Symmetry classification and solutions for Shigesada-Kawasaki-Teramoto system

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Abstract. In this paper, the symmetry classification of the Shigesada-Kawasaki-Teramoto system is presented, then the symmetry reduction and solutions of the classified system are constructed by the generalized conditional symmetry method.
Keywords: symmetry classification, symmetry reduction, solution.

1. Introduction

The reaction-diffusion system

$$
\begin{aligned}
u_t &= [(d_1 + d_2 u + d_3 v)u]_{xx} + u(a_{11} - b_{11}u - c_1 v), \\
v_t &= [(d_4 + d_5 u + d_6 v)v]_{xx} + v(a_{21} - b_{12}u - c_2 v)
\end{aligned} \tag{1}
$$

is the well-known Shigesada-Kawasaki-Teramoto system which was first proposed by Shigesada et al [1]. Where $u = u(x,t), v = v(x,t)$ denote the unknown functions of the space variable x and time t, a_{ij}, b_{ij}, c_i and d_i are arbitrary constants. The generalized forms of system (1) play a significant role in mathematical biology, and some results are shown in [2]-[8] (and references cited therein). In the following sections, we will use the form of generalized conditional symmetry (GCS) [9]-[15]

$$
\begin{aligned}
\eta_1 &= u_{l_1 x} + a_{(l_1-1)} u_{(l_1-1)x} + \cdots + a_1 u, \\
\eta_2 &= v_{l_2 x} + b_{(l_2-1)} + v_{(l_2-1)x} + \cdots + b_1 v
\end{aligned} \tag{2}
$$

to classify system (1) and seek for symmetry reduction of the obtained system, here $l_1 > 2, l_2 \geq 2$.

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The remainder of this paper is organized as follows. In section 2, we perform symmetry classification for system (1), which admits the GCS are presented. In section 3, the symmetry reduction and exact solutions to the classified equations are constructed by the GCS method.

2. Symmetry classification of system (1)

In this section, we display the symmetry classification results of system (1). We will consider the following eight cases.
Case 1. When $l_1 = 3,\ l_2 = 2$.
According to the definition of GCS method and calculation procedure, we obtain the following determining system, then it takes the form

$$\begin{aligned}
F_1 &= 14a_2d_2u_{xx}^2 + [(2a_2^2d_2 + 24a_1d_2 - 6b_{11})u_x + (a_2^2d_3 + 4a_2b_1d_3 + 10b_1^2d_3\\
&+ 2a_1d_3 + 10b_0d_3 - 3c_1)v_x + 20a_0d_2u + (4a_2b_0d_3 + 10b_0b_1d_3)v]u_{xx} + (2a_1a_2d_2\\
&+ 8a_0d_2 + 2a_2b_{11})u_x^2 + [(-4a_2b_1^2d_3 + 5b_1^3d_3 + a_1a_2d_3 + 7a_1b_1d_3 - 4a_2b_0d_3\\
&+ 10b_0b_1d_3 + 3a_0d_3 + 2a_2c_1 - 3b_1c_1)v_x + 2a_0a_2d_2u + (-4a_2b_0b_1d_3 + 5b_0b_1^2d_3\\
&+ 7a_1b_0d_3 + 5b_0^2d_3 - 3b_0c_1)v]u_x + (-a_2b_1^3d_3 + b_1^4d_3 - a_1b_1^2d_3 - 2a_2b_0b_1d_3\\
&+ 3b_0b_1^2d_3 + a_0a_2d_3 + 9a_0b_1d_3 - a_1b_0d_3 + a_2b_1c_1 + b_0^2d_3 - b_1^2c_1 + a_1c_1 - b_0c_1)uv_x\\
&- u^2a_0b_{11} + (-a_2b_0b_1^2d_3 + b_0b_1^3d_3 - a_1b_0b_1d_3 - a_2b_0^2d_3 + 2b_0^2b_1d_3\\
&+ 9a_0b_0d_3 + a_2b_0c_1 - b_0b_1c_1)vu = 0,\\
F_2 &= [(4a_2d_5 + 3b_1d_5)v_x + (a_2^2d_5 - a_2b_1d_5 + a_1d_5 + 5b_0d_5 - b_{12})v]u_{xx}\\
&+ [(b_1^2d_5+4a_1d_5+2b_0d_5-2b_{12})v_x+(a_1a_2d_5-a_1b_1d_5+b_0b_1d_5+a_0d_5+b_1b_{12})v]u_x\\
&+ (8b_1^2d_6 + 6b_0d_6 - 2c_2)v_x^2 + (14b_0b_1d_6v + 4a_0d_5u)v_x + (a_0a_2d_5 - a_0b_1d_5)uv\\
&+ (6b_0^2d_6 - b_0c_2)v^2 = 0.
\end{aligned}$$

