ *and* $\hat{p}(z) = \frac{1+z}{1-qz}$, $q \in (0,\, 1)$.

By definition of subordination, a function $f \in \mathcal{A}$ is said to be in $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ if and only if

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(tz)]} = \frac{(A+1)w(z)+2+(A-1)qw(z)}{(B+1)w(z)+2+(B-1)qw(z)}, \tag{10}$$

where $q \in (0,\, 1)$; $z \in \mathbb{U}$; $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, $|w(z)| < 1$.

Remark 1.1. The classes $\mathcal{V}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$ and $k\text{–}\mathcal{UCV}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$ were motivated by study of [3, 12]. For study of classes associated with Janowski functions on conic domains, refer to [1, 15, 16, 24].

In the present work, we obtain the coefficient estimates, sufficient conditions, inclusion relation and radius of convexity for the functions in the classes $\mathcal{V}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$ and $k-\mathcal{UCV}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$. Also, we obtain the Fekete-Szegö problem for the functions in the class $\mathcal{V}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$.

2. Coefficient estimates and Fekete-Szegö inequality

To prove our results, we make use of the following Lemmas.

Lemma 2.1 ([18]). *let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\mathbb{U}$ and $g(z) = \sum_{n=1}^{\infty} b_z z^n$ is analytic and convex in $\mathbb{U}$. If $f(z) \prec g(z)$, then $|a_n| \leq |b_1|$, for $n = 1, 2, \ldots$.*

Lemma 2.2 ([16]). *Let $p(z) \in \mathcal{P}$ satisfy the subordination condition*

$$p(z) \prec \frac{(A+1)p_{k,\alpha}(z) - (A-1)}{(B+1)p_{k,\alpha}(z) - (B-1)},$$

then

$$|p_n| \leq \frac{(A-B)\delta_1}{2}, \tag{11}$$

where δ_1 is given by (6).

Lemma 2.3 ([23]). *Let $g(z) \in \mathcal{S}^*(\frac{1}{2})$ and $0 < |t| \leq 1$, then $\frac{g(z)g(tz)}{tz} \in \mathcal{S}^*$.*

Using Lemma 2.3 and proceeding on lines similar to Theorem 2.3 of [23], we get the following result.

Lemma 2.4 ([23]). *Let $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, then*

$$\begin{aligned} G(z) &= \frac{[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(tz)]}{tz} \\ &= z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{S}^*, \quad \textit{and} \quad |c_n| \leq n, \end{aligned} \tag{12}$$

where

$$c_n = \left[\Psi_n b_n + \Psi_{n-1}\Psi_2 b_{n-1} b_2 t + \cdots + \Psi_n b_n t^{n-1}\right], \quad \Psi_n = [1 - \lambda + [n]_q \lambda]^m \, \Gamma_n.$$

To start with we prove the following result :

Theorem 2.1. *If $f(z) \in k-\mathcal{UCV}_{q,t}^{\lambda,m}(a_1,\, b_1; A, B)$, then*

$$| \, a_n \, | \leq \frac{1}{|\Gamma_n|[1 - \lambda + [n]_q \lambda]^m} \left(1 + \frac{(A-B)\delta_1(n-1)}{4}\right). \tag{13}$$

Proof. Let $f(z) \in k-\mathcal{UCV}_{q,t}^{\lambda,m}(a_1,\, b_1; A, B)$, then there exist a function $p(z) \in \mathcal{P}$ analytic $\mathbb{U}$ such that

$$p(z) = \frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} \prec \frac{(A+1)p_{k,\alpha}(z) - (A-1)}{(B+1)p_{k,\alpha}(z) - (B-1)}, \tag{14}$$

where $p_{k,\alpha}(z)$ is defined by the equation (4). Clearly $p_{k,\alpha}(z)$ is analytic and maps $\mathbb{U}$ on to a convex domain. From (14), we have

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} = p(z), \quad (p(z) \in \mathcal{P})\,. \tag{15}$$

Using Lemma 2.2 and Lemma 2.4 in (15), we get

$$z + \sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n a_n z^n = (1+\sum_{n=1}^{\infty} p_n z^n)(z+\sum_{n=2}^{\infty} c_n z^n). \tag{16}$$

