Relative relation matrix based approaches for updating approximations in neighborhood multigranulation rough sets

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Abstract. With the revolution of computing and biology technology, data sets containing information could be huge and complex that sometimes are difficult to handle. Dynamic computing is an efficient approach to solve problems. Since neighborhood multigranulation rough sets (NMGRS) have been proposed, few papers focused on how to calculate approximations in NMGRS and how to update approximations in NMGRS dynamically. The purpose of this study is try to propose relative relation matrix based approaches for computing approximations in NMGRS and updating them dynamically. First, static approaches for computing approximations in NMGRS were proposed. Second, relative relation matrix based approaches for updating approximations in NMGRS while decreasing and increasing neighborhood classes were proposed. Third, incremental algorithms for updating approximations in NMGRS while decreasing and increasing neighborhood classes were designed. Finally, the efficiency and the validity of the designed algorithms were verified by experiments.

Keywords: approximation computation, multigranulation rough set, knowledge acquisition, decision making.

Since rough set has been proposed by Pawlak in 1982, it has been widely used in various fields such as pattern recognition [9, 2, 7, 11, 21, 20, 25, 29, 12], machine learning [24, 6, 13, 14, 3, 18, 30, 34, 15], image precessing[16, 17], data mining and other relevant areas. After rough set theory has been proposed, many models were proposed to extend its application, including covering based rough sets [31], variable precision rough sets[8], probabilistic rough sets [30], fuzzy rough sets [26, 23, 25, 6], fuzzy variable precision rough sets [33], and etc.

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Lin et al. proposed NMGRS [10] in 2012. It is a powerful mathematical tool in representing some real life situations and extend the application field of mutigranulation rough sets [19](MGRS) into a broader area. When we apply NMGRS in real life situations, approximation computing is an essential part. Approximations must be computed before decision making. Positive region must be calculated so that attributes can be selected in attribute reduction process. Calculating approximations is necessary when applying rough set theory, thus how to compute the approximations is a significant issue in NMGRS theory.

In the information explosion era, approximation computing becomes more and more difficult: sizes of data sets sometimes are too huge to handled, the structure of data sets are changing to be more and more complex and granular structures often increase or decrease. It is of great importance to calculate approximations of NMGRS by dynamic approaches [28, 5]. In this paper, we attempt to propose relative relation matrix based approaches for computing and updating approximations in NMGRS. In real life applications, neighborhood classes often decreasing and increasing, so it is important to update approximations in NMGRS based on approximations we have computed. We try to design algorithms for updating approximations while decreasing and increasing neighborhood classes. First, we propose relative relation matrix based static approach. Second, we design incremental relative relation matrix based approach for updating approximations in NMGRS need to be proposed.

The rest of this paper is organized as follows. Several basic concepts of NMGRS are introduced in Section 2, and so is relative relation matrix based static algorithm to calculate approximation in NMGRS. In Section 3, dynamic approaches for updating approximations in NMGRS while decreasing and increasing neighborhood classes are proposed. Several algorithms are designed according to the approaches we proposed in Section 4. Experimental evaluations are conducted in Section 5 to verify the efficiency and validity of algorithms we designed. Finally, some conclusions and future work are given in Section 6.

1. Preliminaries

In this section, we review several basic concepts of NMGRS, and then, we propose relative relation matrix based approaches for computing approximations in NMGRS.

1.1 Neighborhood multigranulation rough sets

In this subsection, several basic concepts of NMGRS are reviewed.

Definition 1 ([10]). *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number. $\forall k \in \{1, 2, \ldots, m\}$, A_k^C, $A_k^N \subseteq A_k$ are categorical and numeric attributes, respectively. the neighborhood granules of objects x induced by A_k^C, A_k^N and $A_k^C \cup A_k^N$ are defined as*

- $n_{A_k^C}(x) = \left\{x_i \in U | d_{A_k^C}(x, x_i) = 0\right\};$
- $n_{A_k^N}(x) = \left\{x_i \in U | d_{A_k^N}(x, x_i) \leq \delta\right\};$
- $n_{A_k^C \cup A_k^N}(x) = \left\{x_i \in U | d_{A_k^C}(x, x_i) = 0 \wedge d_{A_k^N}(x, x_i) \leq \delta\right\};$

where d is a distance [22] between two samples, δ is a nonnegative number. Accordingly, we say (U, δ) is a neighborhood approximation space, if there is an attribute subset in the system generating a neighborhood relation on the universe, we can regard this system as a neighborhood information system. Denote by $NIS = (U, AT, \delta)$, where U is a nonempty finite set and AT is a group of attribute set.