The coefficients of $u, v, v_x, u_x, u_{xx}, v_{xx}$ are equal to zero, and the classification results are shown in the following forms.
(i)

$$\begin{aligned}
u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u),\\
v_t &= [(d_4 + d_6v)v]_{xx} + v(a_{21} - 4b_1^2d_6v),\\
\eta_1 &= u_{xxx} - a_1u_x,\\
\eta_2 &= v_{xx} - b_1v_x;
\end{aligned}$$

(ii)

$$\begin{aligned}
u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4b_1^2d_2u - 4b_1^2d_3v),\\
v_t &= [(d_4 + d_6v)v]_{xx} + v(a_{21} - 4b_1^2d_6v),\\
\eta_1 &= u_{xxx} - b_1^2u_x,\\
\eta_2 &= v_{xx} - b_1v_x;
\end{aligned}$$

(iii)

$$\begin{aligned} u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + a_{11}u, \\ v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx}, \\ \eta_2 &= v_{xx}; \end{aligned}$$

(iv)

$$\begin{aligned} u_t &= [(d_1 + d_3v)u]_{xx} + a_{11}u, \\ v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx}, \\ \eta_2 &= v_{xx}; \end{aligned}$$

(v)

$$\begin{aligned} u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u), \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx} - a_1u_x, \\ \eta_2 &= v_{xx} - b_1v_x - b_0v; \end{aligned}$$

(vi)

$$\begin{aligned} u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u), \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx} - a_1u_x, \\ \eta_2 &= v_{xx} - b_0v; \end{aligned}$$

(vii)

$$\begin{aligned} u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4b_0d_2u - 4b_0d_3v), \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx} - b_0u_x, \\ \eta_2 &= v_{xx} - b_0v; \end{aligned}$$

(viii)

$$\begin{aligned} u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + a_{11}u, \\ v_t &= [(d_4 + d_5u)v]_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx}, \\ \eta_2 &= v_{xx}; \end{aligned}$$

(ix)

$$\begin{aligned} u_t &= [(d_1+d_2u)u]_{xx} + u(a_{11} - 16b_0d_2u), \\ v_t &= [(d_4+d_5u)v]_{xx} + v(a_{21} - 9b_0d_5u), \\ \eta_1 &= u_{xxx} - 4b_0u_x, \\ \eta_2 &= v_{xx} - b_0v; \end{aligned}$$

(x)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= [(d_4+d_6v)v]_{xx} + v(a_{21} - 4b_1^2d_6v), \\ \eta_1 &= u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xx} - b_1v_x; \end{aligned}$$

(xi)

$$\begin{aligned} u_t &= [(d_1+d_3v)u]_{xx} + u\left[a_{11} - \left(\frac{1}{9}a_2^2 + \frac{4}{3}a_2b_1 + 4b_1^2\right)d_3v\right], \\ v_t &= [(d_4+d_6v)v]_{xx} + v(a_{21} - 4b_1^2d_6v), \\ \eta_1 &= u_{xxx} - a_2u_{xx} + \left(\frac{1}{3}a_2^2 - b_1^2\right)u_x - \left(\frac{1}{27}a_2^3 - \frac{1}{3}a_2b_1^2\right)u, \\ \eta_2 &= v_{xx} - b_1v_x; \end{aligned}$$

(xii)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xx} - b_1v_x - b_0v; \end{aligned}$$