Equating the coefficients of z^n in (16), we have

$$n\Gamma_n a_n[1-\lambda+[n]_q\lambda]^m = c_n + p_1c_{n-1} + p_2c_{n-2} + \cdots + p_{n-1}. \tag{17}$$

Therefore using (11) and Lemma 2.4 , we get

$$n\Gamma_n[1-\lambda+[n]_q\lambda]^m \mid a_n \mid \leq n\left(1+\frac{(A-B)\delta_1(n-1)}{4}\right). \tag{18}$$

From (18), we get the desired result in (13). □

Theorem 2.2. *If $f(z) \in \mathcal{V}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1; A, B)$, then*

$$\mid a_n \mid \leq \frac{1}{|\Gamma_n|[1-\lambda+[n]_q\lambda]^m}\left(1+\frac{(A-B)(1+q)(n-1)}{4}\right). \tag{19}$$

Proof. It can be easily seen that

$$p(z) = \frac{(A+1)w(z)+2+(A-1)qw(z)}{(B+1)w(z)+2+(B-1)qw(z)} = 1+(A-B)(1+q)z+\cdots$$

is analytic and maps $\mathbb{U}$ on to a convex domain. By Lemma 2.1, we obtain

$$|p_n| \leq \frac{(A-B)(1+q)}{2}, \qquad (n \geq 1).$$

Now, following the steps as in Theorem 2.1, we get the desired result. □

If we let $k = m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \to 1^-$ in Theorem 2.2, we get the following result

Corollary 2.1 ([22])**.** *If $X_t\,(A,\, B)$, then $|a_n| \leq 1+\frac{(n-1)(A-B)}{2}$.*

Letting $A = 1-2\alpha$, $B = -1$ in Corollary 2.1, the following result due to Prajapat [21] becomes obvious.

Corollary 2.2. *If $f \in X_t\,(\alpha)$ then $\mid a_n \mid \leq 1+(n-1)(1-\alpha)$.*

2.1 Fekete-Szegö problem

We recall the following lemmas to prove our results in this subsection:

Lemma 2.5 ([10]). *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part, then for each complex number μ*

$$| p_2 - \mu p_1^2 | \leq 2 \max (1, | 2\mu - 1 |) \tag{20}$$

and the result is sharp for the functions given by $p(z) = \frac{1+z^2}{1-z^2}, p(z) = \frac{1+z}{1-z}$.

Lemma 2.6 ([11]). *If*

$$G(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{S}^*,$$

then for each complex number λ we have $| c_3 - \lambda c_2^2 | \leq \max (1, | 3 - 4\lambda |)$ and the result is sharp for the Koebe functon

$$k(z) = \frac{z}{(1-z)^2} \quad if \quad \left| \lambda - \frac{3}{4} \right| \geq \frac{1}{4}$$

and for

$$k^{\frac{1}{2}} \left(z^2 \right) = \frac{z}{1-z^2} \quad if \quad | \lambda - \frac{3}{4} | \leq \frac{1}{4}.$$

Theorem 2.3. *If $f(z) \in k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then for $\mu \in \mathbb{C}$ we have*

$$\begin{aligned} | a_3 - \mu a_2^2 | \leq & \frac{(A-B)\delta_1}{6\Gamma_3[1-\lambda+[3]_q\lambda]^m} \max (1, | 2\vartheta - 1 |) \\ & + \frac{1}{3} \max (1, | 3 - 4\nu_1 |) + \frac{(A-B)\delta_1}{\Gamma_3[1-\lambda+[3]_q\lambda]^m} \left| \frac{1}{3} - \frac{\mu}{2} \right|, \end{aligned} \tag{21}$$

where

$$\vartheta = \frac{(B+1)\delta_1 + 2\left(1 - \frac{\delta_2}{\delta_1}\right)}{4} + \frac{3(A-B)\delta_1\mu\Gamma_3[1-\lambda+[3]_q\lambda]^m}{16\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}}$$

and $\nu_1 = \frac{3\mu}{4}$.

