A unified inequality of differential polynomials related to small functions

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Abstract. In this paper, we consider the value distribution of the differential polynomials $\varphi f^n f' - 1$ $(n \geq 2)$, where f is a transcendental meromorphic function and φ is a small function, and obtain a precise unified inequality by the reduced counting function.
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1. Introduction and results

Let $f(z)$ be a meromorphic function in the complex plane, we say $a(z)$ is a small function if $a(z)$ is a non-vanishing meromorphic function such that $T(r, a) = S(r, f)$ and $S(r, f)$ denotes $o(T(r, f))(r \to \infty)$, possibly outside a set of r of finite linear measure. We assumed that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [6, 9, 18, 20]).

Definition 1.1. *Let k be a positive integer, for any constant a in the complex plane. We denote by $N_{k)}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $\leq k$, by $N_{(k}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $\geq k$, by $N_k(r, 1/(f-a))$ the counting function of a-points of f with multiplicity of k. and denote the reduced counting function by $\overline{N}_{k)}(r, 1/(f-a))$, $\overline{N}_{(k}(r, 1/(f-a))$ and $\overline{N}_k(r, 1/(f-a))$, respectively.*

In 1979, E. Mues [11] proved that for a transcendental meromorphic function f in the open plane, $f^2 f' - 1$ has infinitely many zeros. This is a qualitative result. In 1992, Q. Zhang [22] has obtained a quantitative result and proved the following theorem.

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Theorem A. *Let f be transcendental meromorphic in the complex plane, then*

$$T(r,f) \leq 6N(r, \frac{1}{f^2 f' - 1}) + S(r,f). \tag{1}$$

In [14], Xu, Yi and Zhang improved Theorem A by the reduced counting function and proved the following:

Theorem B. *Let f be a transcendental meromorphic function. Then*

$$T(r,f) \leq 6\,\overline{N}(r, \frac{1}{f^2 f' - 1}) + S(r,f). \tag{2}$$

In the value distribution theory, it is a very important problem whether we can use the small function to instead of the constant in the counting function (or the reduced counting function)? For example, K. Yamanoi proved the second Nevanlinna main theorem for small functions in [17]. There were some works on the value distribution differential polynomials related to small functions. (see, [1, 2, 13, 14, 15, 16, 19, 21]

In 1993, Q. Zhang [23] studied the zeros of $f^2(z)f'(z) - a(z)$, where $a(z) \not\equiv 0$ is a small function, and improved Theorem A.

Theorem C. *Let $f(z)$ be transcendental meromorphic in the complex plane and $\varphi(z)(\not\equiv 0)$ be a small function, then*

$$T(r,f) \leq 6N(r, \frac{1}{\varphi f^2 f' - 1}) + S(r,f). \tag{3}$$

Corresponding Theorem B, it is naturally to consider the value distribution of $\varphi f^2 f' - 1$ by the reduced counting function. Recently, Xu and Yi proved the following inequality in [16], which generalized Theorem B.

Theorem D. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z)(\not\equiv 0)$ be a small function. Then*

$$T(r,f) \leq 6\overline{N}(r, \frac{1}{\varphi f^2 f' - 1}) + S(r,f). \tag{4}$$

Moreover, noting that from the proof of Theorem 2 in [5], we can obtain the following result.

Theorem E. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z)(\not\equiv 0)$ be a small function. Then for $n \geq 3$,*

$$T(r,f) \leq 2\overline{N}(r, \frac{1}{\varphi f^n f' - 1}) + S(r,f). \tag{5}$$

Naturally, the following question arises: Is it possible to give a unified inequality for $f^n f' - a$ for $n \geq 2$. In fact, we prove

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z)(\not\equiv 0)$ be a small function. If $n \geq 2$, then*

$$T(r,f) \leq \Big(\frac{6}{2n-3}\Big)\overline{N}(r, \frac{1}{\varphi f^n f' - 1}) + S(r,f). \tag{6}$$

Remark 1.1. Obviously, our result gives an affirmative answer. The result improves Theorem E when $n \geq 4$, and generalizes Theorem 1.1 when $k = 1$ and φ is a constant in [8].