(xiii)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx} + \frac{5}{2}b_1u_{xx} - a_1u_x - \left(\frac{91}{216}b_1^2 + \frac{10}{27}a_1 + \frac{43}{54}b_0\right)b_1u, \\ \eta_2 &= v_{xx} - b_1v_x - b_0v; \end{aligned}$$

(xiv)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx} - a_1u_x, \\ \eta_2 &= v_{xx} - b_0v; \end{aligned}$$

(xv)

$$\begin{aligned} u_t &= [(d_1 + d_3 v)u]_{xx} + u(a_{11} - 4b_0 d_3 v), \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxx} - b_0 u_x, \\ \eta_2 &= v_{xx} - b_0 v; \end{aligned}$$

(xvi)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxx} + \frac{5}{2} b_1 u_{xx} - \left(\frac{29}{20} b_1^2 + \frac{77}{5} b_0 \right) u_x - \left(\frac{23}{24} b_1^3 + \frac{351}{54} b_0 b_1 \right) u, \\ \eta_2 &= v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(xvii)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxx} + \frac{5}{2} b_1 u_{xx} + \frac{71}{36} b_1^2 u_x + \frac{35}{72} b_1^3 u, \\ \eta_2 &= v_{xx} - b_1 v_x + \frac{2}{9} b_1^2 v; \end{aligned}$$

(xviii)

$$\begin{aligned} u_t &= [(d_1 + d_3 v)u]_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxx}, \\ \eta_2 &= v_{xx}; \end{aligned}$$

(xix)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_5 u)v]_{xx} + v \left(a_{21} - \frac{1}{64} b_1^2 d_5 u \right), \\ \eta_1 &= u_{xxx} + \frac{3}{4} b_1 u_{xx} + \frac{1}{8} b_1^2 u_x, \\ \eta_2 &= v_{xx} - b_1 v_x + \frac{15}{64} b_1^2 v. \end{aligned}$$

Case 2. When $l_1 = 4,\ l_2 = 2$.
The system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\ \eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xx} - b_1 v_x; \end{aligned}$$

(ii)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(iii)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\ \eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} \\ &\quad - \left(5b_1^3 - \frac{1}{4}a_3^3 - \frac{3}{4}a_3^2 b_1 + \frac{5}{6}a_3 b_1^2 - \frac{5}{6}a_2 a_3 - 2a_2 b_1\right) u_x - a_0 u, \\ \eta_2 &= v_{xx} - b_1 v_x; \end{aligned}$$

(iv)

$$\begin{aligned} u_t &= [(d_1 + d_3 v)u]_{xx} + u\left[a_{11} - \left(\frac{13}{2}b_1^2 d_3 + \frac{1}{2}a_2 d_3\right) v\right], \\ v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\ \eta_1 &= u_{xxxx} + 6b_1 u_{xxx} - a_2 u_{xx} - (27b_1^3 + 3a_2 b_1)u_x + \left(\frac{55}{4}b_1^4 + 4a_2 b_1^2 + \frac{1}{4}a_2^2\right) u, \\ \eta_2 &= v_{xx} - b_1 v_x; \end{aligned}$$

(v)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} + \left(\frac{17}{192}a_3^3 + \frac{1}{3}a_2 a_3 + \frac{5}{3}a_3 b_0\right) u_x - a_0 u, \\ \eta_2 &= v_{xx} + \frac{1}{4}a_3 v_x - b_0 v; \end{aligned}$$

(vi)

$$\begin{aligned} u_t &= [(d_1+d_3v)u]_{xx} + u(a_{11} - \frac{9}{4}b_1^2 d_3 v), \\ v_t &= d_4 v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxx} + 4b_1 u_{xxx} + \frac{7}{2}b_1^2 u_{xx} - b_1^3 u_x - \frac{15}{16}b_1^4 u, \\ \eta_2 &= v_{xx} - b_1 v_x; \end{aligned}$$

(vii)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11}u, \\ v_t &= d_4 v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxx} + \frac{116}{7}b_1 u_{xxx} + \frac{841}{14}b_1^2 u_{xx} - \frac{24389}{343}b_1^3 u_x - \frac{10609215}{38416}b_1^4 u, \\ \eta_2 &= v_{xx} - \frac{29}{7}b_1 v_x; \end{aligned}$$