Proof. Let $h(z) \in \mathcal{P}$ be of the form $h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, we consider

$$h(z) = \frac{1 + w(z)}{1 - w(z)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. On simple computation, we have

$$\begin{aligned} w(z) &= \frac{h(z) - 1}{h(z) + 1} \\ &= \frac{p_1 z + p_2 z^2 + p_3 z^3 + \cdots}{2 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots} \\ &= \frac{1}{2} p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) z^2 + \frac{1}{2}\left(p_3 - p_1 p_2 + \frac{1}{4}p_1^3\right) z^3 + \cdots . \end{aligned} \tag{22}$$

Using (22) in (5), we have

$$\begin{aligned}\hat{p}_{k,\alpha}(w(z)) &= 1+\delta_1 w(z)+\delta_2[w(z)]^2+\delta_3[w(z)]^3+\cdots\\ &= 1+\frac{\delta_1 p_1}{2}z+\frac{\delta_1}{2}\left[p_2-\frac{1}{2}\left(1-\frac{\delta_2}{\delta_1}\right)p_1^2\right]z^2+\cdots.\end{aligned}$$

As $f(z)\in k-\mathcal{UCV}_{q,t}^{\lambda,m}(a_1,\,b_1;A,B)$, by (14) we have

$$\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f(z)]'}{G(z)}=\frac{(A+1)p_{k,\alpha}(w(z))-(A-1)}{(B+1)p_{k,\alpha}(w(z))-(B-1)}. \tag{23}$$

From (23), we obtain

$$\begin{aligned}&1+\Gamma_2[1-\lambda+[2]_q\lambda]^m(2a_2-c_2)z\\ &+\Gamma_3[1-\lambda+[3]_q\lambda]^m(3a_3-2a_2c_2-c_3+c_2^2)z^2+\cdots\\ &=1+\frac{\delta_1 p_1(A-B)}{4}z+\frac{(A-B)\delta_1}{4}\left[p_2-p_1^2\left(\frac{(B+1)\delta_1+2\left(1-\frac{\delta_2}{\delta_1}\right)}{4}\right)\right]z^2+\cdots.\end{aligned}$$

Equating the coefficients at z and z^2 on both sides of the above equation, we get

$$a_2=\frac{p_1(A-B)\delta_1+4c_2\Gamma_2[1-\lambda+[2]_q\lambda]^m}{8\Gamma_2[1-\lambda+[2]_q\lambda]^m}$$

and

$$a_3=\frac{1}{3}\left[c_3+\frac{(A-B)\delta_1}{4\Gamma_3[1-\lambda+[3]_q\lambda]^m}\left(p_1c_2+p_2-p_1^2\frac{(B+1)\delta_1+2\left(1-\frac{\delta_2}{\delta_1}\right)}{4}\right)\right].$$

Therefore, we have

$$\begin{aligned}|\,a_3-\mu a_2^2\,| &\le \frac{(A-B)\delta_1}{12\Gamma_3[1-\lambda+[3]_q\lambda]^m}\,|\,p_2-\vartheta p_1^2\,|+\frac{1}{3}\left|c_3-\nu_1c_2^2\right|\\ &+\frac{(A-B)\delta_1}{4\Gamma_3[1-\lambda+[3]_q\lambda]^m}|c_2|\left|\left(\frac{1}{3}-\frac{\mu}{2}\right)\right||p_1|.\end{aligned} \tag{24}$$

Using Lemma 2.5 and Lemma 2.6, we complete the proof. □

Theorem 2.4. *If $f(z)\in\mathcal{S}_{q,t}^{\lambda,m}(a_1,\,b_1;\,A,\,B)$, then for $\mu\in\mathbb{C}$ we have*

$$\begin{aligned}|\,a_3-\mu a_2^2\,| &\le \frac{(A-B)(1+q)}{6\Gamma_3[1-\lambda+[3]_q\lambda]^m}\max(1,|\,2\eta-1\,|)\\ &+\frac{1}{3}\max(1,|\,3-4\mu_1\,|)+\frac{(A-B)(1+q)}{\Gamma_3[1-\lambda+[3]_q\lambda]^m}\left|\frac{1}{3}-\frac{\mu}{2}\right|,\end{aligned} \tag{25}$$

where

$$\eta=\frac{2+B(1+q)+(1-q)}{4}+\frac{3(A-B)(1+q)\mu\,\Gamma_3[1-\lambda+[3]_q\lambda]^m}{16\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}}$$

and $\mu_1=\frac{3\mu}{4}$.

