On refined Young's inequality

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Abstract. In this paper, we obtain new refinements of the classical Young's inequality for positive real numbers, and by using these results, we establish the corresponding inequalities for Hilbert space operators.
Keywords: operator inequality, Young's inequality, positive operator, operator mean.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with the operator norm $\|\cdot\|$ and the identity $I_{\mathcal{H}}$. If $\dim\mathcal{H} = n$, then we identify $\mathcal{B}(\mathcal{H})$ with the space $\mathcal{M}_n$ of all $n \times n$ complex matrices and denote the identity matrix by I_n. A norm $|||\cdot|||$ on $\mathcal{M}_n$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. For an operator $A \in \mathcal{B}(\mathcal{H})$, we write $A \geq 0$ if A is positive (positive semidefinite for matrices), and $A > 0$ if A is positive and invertible (positive definite for matrices). For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \geq B$ if $A - B \geq 0$. For $A = (a_{ij}) \in \mathcal{M}_n$, the Hilbert-Schmidt (or Frobenius) norm is defined by $\|A\|_2 = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$. $\|\cdot\|_2$ has the unitarily invariant property.

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Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators and $\nu \in [0,1]$. The ν-weighted arithmetic mean of A and B, denoted by $A\nabla_\nu B$, is defined as

$$A\nabla_\nu B = (1-\nu)\,A + \nu B.$$

If A is invertible, ν-geometric mean of A and B, denoted by $A\sharp_\nu B$, is defined by

$$A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}}.$$

When $\nu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The well-known Young's inequality is a classical result attributed to the English mathematician William Henry Young (1863-1942) stating that

$$a^{\nu}b^{1-\nu} \le \nu a + (1-\nu)\,b, \tag{1}$$

where a and b are distinct positive real numbers and $\nu \in [0,1]$.

The inequality (1) was refined by Kittaneh and Mansarah [3] in the following form

$$\left(a^{\nu}b^{1-\nu}\right)^2 + r^2(a-b)^2 \le (\nu a + (1-\nu)\,b)^2, \tag{2}$$

where $r = \min\{\nu, 1-\nu\}$.

The authors of [6] obtained another refinement of the Young inequality as follows:

$$\left(a^{\nu}b^{1-\nu}\right)^2 + r(a-b)^2 \le \nu a^2 + (1-\nu)\,b^2. \tag{3}$$

More recently, Hu in [4] gave the following Young type inequalities:

$$\begin{cases} \left((\nu a)^{\nu}b^{1-\nu}\right)^2 + \nu^2(a-b)^2 \le \nu^2a^2 + (1-\nu)^2b^2, & 0 \le \nu \le \dfrac{1}{2} \\ \left\{\left(a^{\nu}(1-\nu)b\right)^{1-\nu}\right\}^2 + (1-\nu)^2(a-b)^2 \le \nu^2a^2 + (1-\nu)^2b^2, & \dfrac{1}{2} \le \nu \le 1 \end{cases}. \tag{4}$$

When comparing inequalities (4) with the inequalities (2) and (3), it is easy to observe that both the left-hand and the right-hand sides of inequalities (4) are greater than or equal to the corresponding sides in (2) and (3), respectively. It should be noticed that neither inequalities (4) nor (2) and (3) is uniformly better than the other. These inequalities are extended to matrices in various contexts. The original Young's inequality was first extended to $\mathcal{M}_n$ in [1] as follows:

$$|||A^{\nu}B^{1-\nu}||| \le |||\nu A + (1-\nu)\,B|||,$$

where $A, B \in \mathcal{M}_n$ are positive semidefinite and $\nu \in [0,1]$.