(viii)

$$\begin{aligned} u_t &= [(d_1+d_3v)u]_{xx} + a_{11}u, \\ v_t &= d_4 v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxx}, \\ \eta_2 &= v_{xx}; \end{aligned}$$

Case 3. When $l_1 = 3,\ l_2 = 3$, the system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned} u_t &= [(d_1+d_2u)u]_{xx} + u(a_{11} - 4a_1 d_2 u), \\ v_t &= d_4 v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxx} - a_1 u_x, \\ \eta_2 &= v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(ii)

$$\begin{aligned} u_t &= [(d_1+d_2u+d_3v)u]_{xx} + u(a_{11} - 4b_1 d_2 u - 4b_1 d_3 v), \\ v_t &= [(d_4+d_5u+d_6v)v]_{xx} + v(a_{21} - 4b_1 d_5 u - 4b_1 d_6 v), \\ \eta_1 &= u_{xxx} - b_1 u_x, \\ \eta_2 &= v_{xxx} - b_1 v_x; \end{aligned}$$

(iii)

$$\begin{aligned} u_t &= [(d_1+d_2u)u]_{xx} + u(a_{11} - 4a_1 d_2 u), \\ v_t &= [(d_4+d_6v)v]_{xx} + v(a_{21} - 4b_1 d_6 v), \\ \eta_1 &= u_{xxx} - a_1 u_x, \\ \eta_2 &= v_{xxx} - b_1 v_x; \end{aligned}$$

(iv)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1 d_6 v), \\ \eta_1 &= u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xxx} - b_1 v_x; \end{aligned}$$

(v)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(vi)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_5 u)v]_{xx} + \left(a_{21} - \frac{1}{225} b_2^2 d_5 u\right) v, \\ \eta_1 &= u_{xxx} + \frac{2}{5} b_2 u_{xx} + \frac{8}{225} b_2^2 u_x, \\ \eta_2 &= v_{xxx} - b_2 v_{xx} + \frac{71}{225} b_2^2 v_x - \frac{7}{225} b_2^3 v; \end{aligned}$$

(vii)

$$\begin{aligned} u_t &= [(d_1 + d_3 v)u]_{xx} + u(a_{11} - \frac{a_2^2 d_3}{225} v), \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxx} - a_2 u_{xx} + \frac{71}{225} a_2^2 u_x - \frac{7}{225} a_2^3 u, \\ \eta_2 &= v_{xxx} + \frac{2}{5} a_2 v_{xx} + \frac{8}{225} a_2^2 v_x. \end{aligned}$$

Case 4. When $l_1 = 4,\ l_2 = 3$, the system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1 d_6 v), \\ \eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xxx} - b_1 v_x; \end{aligned}$$

(ii)

$$\begin{aligned} u_t &= [(d_1+d_3v)u]_{xx} + u\left(a_{11} - \frac{25}{4}b_1d_3v\right), \\ v_t &= [(d_4+d_6v)v]_{xx} + v(a_{21} - 4b_1d_6v), \\ \eta_1 &= u_{xxxx} - \frac{5}{2}b_1u_{xx} + \frac{9}{16}b_1^2u, \\ \eta_2 &= v_{xxx} - b_1v_x; \end{aligned}$$

(iii)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxx} - a_3u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xxx} - b_2v_{xx} - b_1v_x - b_0v; \end{aligned}$$

(iv)

$$\begin{aligned} u_t &= [(d_1+d_3v)u]_{xx} + u\left(a_{11} - \frac{1}{576}a_3^2d_3v\right), \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxx} - a_3u_{xxx} + \frac{103}{288}a_3^2u_{xx} - \frac{31}{576}a_3^3u_x + \frac{35}{12288}a_3^4u, \\ \eta_2 &= v_{xxx} + \frac{1}{4}a_3v_{xx} + \frac{1}{72}a_3^2v_x. \end{aligned}$$

Case 5. When $l_1 = 5,\ l_2 = 3$, the system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxxx} - a_4u_{xxxx} - a_3u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xxx} - b_2v_{xx} - b_1v_x - b_0v; \end{aligned}$$