In this paper, we consider distance between two objects as Euclidean distance, Euclidean distance is defined as follows:

Definition 2 ([1])**.** *The Euclidean distance between points p and q is the length of the line segment connecting them in Cartesian coordinates, if $p=(p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ are two points in Euclidean n-space, then the distance (d) from p to q, or from q to p is given by the Pythagorean formula:*

$$d(p, q) = d(q, p) = \sqrt{(q_1 - p_1)^2 + q_2 - p_2)^2 + \ldots + (q_n - p_n)^2}.$$

Definition 3 ([10])**.** *(ONMGRS)Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number. For any $X \subseteq U$, the optimistic neighborhood multigranulation lower and upper approximations of X are denoted by $\underline{\sum_{k=1}^m \delta_{A_k}^O}(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^O}(X)$, respectively,*

$$\underline{\sum_{k=1}^m \delta_{A_k}^O}(X) = \Big\{x \in U | n_{A_1^C \cup A_1^N}(x) \subseteq X \vee n_{A_2^C \cup A_2^N}(x) \subseteq X$$
$$\vee \ldots \vee n_{A_m^C \cup A_m^N}(x) \subseteq X\Big\}$$
$$\overline{\sum_{k=1}^m \delta_{A_k}^O}(X) = \sim \underline{\sum_{k=1}^m \delta_{A_k}^O}(\sim X)$$

where $n_{A_k^C \cup A_k^N}(x)$ is the neighborhood class of x in terms of the attribute set A_k and neighborhood radius δ, $\sim X$ is the complement of the set X.

Theorem 1 ([10])**.** *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number. For any $X \subseteq U$, the optimistic neighborhood multigranulation upper approximation of X*

is denoted by $\overline{\sum_{k=1}^{m}\delta_{A_k}^{O}}(X)$,

$$\overline{\sum_{k=1}^{m}\delta_{A_k}^{O}}(X)=\Big\{x\in U|n_{A_1^C\cup A_1^N}(x)\cap X\neq\emptyset\wedge n_{A_2^C\cup A_2^N}(x)\cap X\neq\emptyset \wedge\ldots\wedge n_{A_m^C\cup A_m^N}(x)\cap X\neq\emptyset\Big\}$$

where $n_{A_k^C\cup A_k^N}(x)$ *is the neighborhood class of* x *in terms of the attribute set* A_k *and neighborhood radius* δ, $\sim X$ *is the complement of the set* X.

Definition 4 ([10]). *(PNMGRS)Let* $NIS=(U,AT,\delta)$ *be a neighborhood information system, where* $A_k\in AT$ *for any* $k\in\{1,2,\ldots,m\}$, δ *is a nonnegative number. For any* $X\subseteq U$, *the pessimistic neighborhood multigranulation lower and upper approximation of* X *are denoted by* $\underline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)$ *and* $\overline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)$, *respectively,*

$$\underline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)=\Big\{x\in U|n_{A_1^C\cup A_1^N}(x)\subseteq X\wedge n_{A_2^C\cup A_2^N}(x)\subseteq X \wedge\ldots\wedge n_{A_m^C\cup A_m^N}(x)\subseteq X\Big\}$$

$$\overline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)=\sim\underline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(\sim X)$$

where $n_{A_k^C\cup A_k^N}(x)$ *is the neighborhood class of* x *in terms of the attribute set* A_k *and neighborhood radius* δ, $\sim X$ *is the complement of the set* X.

Theorem 2 ([10]). *Let* $NIS=(U,AT,\delta)$ *be a neighborhood information system, where* $A_k\in AT$ *for any* $k\in\{1,2,\ldots,m\}$, δ *is a nonnegative number and for any* $X\subseteq U$. *The pessimistic neighborhood multigranulation upper approximation of* X *is denoted by* $\overline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)$,

$$\overline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X) =\Big\{x\in U|n_{A_1^C\cup A_1^N}(x)\cap X\neq\emptyset\vee n_{A_2^C\cup A_2^N}(x)\cap X\neq\emptyset\vee\ldots\vee n_{A_m^C\cup A_m^N}(x)\cap X\neq\emptyset\Big\},$$

where $n_{A_k^C\cup A_k^N}(x)$ *is the neighborhood class of* x *in terms of the attribute set* A_k *and neighborhood radius* δ, $\sim X$ *is the complement of the set* X.