2. Lemmas

In order to prove our result, we need the following lemmas.

Lemma 2.1. *Let f be a transcendental meromorphic function, and let $\varphi(z)(\not\equiv 0)$ be a small function. Then*

$$\begin{aligned}(n+1)T(r,f) \leq\ & \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + N_{1)}(r,\frac{1}{f}) + 2N_{(2}(r,\frac{1}{f}) \\ & + \overline{N}(r,\frac{1}{\varphi f^n f' - 1}) - N_0(r,\frac{1}{(\varphi f^n f')'}) + S(r,f),\end{aligned} \tag{7}$$

where $N_0(r, \frac{1}{(\varphi f^n f')'})$ denotes the counting function of the zeros of $(\varphi f^n f')'$, not of $f(\varphi f^n f' - 1)$.

Proof. We know that $\varphi f^n f' \not\equiv$ constant by Lemma 3.4 in [3].

Let

$$\frac{1}{f^{n+1}} \equiv \frac{\varphi f^n f'}{f^{n+1}} - \frac{(\varphi f^n f')'}{f^{n+1}}\frac{\varphi f^n f' - 1}{(\varphi f^n f')'},$$

we have

$$\begin{aligned}(n+1)m(r,\frac{1}{f}) &= m(r,\frac{1}{f^{n+1}}) \\ &\leq m(r,\frac{\varphi f^n f' - 1}{(\varphi f^n f')'}) + m(r,\varphi\frac{f'}{f}) + m(r,\frac{(\varphi f^n f')'}{f^{n+1}}) + O(1) \\ &\leq N(r,\frac{(\varphi f^n f')'}{\varphi f^n f' - 1}) - N(r,\frac{\varphi f^n f' - 1}{(\varphi f^n f')'}) + S(r,f) \\ &= N(r,(\varphi f^n f')') + N(r,\frac{1}{\varphi f^n f' - 1}) \\ &\quad - N(r,\frac{1}{(\varphi f^n f')'}) - N(r,\varphi f^n f') + S(r,f) \\ &= \overline{N}(r,f) + N(r,\frac{1}{\varphi f^n f' - 1}) - N(r,\frac{1}{(\varphi f^n f')'}) + S(r,f).\end{aligned} \tag{8}$$

Hence

$$\begin{aligned}(n+1)T(r,f) &= (n+1)m(r,\frac{1}{f})+(n+1)N(r,\frac{1}{f})+O(1)\\ &= \overline{N}(r,f)+(n+1)N(r,\frac{1}{f})\\ &+N(r,\frac{1}{\varphi f^nf'-1})-N(r,\frac{1}{(\varphi f^nf')'})+S(r,f).\end{aligned} \tag{9}$$

Let

$$N(r,\frac{1}{(\varphi f^nf')'}) = N_{000}(r,\frac{1}{(\varphi f^nf')'})+N_{00}(r,\frac{1}{(\varphi f^nf')'})+N_0(r,\frac{1}{(\varphi f^nf')'}), \tag{10}$$

where $N_{000}(r,\frac{1}{(\varphi f^nf')'})$ denotes the counting function of the zeros of $(\varphi f^nf')'$, which come from the zeros of $\varphi f^nf'-1$, $N_{00}(r,\frac{1}{(\varphi f^nf')'})$ denotes the counting function of the zeros of $(\varphi f^nf')'$, which come from the zeros of f. Hence we have

$$N(r,\frac{1}{\varphi f^nf'-1})-N_{000}(r,\frac{1}{(\varphi f^nf')'}) = \overline{N}(r,\frac{1}{\varphi f^nf'-1}). \tag{11}$$

Supposed that z_0 is a zero of f with multiplicity q and the pole of φ with multiplicity of t.

Case I. Supposed that $t\le nq-1$. If $q=1$, then z_0 is a zero of $(\varphi f^nf')'$ with multiplicity at least $nq-1-t$; if $q\ge 2$, then z_0 is a zero of $(\varphi f^nf')'$ with multiplicity at least $(n+1)q-2-t$.