The Frobenius norm version of Young's inequality was established by Kosaki [7] and also by Bhatia and Parthasarathy [2] as

$$\left\|A^{\nu}XB^{1-\nu}\right\|_2 \le \left\|\nu AX + (1-\nu)\,XB\right\|_2,$$

for $A, B, X \in \mathcal{M}_n(\mathbb{C})$, with A and B positive semidefinite and $\nu \in [0,1]$. Then, each refinement of the scalar Young's inequality accompanies a corresponding refinement of the matrix inequality. For example, let $A, B, X \in \mathcal{M}_n$, such that A and B are positive semidefinite matrices. Then, the matrix versions of (4) are (see [4])

$$\nu^2 \|AX - XB\|_2^2 + \nu^{2\nu} \left\|A^\nu X B^{1-\nu}\right\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|_2^2 \le \|\nu AX + (1-\nu) XB\|_2^2 \tag{5}$$

for $0 \le \nu \le \frac{1}{2}$, and

$$(1-\nu)^2 \|AX - XB\|_2^2 + (1-\nu)^{2\nu} \left\|A^\nu X B^{1-\nu}\right\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|_2^2 \le \|\nu AX + (1-\nu) XB\|_2^2, \tag{6}$$

for $\frac{1}{2} \le \nu \le 1$.

The primary objective of this paper is to present new inequalities of Young's-type. Here we employ the analysis, which is utilized in [5]. In section 2, we first propose a refinement of the inequalities in (4). Furthermore, we establish the operator analogue of the new scalar inequalities.

2. Main results

2.1 Scalar inequalities

The main aim in this section is the improvement of Young's inequality established in [4].

Throughout the paper we will use the following notations:

$$S_N(\nu; a, b) = \sum_{j=1}^{N} s_j(\nu) \left(\sqrt[2^j]{b^{2^{j-1}-k_j(\nu)} a^{k_j(\nu)}} - \sqrt[2^j]{a^{k_j(\nu)+1} b^{2^{j-1}-k_j(\nu)-1}} \right)^2$$

with

$$k_j(\nu) = \left[2^{j-1}\nu\right], \; r_j(\nu) = \left[2^j \nu\right] \text{ and } s_j(\nu) = (-1)^{r_j(\nu)} 2^{j-1}(\nu) + (-1)^{r_j(\nu)+1} \left[\frac{r_j(\nu)+1}{2}\right],$$

for $N \in \mathbb{N}$ and $j = 1, 2, \ldots, N$. Notice that $[x]$ is the greatest integer less than or equal to x.

Recall that, in [8], it has been shown that:

$$a^\nu b^{1-\nu} + S_N(\nu; a, b) \le \nu a + (1-\nu) b. \tag{7}$$

Now, we start with some numerical results.

Theorem 2.1. *Let $a, b > 0$ and $N \in \mathbb{N}$:*

(i) If $0 \le \nu \le \frac{1}{2}$*, then*

$$bS_N\left(2\nu;\nu a,b\right)+\left((\nu a)^{\nu}b^{1-\nu}\right)^2+\nu^2(a-b)^2\le \nu^2a^2+(1-\nu)^2b^2. \tag{8}$$

(ii) If $\frac{1}{2} \le \nu \le 1$*, then*

$$\begin{aligned}&aS_N\left(2\nu-1;a,(1-\nu)\,b\right)+\left\{a^{\nu}((1-\nu)\,b)^{1-\nu}\right\}^2+(1-\nu)^2(a-b)^2\\&\le \nu^2a^2+(1-\nu)^2b^2.\end{aligned} \tag{9}$$

Proof of Theorem 2.1. We consider the case $0 \le \nu \le \frac{1}{2}$ at first. By (7), we have

$$\begin{aligned}&\nu^2a^2+(1-\nu)^2b^2-\nu^2(a-b)^2\\&=b\left[2\nu\left(\nu a\right)+\left(1-2\nu\right)b\right]\\&\ge b\left[(\nu a)^{2\nu}b^{1-2\nu}+S_N\left(2\nu;\nu a,b\right)\right]\\&=\left((\nu a)^{\nu}b^{1-\nu}\right)^2+bS_N\left(2\nu;\nu a,b\right).\end{aligned} \tag{10}$$