Proof. As $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, by (10) we have

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} = \frac{(A+1)w(z) + 2 + (A-1)qw(z)}{(B+1)w(z) + 2 + (B-1)qw(z)}.$$

Let

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots,$$

then $Re(h(z)) > 0$ and $h(0) = 1$. Hence,

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} = \frac{(A+1)(1+q)\,[h(z)-1] + 2\,[h(z)+1-q(h(z)-1)]}{(B+1)(1+q)\,[h(z)-1] + 2\,[h(z)+1-q(h(z)-1)]}. \tag{26}$$

From (26), we obtain

$$\begin{aligned} &1 + \Gamma_2[1-\lambda+[2]_q\lambda]^m(2a_2 - c_2)z \\ &+ \Gamma_3[1-\lambda+[3]_q\lambda]^m(3a_3 - 2a_2c_2 - c_3 + c_2^2)z^2 + \cdots \\ &= 1 + \frac{p_1(A-B)(1+q)}{4}z \\ &+ \frac{(A-B)(1+q)}{4}\left[p_2 - p_1^2\left(\frac{2 + B(1+q) + (1-q)}{4}\right)\right]z^2 + \cdots. \end{aligned}$$

Equating the corresponding coefficients of the above equation, we get

$$a_2 = \frac{p_1(A-B)(1+q) + 4c_2\Gamma_2[1-\lambda+[2]_q\lambda]^m}{8\Gamma_2[1-\lambda+[2]_q\lambda]^m}$$

and

$$a_3 = \frac{1}{3}\left[c_3 + \frac{(A-B)(1+q)}{4\Gamma_3[1-\lambda+[3]_q\lambda]^m}\left(p_1c_2 + p_2 - \frac{p_1^2[2 + B(1+q) + (1-q)]}{4}\right)\right].$$

Therefore, we have

$$\begin{aligned} |\, a_3 - \mu a_2^2 \,| \;\leq\; & \frac{(A-B)(1+q)}{12\Gamma_3[1-\lambda+[3]_q\lambda]^m}\,|\, p_2 - \eta p_1^2 \,| \\ +\; & \frac{|\, c_3 - \mu_1 c_2^2 \,|}{3} + \frac{(A-B)(1+q)}{4\Gamma_3[1-\lambda+[3]_q\lambda]^m}|c_2|\left|\left(\frac{1}{3} - \frac{\mu}{2}\right)\right||p_1|. \end{aligned} \tag{27}$$

Using Lemma 2.5 and Lemma 2.6, we complete the proof. □

If we let $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \to 1^-$ in Theorem 2.4, we get the following result.

Corollary 2.3 ([22]). *If $f(z) \in X_t(A, B)$, then for $\mu \in \mathbb{C}$ we have*

$$\begin{aligned} |a_3 - \mu a_2^2| \leq \frac{A-B}{3} \max(1, |2\beta - 1|) + \frac{1}{3} \max(1, |3 - 4\mu_1|) \\ + 2(A-B)\left|\frac{1}{3} - \frac{\mu}{2}\right|, \end{aligned} \tag{28}$$

where

$$\beta = \frac{1+B}{2} + \frac{3(A-B)\mu}{8}, \ \mu_1 = \frac{3\mu}{4}.$$

For $A = 1 - 2\alpha, B = -1$, Theorem 2.4 gives the following result which is analogous to one obtained by Prajapat in [21].

Corollary 2.4. *If $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z) \in X_t(\alpha)$, then for $\mu \in \mathbb{C}$,*

$$\begin{aligned} |a_3 - \mu a_2^2| \leq \frac{2(1-\alpha)}{3\Gamma_3[1-\lambda+[3]_q\lambda]^m} \max(1, |2\beta - 1|) \\ + \frac{1}{3} \max(1, |3 - 4\mu_1|) + \frac{4(1-\alpha)}{\Gamma_3[1-\lambda+[3]_q\lambda]^m}\left|\frac{1}{3} - \frac{\mu}{2}\right| \end{aligned} \tag{29}$$

where

$$\beta = \frac{3(1-\alpha)\mu\Gamma_3[1-\lambda+[3]_q\lambda]^m}{4\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}} \quad \text{and} \quad \mu_1 = \frac{3\mu}{4}.$$

3. Characterization properties of $k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$

Theorem 3.1. *Let the function $f(z)$ defined by (1) and let*

$$\sum_{n=2}^{\infty} \frac{2(k+1)}{1-\alpha}|na_n - b_n| - |n(B+1)a_n - (A+1)b_n| < (A-B) \tag{30}$$

holds, then $f(z)$ belongs to $k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$.