(ii)

$$\begin{aligned} u_t &= [(d_1+d_3v)u]_{xx} + u(a_{11} - 9b_1d_3v), \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxxx} - 5b_1u_{xxx} + 4b_1^2u_x, \\ \eta_2 &= v_{xxx} - b_1v_x; \end{aligned}$$

(iii)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxxx} - a_4u_{xxxx} + (\frac{86}{245}a_4^2 - 5b_1)u_{xxx} + (5a_4b_1 - \frac{46}{1225}a_4^3)u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xxx} + \frac{6}{35}a_4v_{xx} - b_1v_x; \end{aligned}$$

(iv)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= [(d_4 + d_6v)v]_{xx} + (a_{21} - 4b_1d_6v)v, \\ \eta_1 &= u_{xxxxx} - a_4u_{xxxx} - a_3u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xxx} - b_1v_x; \end{aligned}$$

(v)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= [(d_4 + d_6v)v]_{xx} + (a_{21} - 4b_1d_6v)v, \\ \eta_1 &= u_{xxxxx} - 5b_1u_{xxx} - a_1u_x, \\ \eta_2 &= v_{xxx} - b_1v_x; \end{aligned}$$

(vi)

$$\begin{aligned} u_t &= [(d_1 + d_3v)u]_{xx} + (a_{11} - 9b_1d_3v)u, \\ v_t &= [(d_4 + d_6v)v]_{xx} + (a_{21} - 4b_1d_6v)v, \\ \eta_1 &= u_{xxxxx} - 5b_1u_{xxx} + 4b_1^2u_x, \\ \eta_2 &= v_{xxx} - b_1v_x. \end{aligned}$$

Case 6. When $l_1 = 4,\ l_2 = 4$, the system (1) is equivalent to the following case.
(i)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxx} - a_3u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xxxx} - b_3v_{xxx} - b_2v_{xx} - b_1v_x - b_0v. \end{aligned}$$

Case 7. When $l_1 = 5,\ l_2 = 4$, the system (1) is equivalent to the following case.
(i)

$$\begin{aligned} u_t &= d_1u_{xx} + a_{11}u, \\ v_t &= d_4v_{xx} + a_{21}v, \\ \eta_1 &= u_{xxxxx} - a_4u_{xxxx} - a_3u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\ \eta_2 &= v_{xxxx} - b_3v_{xxx} - b_2v_{xx} - b_1v_x - b_0v. \end{aligned}$$

Case 8. When $l_1 = 5,\ l_2 = 5$, the system (1) is equivalent to one of the following cases.
(i)

$$\begin{aligned} u_t &= d_1 u_{xx} + (a_{21} - d_1 a_4^2 + d_4 a_4^2)u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxxx} - a_4 u_{xxxx}, \\ \eta_2 &= v_{xxxxx} - b_4 v_{xxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(ii)

$$\begin{aligned} u_t &= d_1 u_{xx} + (a_{21} - d_1 a_3 + d_4 a_3)u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxxx} - a_3 u_{xxx}, \\ \eta_2 &= v_{xxxxx} - b_4 v_{xxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(iii)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxxx} - a_4 u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xxxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(iv)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_1 v_{xx} + a_{11} v, \\ \eta_1 &= u_{xxxxx} - a_4 u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xxxxx} - b_4 v_{xxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v. \end{aligned}$$

3. Example

In this section, we show two examples to illustrate the main feature of the symmetry reduction procedure and solutions.
Example 1. The system

$$\begin{aligned} u_t &= [(d_1 + d_2 u + d_3 v)u]_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_5 u + d_6 v)v]_{xx} + a_{21} v, \end{aligned} \tag{3}$$

admits the GCS

$$\begin{aligned} \eta_1 &= u_{xxx}, \\ \eta_2 &= v_{xx}. \end{aligned} \tag{4}$$

By integrating the system (4), the corresponding solutions are given as below.

$$
\begin{aligned}
u &= \phi_1(t)x^2 + \phi_2(t)x + \phi_3(t), \\
v &= \psi_1(t)x + \psi_2(t).
\end{aligned} \tag{5}
$$