Case II. Supposed that $t\ge 2q$, z_0 is at most the pole of φ^2. Hence, we have

$$\begin{aligned}&(n+1)N(r,\frac{1}{f})-N_{00}(r,\frac{1}{(\varphi f^nf')'})\\ &\le nN(r,\frac{1}{f})+\overline{N}_{1)}(r,\frac{1}{f})+2\overline{N}_{(2}(r,\frac{1}{f})\\ &+N(r,\varphi^2)-N_{00}(r,\frac{1}{(\varphi f^nf')'})\\ &= N_{1)}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f})+\overline{N}_{(2}(r,\frac{1}{f})+S(r,f).\end{aligned} \tag{12}$$

Combining (9)-(12), we have

$$\begin{aligned}(n+1)T(r,f) &\le \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+N_{1)}(r,\frac{1}{f})+2\overline{N}_{(2}(r,\frac{1}{f})\\ &+\overline{N}(r,\frac{1}{\varphi f^nf'-1})-N_0(r,\frac{1}{(\varphi f^nf')'})+S(r,f).\end{aligned}$$

This completes the proof of the lemma. □

If the coefficients of differential polynomials $M[f]$ are $a_j, j = 0, 1, \cdots, n$, which satisfy $m(r, a_j) = S(r, f)$, then differential polynomials $M[f]$ is called a quasi-differential polynomials in f. The following lemma is nothing but an easy variant of standard Clunie lemma [4], Lemma 1.

Lemma 2.2 ([14]). *Let f be a non-constant meromorphic in the complex plane, $Q_1[f], Q_2[f]$ are quasi-differential polynomials in f, satisfy $f^nQ_1[f] = Q_2[f]$, if the total degree of $Q_2 \leq n$, then $m(r, Q_1[f]) = S(r, f)$.*

Lemma 2.3. *Let f be a transcendental meromorphic function and $n \geq 3$, $F(z) = \varphi(z)f^n(z)f'(z) - 1$ and*

$$h(z) = \frac{F'(z)}{f^{n-1}(z)} = \varphi'(z)f(z)f'(z) + \varphi(z)f(z)f''(z) + n\varphi(z)f'^2(z), \phi(z) = \frac{h(z)}{F(z)}.$$

Also, let

$$G(z)=a_1\Big(\frac{F'(z)}{F(z)}\Big)^2 + a_2\Big(\frac{F'(z)}{F(z)}\Big)'+a_3\Big(\frac{F'(z)}{F(z)}\frac{h'(z)}{h(z)}\Big)+a_4\Big(\frac{h'(z)}{h(z)}\Big)^2+a_5\Big(\frac{h'(z)}{h(z)}\Big)'$$
$$+ a_6\Big(\frac{\varphi'(z)}{\varphi(z)}\frac{F'(z)}{F(z)}\Big) + a_7\Big(\frac{\varphi'(z)}{\varphi(z)}\frac{h'(z)}{h(z)}\Big) + a_8\Big(\frac{\varphi'(z)}{\varphi(z)}\Big)' + a_9\Big(\frac{\varphi'(z)}{\varphi(z)}\Big)^2.$$

Where

$$\begin{cases} a_1 = 2(4n^2 + 8n + 7), \\ a_2 = 2(n + 2)(2n + 1)(2n - 1), \\ a_3 = -2(n + 2)(2n^2 + 3n + 4), \\ a_4 = (n + 1)(n + 2)^2, \\ a_5 = -(n + 2)^2(2n - 1), \\ a_6 = 2(n + 2)(n + 1)(2n - 5), \\ a_7 = 3(n + 2)^2, \\ a_8 = -(n + 2)^2(2n - 1), \\ a_9 = -(n + 2)^2(n - 2). \end{cases}$$

Then, $G(z) \not\equiv 0$ and the simple pole of $f(z)$ is the zero of $G(z)$.