Similarly, if $\frac{1}{2} \le \nu \le 1$, then we have

$$\begin{aligned}&\nu^2a^2+(1-\nu)^2b^2-(1-\nu)^2(a-b)^2\\&=a\left[\left(2\nu-1\right)a+2(1-\nu)^2b\right]\\&\ge a\left[a^{2\nu-1}[(1-\nu)\,b]^{2-2\nu}+S_N\left(2\nu-1;a,(1-\nu)\,b\right)\right]\\&=\left\{a^{\nu}((1-\nu)\,b)^{1-\nu}\right\}^2+aS_N\left(2\nu-1;a,(1-\nu)\,b\right).\end{aligned} \tag{11}$$

This completes the proof.

Remark 2.1. Since $bS_N\left(2\nu;\nu a,b\right),\ aS_N\left(2\nu-1;a,(1-\nu)\,b\right)\ge 0$, so Theorem 2.1 improves the inequalities in (4).

As a direct consequence of Theorem 2.1, we have:

Corollary 2.1. *Let* $a,b>0$ *and* $N\in\mathbb{N}$*:*
(i) If $0 \le \nu \le \frac{1}{2}$*, then*

$$\sqrt{b}S_N\left(2\nu;\nu\sqrt{a},\sqrt{b}\right)+\nu^{2\nu}\left(a^{\nu}b^{1-\nu}\right)+\nu^2\left(\sqrt{a}-\sqrt{b}\right)^2\le\nu^2a+(1-\nu)^2b. \tag{12}$$

(ii) If $\frac{1}{2} \le \nu \le 1$*, then*

$$\begin{aligned}&\sqrt{a}S_N\left(2\nu-1;\sqrt{a},(1-\nu)\sqrt{b}\right)+(1-\nu)^{2(1-\nu)}\left(a^{\nu}b^{1-\nu}\right)\\&+(1-\nu)^2\left(\sqrt{a}-\sqrt{b}\right)^2\le\nu^2a+(1-\nu)^2b.\end{aligned} \tag{13}$$

Corollary 2.2. *Assume that* $a, b \geq 1$*:*

(*i*) *If* $0 \leq \nu \leq \frac{1}{2}$*, then* $S_N(2\nu; \nu a, b) + \left((\nu a)^{\nu} b^{1-\nu}\right)^2 + \nu^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2$.

(*ii*) *If* $\frac{1}{2} \leq \nu \leq 1$*, then* $S_N(2\nu-1; a, (1-\nu)b) + \left(a^{\nu}((1-\nu)b)^{1-\nu}\right)^2 + (1-\nu)^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2$.

2.2 Operator versions

Here, the operator versions of the inequalities proved in the previous section are established.

Theorem 2.2. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be two positive and invertible operators, and* $N \in \mathbb{N}$*. If* $0 \leq \nu \leq \frac{1}{2}$ *and* $\alpha_j(2\nu) = \frac{k_j(2\nu)}{2^j}$*, then*

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2\nu) \Big(\nu^{2\alpha_j(2\nu)} A \sharp_{\alpha_j(2\nu)} B + \nu^{2\alpha_j(2\nu)+2^{1-j}} A \sharp_{\alpha_j(2\nu)+2^{-j}} B \\
&-2\nu^{2\alpha_j(2\nu)+2^{-j}} A \sharp_{\alpha_j(2\nu)+2^{-(j+1)}} B \Big) \\
&+ \nu^{2\nu} A \sharp_{\nu} B + 2\nu^2 (A \nabla B - A \sharp B) \\
&\leq ((1-\nu) A) \nabla_{\nu} (\nu B).
\end{aligned} \tag{14}$$

On the other hand, if $\frac{1}{2} \leq \nu \leq 1$ *and* $\beta_j(2\nu-1) = \frac{k_j(2\nu-1)}{2^j}$*, then*