Inserting (5) into system (3) yields the following ordinary differential equations(ODEs).

$$
\begin{aligned}
\frac{d\phi_1}{dt} &= 12\phi_1(t)^2 d_2 + \phi_1(t)a_{11}, \\
\frac{d\phi_2}{dt} &= (6d_3\psi_1(t) + 12\phi_2(t)d_2)\phi_1(t) + \phi_2(t)a_{11}, \\
\frac{d\phi_3}{dt} &= (4d_2\phi_3(t) + 2d_3\psi_2(t) + 2d_1)\phi_1(t) + 2\psi_1(t)\phi_2(t)d_3 + 2\phi_2(t)^2 d_2 \\
&\quad +\phi_3(t)a_{11}, \\
\frac{d\psi_1}{dt} &= \psi_1(t)(6d_5\phi_1(t) + a_{21}), \\
\frac{d\psi_2}{dt} &= 2\psi_1(t)^2 d_6 + 2\psi_1(t)\phi_2(t)d_5 + (2d_5\phi_1(t) + a_{21})\psi_2(t).
\end{aligned} \tag{6}
$$

By solving the above system, we obtain

$$
\begin{aligned}
\phi_1(t) &= \frac{a_{11}}{a_{11}N_5 e^{-a_{11}t} - 12d_2}, \\
\phi_2(t) &= \left[\int 6d_3\phi_1(t)\psi_1(t)e^{-\int(12d_2\phi_1(t)+a_{11})dt}dt + N_3\right] e^{\int(12d_2\phi_1(t)+a_{11})dt}, \\
\phi_3(t) &= \left[\int 2(d_3\phi_1(t)\psi_2(t) + d_3\psi_1(t)\phi_2(t) + d_2\phi_2(t)^2 \right. \\
&\quad \left. +d_1\phi_1(t))e^{-\int(4d_2\phi_1(t)+a_{11})dt}dt + N_1\right] \\
&\quad e^{\int(4d_2\phi_1(t)+a_{11})dt}, \\
\psi_1(t) &= N_4 e^{\int(6d_5\phi_1(t)+a_{21})dt}, \\
\psi_2(t) &= \left[\int 2\psi_1(t)(d_6\psi_1(t) + d_5\phi_2(t))e^{-\int(2d_5\phi_1(t)+a_{21})dt}dt + N_2\right] e^{\int(2d_5\phi_1(t)+a_{21})dt}.
\end{aligned}
$$

The solution is obtained by substituting the above system into (5).
Example 2. The system

$$
\begin{aligned}
u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4d_2b_1u - 4d_3b_1v), \\
v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + v(a_{21} - 4d_5b_1u - 4d_6b_1v)
\end{aligned} \tag{7}
$$

admits GCS

$$
\begin{aligned}
\eta_1 &= u_{xxx} - b_1u_x, \\
\eta_2 &= v_{xxx} - b_1v_x.
\end{aligned} \tag{8}
$$

The solutions of (8) depend on the sign of the parameter b_1, so we can give three cases, then the solutions are

$$(9)\qquad \begin{aligned} u &= \phi_1(t)x^2 + \phi_2(t)x + \phi_3(t), b_1 = 0;\\ v &= \psi_1(t)x^2 + \psi_2(t)x + \psi_3(t); \end{aligned}$$

$$(10)\qquad \begin{aligned} u &= \phi_1(t) + \phi_2(t)e^{bx} + \phi_3(t)e^{-bx}, b_1 = b^2 > 0;\\ v &= \psi_1(t) + \psi_2(t)e^{bx} + \psi_3(t)e^{-bx}; \end{aligned}$$

$$(11)\qquad \begin{aligned} u &= \phi_1(t) + \phi_2(t)\sin bx + \phi_3(t)\cos bx, b_1 = -b^2 < 0;\\ v &= \psi_1(t) + \psi_2(t)\sin bx + \psi_3(t)\cos bx; \end{aligned}$$