Proof. Suppose that $G(z) \equiv 0$. We first give the following inequalities in a similar way to Lemma 4 and 5 in [13]. For any $z_0 \in \mathbb{C}$, we have

$$(n-1)\omega(\frac{1}{f}, z_0) + \omega(\frac{1}{h}, z_0) \leq \omega(\frac{1}{f^{n-1}h}, z_0) + \omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0), \tag{13}$$

$$\omega(\phi, z_0) \leq 2\omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0), \tag{14}$$

$$\omega(\frac{1}{F}, z_0) \leq \omega(\frac{1}{h}, z_0) + 2\omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0). \tag{15}$$

We omit the proof of (13) and (14) here. We just prove (15). One can claim that the zeros of $F(z)$ must be the zeros or poles of $\varphi(z)$. This is

$$\overline{\omega}(\frac{1}{F}, z_0) \leq \omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0). \tag{16}$$

Let z_1 be a zero of $F(z)$ of multiplicity $l(l \geq 1)$ and $\varphi(z_1) \neq 0, \infty$. Then $F(z_1) = 0$ and $\varphi(z_1)f^n(z_1)f'(z_1) - 1 = 0$ such that $f(z_1) \neq 0, \infty$. From $h = \frac{F'}{f^{n-1}}$, we get that z_1 is a zero of h with multiplicity $l - 1$. Using the Laurent series of G at the point z_1, we obtain the coefficient of $(z - z_1)^{-2}$:

$$A(l) = (a_1 + a_3 + a_4)l^2 - (a_2 + a_3 + 2a_4 + a_5)l + (a_4 + a_5).$$

Therefore,

$$A(l) = -(3n^2 + n^2 - 4n - 2)l^2 - (n + 2)(4n^2 - 3n - 4)l - (n - 2)(n + 2)^2.$$

Obviously, $A(l) < 0$ for all positive integers l. Hence z_1 is a pole of $G(z)$ which contradicts $G(z) \equiv 0$. Hence the zeros of $F(z)$ must be the zeros or poles of $\varphi(z)$. Therefore, (16) holds. In the following we begin to prove (15).

If $F(z_0) \neq 0$, then the inequality (15) obviously holds. If $F(z_0) = 0$, then from (16) we obtain

$$\begin{aligned}\omega(\frac{1}{F}, z_0) - \omega(\frac{1}{h}, z_0) &= [\omega(\frac{1}{F}, z_0) - \omega(\frac{1}{F'}, z_0)] + [\omega(\frac{1}{f^{n-1}h}, z_0) - \omega(\frac{1}{h}, z_0)] \\ &\leq \overline{\omega}(\frac{1}{F}, z_0) + (n-1)\omega(\frac{1}{f}, z_0) \\ &\leq (n-1)\omega(\frac{1}{f}, z_0) + \omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0).\end{aligned}$$

If $f(z_0) \neq 0$, then we have $\omega(\frac{1}{f}, z_0) = 0$.

If $f(z_0) = 0$, then from $\varphi(z_0)f^n(z_0)f'(z_0) = F(z_0) + 1 = 1$, we have $n\omega(\frac{1}{f}, z_0) \leq \omega(\varphi, z_0)$. Hence

$$\omega(\frac{1}{F}, z_0) - \omega(\frac{1}{h}, z_0) \leq 2\omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0).$$

Thus, the inequality (15) holds. From (14) and (15), we have

$$N(r, \phi) \leq 2N(r, \varphi) + N(r, \frac{1}{\varphi}) = S(r, f), \tag{17}$$

and

$$N(r, \frac{1}{F}) - N(r, \frac{1}{h}) \leq 2N(r, \varphi) + N(r, \frac{1}{\varphi}) = S(r, f). \tag{18}$$

By (8), we have

$$(n+1)m(r, \frac{1}{f}) \leq \overline{N}(r, f) + N(r, \frac{1}{F}) - N(r, \frac{1}{f^{n-1}h}) + S(r, f). \tag{19}$$

By (13), we have

$$(n-1)N(r,\frac{1}{f})+N(r,\frac{1}{h})\le N(r,\frac{1}{f^{n-1}h})+N(r,\varphi)+N(r,\frac{1}{\varphi}). \tag{20}$$