Proof. Assume that (30) holds. It is sufficient to show that

$$\begin{aligned} k\left|\frac{(B-1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)} - (A-1)}{(B+1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)} - (B-1)} - 1\right| \\ - \Re\left(\frac{(B-1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)} - (A-1)}{(B+1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)} - (B-1)} - 1\right) < 1 - \alpha. \end{aligned}$$

Now, consider

$$
\begin{aligned}
& k\left|\frac{(B-1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)}-(A-1)}{(B+1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)}-(B-1)}-1\right| \\
& -\Re\left(\frac{(B-1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)}-(A-1)}{(B+1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)}-(B-1)}-1\right) \\
& \leq (k+1)\left|\frac{(B-1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)}-(A-1)}{(B+1)\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)}-(B-1)}-1\right| \\
& \leq \frac{2(k+1)\sum_{n=2}^{\infty}|na_n-b_n|}{(A-B)-\sum_{n=2}^{\infty}|n(B+1)a_n-(A+1)b_n|}
\end{aligned}
$$

The last inequality is bounded by $1-\alpha$ if

$$\sum_{n=2}^{\infty}\frac{2(k+1)}{1-\alpha}|na_n-b_n|-|n(B+1)a_n-(A+1)b_n|<(A-B). \qquad \square$$

Theorem 3.2. *Let the function $f(z)$ defined by (1) and let*

$$
\begin{aligned}
& \left(1+\frac{|B(1+q)+(1-q)|}{2}\right)\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n|a_n| \\
& +\left(1+\frac{|A(1+q)+(1-q)|}{2}\right)\sum_{n=2}^{\infty}|c_n|<\frac{(A-B)(1+q)}{2}
\end{aligned} \tag{31}
$$

holds, then $f(z)$ belongs to $\mathcal{V}_{q,t}^{\lambda,m}(a_1,b_1;A,B)$.

Proof. Let us assume that (31) holds. Now, consider

$$
\begin{aligned}
& \left|z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'-G(z)\right| \\
& -\left|\frac{A(1+q)+(1-q)}{2}G(z)-\frac{B(1+q)+(1-q)}{2}z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'\right| \\
& =\left|\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n a_n z^n-\sum_{n=2}^{\infty}c_n z^n\right| \\
& -\left|\frac{A(1+q)+(1-q)}{2}\left(z+\sum_{n=2}^{\infty}c_n z^n\right)\right. \\
& \left.-\frac{B(1+q)+(1-q)}{2}\left(z+\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n a_n z^n\right)\right| \\
& \leq\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n|a_n||z|^n+\sum_{n=2}^{\infty}|c_n||z|^n-\left(\frac{(A-B)(1+q)}{2}|z|\right.
\end{aligned}
$$

$$- \frac{A(1+q)+(1-q)}{2}\sum_{n=2}^{\infty}|c_n||z|^n$$
$$\left. - \frac{B(1+q)+(1-q)}{2}\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n|a_n||z^n|\right)$$
$$= -\frac{(A-B)(1+q)}{2}|z|$$
$$+\left(1+\frac{B(1+q)+(1-q)}{2}\right)\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n|a_n||z|^n$$
$$+\left(1+\frac{A(1+q)+(1-q)}{2}\right)\sum_{n=2}^{\infty}|c_n||z|^n \le 0.$$

From the above inequality, we have established that

$$\left|z[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f(z)]'-G(z)\right|$$
$$<\left|\frac{A(1+q)+(1-q)}{2}G(z)-\frac{B(1+q)+(1-q)}{2}z[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f(z)]'\right|,$$

which is equivalent to (10). Thus $f(z)\in\mathcal{V}_{q,t}^{\lambda,m}(a_1,\,b_1;A,B)$. □

3.1 Inclusion relation

The following lemma is useful in the proof of the main result in this subsection.