• For $b_1 = 0$, the system (7) can be reduced to the following ODEs

$$\begin{aligned}
\frac{d\phi_1}{dt} &= 12\phi_1(t)^2 d_2 + (12\psi_1(t)d_3 + a_{11})\phi_1(t),\\
\frac{d\phi_2}{dt} &= (6\psi_2(t)d_3 + 12\phi_2(t)d_2)\phi_1(t) + (6\psi_1(t)d_3 + a_{11})\phi_2(t),\\
\frac{d\phi_3}{dt} &= (4\phi_3(t)d_2 + 2\psi_3(t)d_3 + 2d_1)\phi_1(t) + 2\phi_2(t)^2 d_2 + 2\psi_2(t)\phi_2(t)d_3\\
&\quad +(2\psi_1(t)d_3 + a_{11})\phi_3(t)\\
\frac{d\psi_1}{dt} &= 12\psi_1(t)^2 d_6 + (12\phi_1(t)d_5 + a_{21})\psi_1(t),\\
\frac{d\psi_2}{dt} &= (12\psi_2(t)d_6 + 6\phi_2(t)d_5)\psi_1(t) + (6\phi_1(t)d_5 + a_{21})\psi_2(t),\\
\frac{d\psi_3}{dt} &= (2\phi_3(t)d_5 + 4\psi_3(t)d_6 + 2d_4)\psi_1(t) + 2\psi_2(t)^2 d_6 + 2\psi_2(t)\phi_2(t)d_5\\
&\quad +(2\phi_1(t)d_5 + a_{21})\psi_3(t).
\end{aligned}$$

• For $b_1 = b^2 > 0$, the system (7) can be reduced to the following ODEs

$$\begin{aligned}
\frac{d\phi_1}{dt} &= -4\phi_1(t)^2 b^2 d_2 + (-4\psi_1(t)b^2 d_3 + a_{11})\phi_1(t) + (-8\phi_3(t)b^2 d_2 - 4\psi_3(t)b^2 d_3)\phi_2(t)\\
&\quad - 4\phi_3(t)\psi_2(t)b^2 d_3,\\
\frac{d\phi_2}{dt} &= (-6\phi_2(t)b^2 d_2 - 3\psi_2(t)b^2 d_3)\phi_1(t) + (-3\psi_1(t)b^2 d_3 + b^2 d_1 + a_{11})\phi_2(t),\\
\frac{d\phi_3}{dt} &= (-6\phi_3(t)b^2 d_2 - 3\psi_3(t)b^2 d_3)\phi_1(t) + (-3\psi_1(t)b^2 d_3 + b^2 d_1 + a_{11})\phi_3(t),\\
\frac{d\psi_1}{dt} &= -4\psi_1(t)^2 b^2 d_6 + (-4\phi_1(t)b^2 d_5 + a_{21})\psi_1(t) + (-4\phi_3(t)b^2 d_5 - 8\psi_3(t)b^2 d_6)\psi_2(t)\\
&\quad - 4\phi_2(t)\psi_3(t)b^2 d_5,\\
\frac{d\psi_2}{dt} &= (-3\phi_2(t)b^2 d_5 - 6\psi_2(t)b^2 d_6)\psi_1(t) + (-3\phi_1(t)b^2 d_5 + b^2 d_4 + a_{21})\psi_2(t),\\
\frac{d\psi_3}{dt} &= (-3\phi_3(t)b^2 d_5 - 6\psi_3(t)b^2 d_6)\psi_1(t) + (-3\phi_1(t)b^2 d_5 + b^2 d_4 + a_{21})\psi_3(t).
\end{aligned}$$