From (19) and (20), we have

$$(n+1)m(r,\frac{1}{f})\le \overline{N}(r,f)+N(r,\frac{1}{F})-(n-1)N(r,\frac{1}{f})-N(r,\frac{1}{h})+S(r,f). \tag{21}$$

From (18) and (21), we have

$$nm(r,\frac{1}{f})\le N(r,\frac{1}{F})-N(r,\frac{1}{h})+S(r,f)=S(r,f). \tag{22}$$

From (17) and (22), we have

$$\begin{aligned} T(r,\phi)&=m(r,\phi)+N(r,\phi)=m(r,\frac{1}{f^{n-1}}\cdot\frac{F'}{F})+N(r,\phi)\\ &\le (n-1)m(r,\frac{1}{f})+m(r,\frac{F'}{F})+N(r,\phi)=S(r,f). \end{aligned} \tag{23}$$

Note that $\frac{F'}{F}=f^{n-1}\phi$ and

$$\frac{h'}{h}=\frac{F'}{F}+\frac{\phi'}{\phi}=f^{n-1}\phi+\frac{\phi'}{\phi}.$$

Substituting the above expressions into $G(z)$, we have

$$\begin{aligned} &(a_1+a_3+a_4)f^{2n-2}\phi^2+\big[(a_2+a_3+2a_4+a_5)\frac{\phi'}{\phi}+(a_6+a_7)\frac{\varphi'}{\varphi}\big]\phi f^{n-1}\\ &+(n-1)(a_2+a_5)f^{n-2}f'\phi+\big[a_4\big(\frac{\phi'}{\phi}\big)^2\\ &+a_5(\frac{\phi'}{\phi})'+a_7\frac{\varphi'}{\varphi}\frac{\phi'}{\phi}+a_8(\frac{\varphi'}{\varphi})'+a_9(\frac{\varphi'}{\varphi})^2\big]=0. \end{aligned} \tag{24}$$

Note that $(n-1)(a_2+a_5)=3n(2n^3+n^2-5n+2)\ne 0$. By (24), we have

$$f'\equiv\frac{1}{\phi f^{n-2}}c_1(z)+fc_2(z)+f^2\phi c_3(z), \tag{25}$$

where $c_1(z)=\big[a_4\big(\frac{\phi'}{\phi}\big)^2+a_5(\frac{\phi'}{\phi})'+a_7\frac{\varphi'}{\varphi}\frac{\phi'}{\phi}+a_8(\frac{\varphi'}{\varphi})'+a_9(\frac{\varphi'}{\varphi})^2\big]/(n-1)(a_2+a_5)$, $c_2(z)=[(a_2+a_3+2a_4+a_5)\frac{\phi'}{\phi}+(a_6+a_7)\frac{\varphi'}{\varphi}]/(n-1)(a_2+a_5)$, $c_3(z)=-[(a_1+a_3+a_4)]/(n-1)(a_2+a_5)$.

From (24), we can obtain

$$F=\varphi f^n f'-1=\frac{\varphi}{\phi}c_1f^2+c_2\varphi f^{n+1}+c_3\phi\varphi f^{n+2}-1. \tag{26}$$

$$\begin{aligned} F' &= (n+2)c_3^2\varphi\phi^2 f^{n+3} + [c_3\phi\varphi' + c_3\phi'\varphi + (2n+3)c_2c_3\phi\varphi]f^{n+2} \\ &\quad + [c_2'\phi + c_2\phi' + 2\phi c_1c_3 + (n+1)c_2^2\phi]f^{n+1} + [(\frac{\phi}{\varphi})'c_1 + \frac{\phi}{\varphi}c_1' \\ &\quad + (n+3)\frac{\phi}{\varphi}c_1c_2]f^2 + 2\frac{\phi}{\varphi^2}c_1^2 f^{3-n}. \end{aligned} \tag{27}$$