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2\nu-1) \Big((1-\nu)^{1-2\beta_j(2\nu-1)} B \sharp_{\frac{1}{2}-\beta_j(2\nu-1)} A \\
&+ (1-\nu)^{1-2\beta_j(2\nu-1)-2^{1-j}} B \sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-j}} A \\
&-2(1-\nu)^{1-2\beta_j(2\nu-1)-2^{-j}} B \sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-(j+1)}} A \Big) \\
&+ (1-\nu)^{2(1-\nu)} A \sharp_{\nu} B + 2(1-\nu)^2 (A \nabla B - A \sharp B) \\
&\leq ((1-\nu) A) \nabla_{\nu} (\nu B).
\end{aligned} \tag{15}$$

Proof of Theorem 2.2. Let $b = 1$ in (12) and expand the summand to get

$$S_N\left(2\nu; \nu\sqrt{a}, 1\right) + \nu^{2\nu} a^{\nu} + \nu^2 \left(a + 1 - 2\sqrt{a}\right) \leq \nu^2 a + (1-\nu)^2,$$

where

$$\begin{aligned}
S_N\left(2\nu; \nu\sqrt{a}, 1\right) = \sum_{j=1}^{N} s_j(2\nu) \Bigg(&\sqrt[2^j]{\nu^{2k_j(2\nu)}} a^{\frac{k_j(2\nu)}{2^j}} + \sqrt[2^j]{\nu^{2k_j(2\nu)+2}} a^{\frac{k_j(2\nu)+1}{2^j}} \\
&-2 \sqrt[2^j]{\nu^{2k_j(2\nu)+1}} a^{\frac{2k_j(2\nu)+1}{2^{j+1}}} \Bigg).
\end{aligned}$$

Now, replacing a with the positive operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we get

$$S_N\left(2\nu;\nu X^{\frac{1}{2}},1\right)+\nu^{2\nu}X^{\nu}+\nu^2\left(X+I_{\mathcal{H}}-2X^{\frac{1}{2}}\right)\leq \nu^2X+(1-\nu)^2I_{\mathcal{H}},$$

where

$$S_N\left(2\nu;\nu X^{\frac{1}{2}},1\right)=\sum_{j=1}^{N}s_j\left(2\nu\right)\left(\sqrt[2^j]{\nu^{2k_j(2\nu)}}X^{\frac{k_j(2\nu)}{2^j}}+\sqrt[2^j]{\nu^{2k_j(2\nu)+2}}X^{\frac{k_j(2\nu)+1}{2^j}}\right.$$
$$\left.-2\sqrt[2^j]{\nu^{2k_j(2\nu)+1}}X^{\frac{2k_j(2\nu)+1}{2^{j+1}}}\right).$$

Multiplying both sides by $A^{\frac{1}{2}}$, we deduce the desired inequality (14).

Likewise, if $\frac{1}{2}\leq\nu\leq 1$, by (13), we can write

$$S_N\left(2\nu-1;1,(1-\nu)\sqrt{b}\right)+(1-\nu)^{2(1-\nu)}b^{1-\nu}+(1-\nu)^2\left(1+b-2\sqrt{b}\right)$$
$$\leq\nu^2+(1-\nu)^2b,$$

where

$$S_N\left(2\nu-1;1,(1-\nu)\sqrt{b}\right)$$
$$=\sum_{j=1}^{N}s_j\left(2\nu-1\right)\left((1-\nu)^{\frac{2\left(2^{j-1}-k_j(2\nu-1)\right)}{2^j}}b^{\frac{2^{j-1}-k_j(2\nu-1)}{2^j}}\right.$$
$$\left.+(1-\nu)^{\frac{2\left(2^{j-1}-k_j(2\nu-1)-1\right)}{2^j}}b^{\frac{2^{j-1}-k_j(2\nu-1)-1}{2^j}}-2(1-\nu)^{\frac{2^j-2k_j(2\nu-1)-1}{2^j}}b^{\frac{2^j-2k_j(2\nu-1)-1}{2^{j+1}}}\right).$$

Replacing b with the positive operator $Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$, and then multiplying both sides of the inequality by $B^{\frac{1}{2}}$, we get (15). This completes the proof.

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Accepted: August 28, 2020