• For $b_1 = -b^2 < 0$, the system (7) can be reduced to the following ODEs

$$\begin{aligned}
\frac{d\phi_1}{dt} &= 4\phi_1(t)^2 b^2 d_2 + (4\psi_1(t) b^2 d_3 + a_{11})\phi_1(t) + 2\phi_2(t)^2 b^2 d_2 + 2\phi_2(t)\psi_2(t) b^2 d_3 \\
&\quad + 2\phi_3(t)^2 b^2 d_2 + 2\phi_3(t)\psi_3(t) b^2 d_3, \\
\frac{d\phi_2}{dt} &= (6\phi_2(t) b^2 d_2 + 3\psi_2(t) b^2 d_3)\phi_1(t) + (3\psi_1(t) b^2 d_3 - b^2 d_1 + a_{11})\phi_2(t), \\
\frac{d\phi_3}{dt} &= (6\phi_3(t) b^2 d_2 + 3\psi_3(t) b^2 d_3)\phi_1(t) + (3\psi_1(t) b^2 d_3 - b^2 d_1 + a_{11})\phi_3(t), \\
\frac{d\psi_1}{dt} &= 4\psi_1(t)^2 b^2 d_6 + (4\phi_1(t) b^2 d_5 + a_{21})\psi_1(t) + 2\phi_2(t)\psi_2(t) b^2 d_5 + 2\phi_3(t)\psi_3(t) b^2 d_5, \\
&\quad + 2\psi_2(t)^2 b^2 d_6 + 2\psi_3(t)^2 b^2 d_6, \\
\frac{d\psi_2}{dt} &= (3\phi_2(t) b^2 d_5 + 6\psi_2(t) b^2 d_6)\psi_1(t) + (3\phi_1(t) b^2 d_5 - b^2 d_4 + a_{21})\psi_2(t), \\
\frac{d\psi_3}{dt} &= (3\phi_3(t) b^2 d_5 + 6\psi_3(t) b^2 d_6)\psi_1(t) + (3\phi_1(t) b^2 d_5 - b^2 d_4 + a_{21})\psi_3(t).
\end{aligned}$$

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References

[1] N. Shigasada, K. Kawadakik, E. Teramoto, *Spatial Segregation of interacting species*, J. Theor. Biology, 79 (1979), 83-99.

[2] Y. Lou, M. Winkler,*Global existence and uniform boundedness of smooth solutions to a cross-diffusion system with equal diffusion rates*, Commun, Part. Diff. Eqs, 40 (2015), 1905-1941.

[3] L. Desvilletter, A. Trescases, *New results for triangular reaction cross diffusion system*, J. Math. Anal. Appl., 430 (2015), 32-59.

[4] R. Cherniha, V. Davydovych, *Nonlinear reaction-diffusion systems, conditional symmetry, exact solutions and their applications in biology*, Springer, 2017.

[5] R. Cherniha, *New exact solutions of a nonlinear cross-diffusion system*, J. Phys. A., 41 (2008), 395204.

[6] R. Cherniha, V. Davydovych, *Lie and conditional symmetries of the three-component diffusive lotka-volterra system*, J. Phys. A, 46 (2013), 185204.

[7] R. Cherniha, *New results solutions and their properties of a nonliear equation of mathematical bilolgy*, UKr. Math. J., 53 (2001), 1409-1421.

[8] R. Cherniha, V. Davydovych, L. Muzyka, *Lie symmetries of the shigesada-Kawasaki-Teramoto system*, Commun. Nonlinear. Sci. Numer. Simulat., 45 (2017), 81-92.

[9] S. Lie, *Uber die integration durch bestimmte integrale von einer klasse linearer partieller differential gleichungen*, Arch. Math., 6 (1881), 328-368.

[10] L.V. Ovsiannikov, *Group analysis of differential equations*, Academic Press, New York, 1982.

[11] R.Z. Zhdanov, V. I. Lahno, *Group classification of heat conductivity equations with a nonlinear source*, J. Phys. A: Math. Gen., 32 (1999), 7405-7418.

[12] C.Z. Qu, *Group classification and generalized conditional symmetry reduction of the nonlinear diffusion-convection equation with a nonlinear source*, Stud. Appl. Math., 99 (1997), 107-136.

[13] H.Z. Liu, *Invariant subspace classification and exact solutions to the generalized nonlinear D-C equation*, Appl. Math. Lett., 83 (2018), 164-168.

[14] J.N. Li, S.L. Zhang, *Symmetry reduction and cauchy problems for a class of fourth-order evolution equations*, Commun. Theor. Phys., 50 (2008), 31-38.

[15] J.N. Li, X.H. Ji, *Numerical simulation and symmetry reduction of a two-component reaction-diffusion system*, Adv. Math. Phys., 2020, ID8949263, 5 pages.

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