Substituting (26) and (27) into $F' = f^{n-1}\varphi F$, we have

$$\begin{aligned} & c_3\phi\varphi^2 f^{3n+1} + c_2\varphi f^{3n} - (n+2)c_3^2\phi^2\varphi f^{2n+3} \\ & - [c_3\phi\varphi' + c_3\phi'\varphi + (2n+3)c_2c_3\phi\varphi] \\ & f^{2n+2} + [\frac{\varphi^2}{\phi}c_1 - c_2'\phi - c_2\phi' - (n+1)c_2^2\phi]f^{2n+1} - \varphi f^{2n-1} - (n+4)c_1c_3\varphi \\ & f^{3+n} - [(\frac{\varphi}{\phi})'c_1 + \frac{\varphi}{\phi}c_1' + (n+3)\frac{\phi}{\varphi}c_1c_2]f^{2+n} - 2\frac{\phi}{\varphi^2}c_1^2 f^3 = 0. \end{aligned}$$

By (23) and Lemma 3 in [7], we have $c_3\phi\varphi^2 \equiv 0$, note that $c_3 \neq 0$, therefore $\phi \equiv 0$ or $\varphi \equiv 0$. It is impossible. Hence $G(z) \not\equiv 0$.

Suppose that, z_0 is a simple pole of $f(z)$. Then

$$\begin{aligned} \varphi(z) &= A\left\{1 + x(z-z_0) + y(z-z_0)^2 + O[(z-z_0)^3]\right\} (A \neq 0), \\ f(z) &= \frac{a}{(z-z_0)}\left\{1 + b_0(z-z_0) + b_1(z-z_0)^2 + O[(z-z_0)^3]\right\} (a \neq 0), \end{aligned}$$

and

$$\begin{aligned} h(z) = \frac{Aa^2}{(z-z_0)^4}\Big\{ & (n+2) + [(n+1)x + 2b_0](z-z_0) \\ & + [b_0x + ny - 2(n-1)b_1](z-z_0)^2 + O[(z-z_0)^3]\Big\}, \end{aligned}$$

where $a \neq 0, b_0, b_1, b_2$ are constants. Therefore, we have

$$\begin{aligned} F(z) &= \frac{-Aa^{n+1}}{(z-z_0)^{n+2}}\Big\{1 + (x+nb_0)(z-z_0) + [y + nxb_0 + \frac{1}{2}n(n-1)b_0^2 \\ &\quad + (n-1)b_1](z-z_0)^2 + O[(z-z_0)^3]\Big\}, \\ \frac{F'(z)}{F(z)} &= -\frac{n+2}{(z-z_0)} + (x+nb_0) + [2y - x^2 - nb_0^2 + 2(n-1)b_1](z-z_0) \\ &\quad + O[(z-z_0)^2], \\ \frac{\varphi'(z)}{\varphi(z)} &= x + (2y - x^2)(z-z_0) + O[(z-z_0)^2], \\ \frac{h'(z)}{h(z)} &= \frac{-4}{(z-z_0)} + [\frac{(n+1)}{(n+2)}x + \frac{2}{(n+2)}b_0] + [\frac{2n}{(n+2)}y - \frac{2n}{(n+2)^2}xb_0 \\ &\quad - \frac{4(n-1)}{(n+2)}b_1 - \frac{(n+1)^2}{(n+2)^2}x^2 - \frac{4}{(n+2)^2}b_0^2](z-z_0) + O[(z-z_0)^2], \end{aligned}$$

$$\left(\frac{F'(z)}{F(z)}\right)^2 = \frac{(n+2)^2}{(z-z_0)^2} + \frac{-2(n+2)x - 2(n+2)nb_0}{(z-z_0)} + [-4(n+2)y$$
$$+ (2n+5)x^2 + 2nxb_0 + (3n^2+4n)b_0^2 - 4(n-1)(n+2)b_1] + O(z-z_0), \tag{28}$$

$$\left(\frac{F'(z)}{F(z)}\right)' = \frac{(n+2)}{(z-z_0)^2} + [2y - x^2 - nb_0^2 + 2(n-1)b_1] + O(z-z_0), \tag{29}$$