Lemma 3.1 ([23])**.** *If* $-1\le B_2\le B_1<A_1\le A_2\le 1$, *then*

$$\frac{1+A_1z}{1+B_1z}\prec\frac{1+A_2z}{1+B_2z}.$$

Theorem 3.3. *Let* $-1\le B_2\le B_1<A_1\le A_2\le 1$, *then*

$$\mathcal{V}_{q,t}^{\lambda,m}(a_1,\,b_1;A_1,B_1)\subset\mathcal{V}_{q,t}^{\lambda,m}(a_1,\,b_1;A_2,B_2).$$

Proof. As $f(z)\in\mathcal{V}_{q,t}^{\lambda,m}(a_1,\,b_1;A_1,\,B_1)$, therefore

$$p(z)=\frac{tz^2[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)g(z)]g(tz)}\prec\frac{(A_1+1)z+2+(A_1-1)qz}{(B_1+1)z+2+(B_1-1)qz}. \tag{32}$$

Since $-1\le B_2\le B_1<A_1<A_2\le 1$, by Lemma 3.1, we have

$$p(z)\prec\frac{(A_1+1)z+2+(A_1-1)qz}{(B_1+1)z+2+(B_1-1)qz}\prec\frac{(A_2+1)z+2+(A_2-1)qz}{(B_2+1)z+2+(B_2-1)qz}. \tag{33}$$

This yields that $f(z)\in\mathcal{V}_{q,t}^{\lambda,m}(a_1,\,b_1;A_2,B_2)$ and this proves the inclusion relation. □

For completeness, we just state the following.

Theorem 3.4. *Let* $-1\le B_2\le B_1<A_1\le A_2\le 1$, *then*

$$k-\mathcal{UCV}_{q,t}^{\lambda,m}(a_1,\,b_1;A_1,B_1)\subset k-\mathcal{UCV}_{q,t}^{\lambda,m}(a_1,\,b_1;A_2,B_2).$$

3.2 Radius of convexity

Theorem 3.5. *If* $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1)$, *then* $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ *is convex in* $|z| = r_1$, *where* r_1 *is the smallest positive root in* $(0, 1)$ *of the equation*

$$Qr^3 - Rr^2 - 4Sr - 4 = 0, \tag{34}$$

where $Q = AB(1+q)^2 + (A+B)(1-q^2) + (1-q)^2$, $R = [A(1+q)+(1-q)][B(1+q)-(3+q)]$ *and* $S = [B(1+q) - q]$.

Proof. As $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1)$, we have

$$z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' = G(z)p(z). \tag{35}$$

By applying logarithmic differentiation in (35), we get

$$1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'} = \frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)}. \tag{36}$$

Now, for $G(z) \in \mathcal{S}^*$, we have

$$\Re\left(\frac{zG'(z)}{G(z)}\right) \geq \frac{1-r}{1+r}.$$

Therefore, (36) yields that

$$\Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \geq \frac{1-r}{1+r} - \left|\frac{zp'(z)}{p(z)}\right|.$$

Further, we have

$$\Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \geq \frac{1-r}{1+r} - \frac{2r(A-B)(1+q)}{(2 + A(1+q)r + (1-q)r)(2 + B(1+q)r + (1-q)r)}.$$

By a straightforward computation, we have

$$\Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \geq \frac{-Qr^3 + Rr^2 + 4Sr + 4}{4(1+r)(2 + A(1+q)r + (1-q)r)(2 + B(1+q)r + (1-q)r)}.$$

Hence, the function $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ is convex in $|z| < r_1$, where r_1 is the smallest positive root in $(0, 1)$ of the equation $Qr^3 - Rr^2 - 4Sr - 4 = 0$. □

If we let $A = 1 - 2\alpha$, $B = -1$, $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \to 1^-$ in Theorem 3.5 gives the following result obtained by Prajapat [21].

Corollary 3.1. *If* $f(z) \in X_t(\alpha)$, *then* $f(z)$ *is convex in* $|z| < r_0 = 2 - \sqrt{3}$.

Conclusion

The motivation of the classes $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $k-\mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, was from the study of univalent functions with respect to symmetric points. Many authors introduced and defined various classes of analytic functions with respect to symmetric points. For appropriate choice of parameters involved in the classes $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $k-\mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, several well-known and new results can be obtained as special case of our results.

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