$$\frac{F'(z)}{F(z)}\frac{h'(z)}{h(z)} = \frac{4(n+2)}{(z-z_0)^2} + \frac{-(n+5)x - (4n+2)b_0}{(z-z_0)} + [-(2n+8)y$$
$$+ (n+5)x^2 + (n+1)xb_0 + (4n+2)b_0^2 + (4-4n)b_1] + O(z-z_0), \tag{30}$$

$$\left(\frac{h'(z)}{h(z)}\right)^2 = \frac{16}{(z-z_0)^2} + [\frac{-8(n+1)}{(n+2)}x + \frac{-16}{(n+2)}b_0]\frac{1}{(z-z_0)} + [\frac{-16n}{(n+2)}y$$
$$+ \frac{9(n+1)^2}{(n+2)^2}x^2 + \frac{(20n+4)}{(n+2)^2}xb_0 + \frac{36}{(n+2)^2}b_0^2 + \frac{32(n-1)}{(n+2)}b_1] + O(z-z_0), \tag{31}$$

$$\left(\frac{h'(z)}{h(z)}\right)' = \frac{4}{(z-z_0)^2} + [\frac{2n}{(n+2)}y - \frac{(n+1)^2}{(n+2)^2}x^2 - \frac{2n}{(n+2)^2}xb_0$$
$$- \frac{4}{(n+2)^2}b_0^2 - \frac{4(n-1)}{(n+2)}b_1] + O(z-z_0), \tag{32}$$

$$\frac{\varphi'(z)}{\varphi(z)}\frac{F'(z)}{F(z)} = \frac{-(n+2)x}{(z-z_0)} + [-2(n+2)y + (n+3)x^2 + nxb_0] + O(z-z_0), \tag{33}$$

$$\frac{\varphi'(z)}{\varphi(z)}\frac{h'(z)}{h(z)} = \frac{-4x}{(z-z_0)} + (-8y + \frac{5n+9}{n+2}x^2 + \frac{2}{n+2}xb_0) + O(z-z_0), \tag{34}$$

$$\left(\frac{\varphi'(z)}{\varphi(z)}\right)' = (2y - x^2) + O(z-z_0), \tag{35}$$

$$\left(\frac{\varphi'(z)}{\varphi(z)}\right)^2 = x^2 + 2x(2y-x^2)(z-z_0) + O(z-z_0)^2. \tag{36}$$

By substituting (28)-(36) in the expression of $G(z)$ and performing some simple calculations, it can be easily seen that $G(z) = O[(z-z_0)]$. Hence the simple pole of $f(z)$ is the zero of $G(z)$. □

Remark 2.1. Obviously, the lemma does not hold for $n = 2$. In fact, Lemma 2.6 also holds just for $n \geq 3$ in [8]. If $n = 2$, the result has been proved by Zhang in [23], and the similar method can be found in [13].

3. The proof of the Theorem 1.1

If $n=2$, we can obtain the conclusion by Theorem D. In the following, we only proved the case of $n\geq 3$.

By differentiating the equation $F=\varphi f^n f'-1$, we get

$$f^{n-1}\beta=-\frac{F'}{F},$$

where

$$\beta=\varphi' f f'+n\varphi f'^2+\varphi f f''-\varphi f f'\frac{F'}{F},\quad h=-\beta F.$$

Note that the poles of $G(z)$ whose multiplicity are at most two come from the multiple poles of $f(z)$, $F(z)$ or the zeros of $h(z)$.

We consider the poles of $\beta^2 G$. We can see the zeros of h either are the zeros of F, or the zeros of β. From the above we know that the multiple poles of f with the multiplicity $q(\geq 2)$ are the zeros of β with the multiplicity of $(n-1)q-1$. Hence the poles of $\beta^2 G$ only come from the zeros of F, and the multiplicity is at most 4. Hence,

$$N(r,\beta^2 G)\leq 4\overline{N}(r,1/F).$$

By $m(r,G)=S(r,f)$ and Lemma 2.2, we have $m(r,\beta^2 G)=S(r,f)$. Hence

$$T(r,\beta^2 G)\leq 4\overline{N}(r,1/F).$$

Since the multiple zeros of f with the multiplicity $p(\geq 2)$ are the multiple zeros of β with multiplicity at least $2p-2$, therefore, are at least the zeros of $\beta^2 G$ with the multiplicity $2(2p-2)-2=4p-6$. Also note that the simple poles of f are the zeros of $\beta^2 G$. Hence we have

$$N(r,f)+2N(r,\frac{1}{f})-2\overline{N}(r,\frac{1}{f})\leq N(r,\frac{1}{\beta^2 G})\leq T(r,\beta^2 G)\leq 4\overline{N}(r,\frac{1}{F}). \tag{37}$$

From (7) and (37), we have

$$\begin{aligned}&2(n+1)T(r,f)+N_1(r,f)+2N(r,\frac{1}{f})-2\overline{N}(r,\frac{1}{f})\\&\leq 2\overline{N}(r,f)+4\overline{N}(r,\frac{1}{f})+6\overline{N}(r,\frac{1}{\varphi f^n f'-1})-2N_0(r,\frac{1}{(\varphi f^n f')'})+S(r,f).\end{aligned}$$

i.e.,

$$\begin{aligned}&2(n+1)T(r,f)+N_1(r,f)+2N(r,\frac{1}{f})-6\overline{N}(r,\frac{1}{f})\\&\leq 2N_1(r,f)+2\overline{N}_{(2}(r,\frac{1}{f})+6\overline{N}(r,\frac{1}{\varphi f^n f'-1})-2N_0(r,\frac{1}{(\varphi f^n f')'})+S(r,f),\end{aligned}$$

which gives

$$(2n-3)T(r,f)+m(r,f)+N(r,f)+4m(r,\frac{1}{f})+6N(r,\frac{1}{f})$$
$$-2\overline{N}_{(2}(r,f)-6\overline{N}(r,\frac{1}{f})\le N_1(r,f)+6\overline{N}(r,\frac{1}{\varphi f^n f'-1})+S(r,f).$$

Therefore,

$$(2n-3)T(r,f)+N_{(2}(r,f)-2\overline{N}_{(2}(r,f)+m(r,f)+4m(r,\frac{1}{f})+6N_1(r,\frac{1}{f})$$
$$\le 6\overline{N}(r,\frac{1}{F})+S(r,f),$$

Hence, we have

$$T(r,f)<\Big(\frac{6}{2n-3}\Big)\overline{N}(r,\frac{1}{\varphi f^n f'-1})+S(r,f).$$

4. Conclusions

In our present investigation, we applied several methods which appeared in [8, 15, 23] and gave a unified inequality for $\varphi f^n f'-1$, where $n\ge 2$. The result also improved the coefficient in Theorem E. The Lemma 2.3 plays a key role in the study. Obviously, the lemma does not hold for $n=2$. In fact, Lemma 2.6 also does not hold for $n=2$ in [8]. When $n=2$, the result has been proved by Zhang in [23]. The reader can find a similar method used in [13]. For further study of related problems, we would like to pose the following question.

Question 4.1. *If f' is replaced by $f^{(k)}$ in Theorem 1.1, whether the inequality is still true?*

Remark 4.1. *There are some complete generalizations of the Hayman result about the zeros of $f^n f'-1$ where f is a transcendental meromorphic function. For example, in 1981 N. Steinmetz [12] proved that for a transcendental meromorphic function f, $\Psi = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}-1$ has infinitely many zeros, where $n_0\ge 2$, $n_1+n_2+\cdots+n_k\ge 1$. In 1982, W. Doeringer [5] proved that for a transcendental meromorphic function f, let $Q(f), P(f)$ be differential polynomials in f satisfying $P(f)\not\equiv 0$, $Q(f)\not\equiv 0$. Then $\Psi = f^nP(f)+Q(f)$ has infinitely many zeros, where $n\ge 3+\Gamma_p$. For Question 4.1, if φ is a constant, there are some results obtained in [3] and [8].*

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