1.2 Relative relation matrix-based algorithm for computing approximations in multigranulation rough set

Definition 5 ([4]). *Let* U *be an universe and* $|U|=n$, *the matrix representation of* $X\subseteq U$ *is denoted as* $G^U(X)=[g_1^U(X),\ldots,g_{|U|}^U(X)]^T$, *where*

$$g_i^U(X)=\begin{cases}1, & x_i\in X\\ 0, & x_i\notin X\end{cases},\qquad i\in\{1,2,\ldots,|U|\}$$

and "T" denotes the transpose operation.

Definition 6. *Let U be an universe and $|U| = n$, the set representation of column matrix $P = [p_1, p_2, \ldots, p_n]^T$ is denoted as $G_{-1}^{U}(P) = \{x_{r_1}, x_{r_2}, \ldots, x_{r_t}\}$, where*

$$x_{r_s} = \begin{cases} x_i, & p_i = 1 \\ None, & p_i = 0 \end{cases}, \qquad i \in \{1, 2, \ldots, |U|\}.$$

Definition 7. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, R_k denotes the neighborhood relation of the granular structure A_k on U, the relative approximation space which is relative to X is denoted by $V^k = \{y_1^k, y_2^k, \ldots, y_r^k\}$, where for any $x_i, x_j \in U$, if $(x_i, x_j) \in R_k$ and $x_i \in X$, exists $y_p^k, y_q^k \in V^k$ satisfied that $y_p^k = x_i, y_q^k = x_j$. Further more, we view $RNIS = (VT, AT, \delta)$ as a relative neighborhood information system where $VT = \{V^1, V^2, \ldots, V^m\}$.*

Definition 8. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \ldots, m\}$, if $R_k(X)\{x_{s_1}, x_{s_2}, \ldots, x_{s_{r_k}}\}$ satisfied that $(x_{s_p}, x_{s_q}) \in R_k$ and $x_{s_p} \in X$, then $y_1 = x_{s_1}, \ldots, y_2 = x_{s_2}, y_r = x_{s_{r_k}}$, where $s_1 < s_2 < \ldots < s_{r_k}$, then exists bijective mapping $f : U \to V^k$ and its inverse mapping f^{-1} satisfied that*

$$f(x_{s_p}) = y_p \wedge f^{-1}(y_p) = x_{s_p},$$

for all $p \in \{1, 2, \ldots, r_k\}$.

Definition 9. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \ldots, m\}$. For any $Y \subseteq V^k$, if the boolean column matrix of Y in U and V is denoted by $G^{V^k}(Y)$ and $G^U(f^{-1}(Y))$, then exists bijective mapping F and its inverse mapping F^{-1} satisfied that*

$$F(G^U(f^{-1}(Y))) = G^{V^k}(Y) \wedge F^{-1}(G^{V^k}(Y)) = G^U(f^{-1}(Y).$$

Definition 10 ([27])**.** *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure*

Table 1: A neighborhood information system.

U	a_1	a_2	a_3	a_4
x_1	1	3	1.5	0.5
x_2	2	2	0.5	1
x_3	3	1	2	2.5
x_4	2	3	1.5	1
x_5	1	1	2	0.5
x_6	3	2	0.5	2

A_k on U and V^k for any $k \in \{1, 2, \ldots, m\}$, $MX_{A_k} = (mx_{ij}^{A_k})_{n\times n}$ denote the relative neighborhood relation matrix of the granular structure A_k which is relative to X, where

$$(1) \qquad mx_{ij}^{A_k} = \begin{cases} 1, & (y_p, y_q) \in R_k \\ 0, & (y_p, y_q) \notin R_k \end{cases}, \qquad p, q \in \{1, 2, \ldots, r\}$$

and the elements that relative relation matrix perform their relations are $R_k(X) = \{x_{s_1}, x_{s_2}, \ldots, x_{s_r}\} = \{y_1^k, y_2^k, \ldots, y_r^k\}$, $|R(X)| = \max_{k=1}^{m} |R_{A_k}(X)|$.

Example 1. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, as shown in Table 1, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $A_1 = A_1^C \cup A_1^N$, $A_1^C = \{a_1\}$ and $A_1^N = \{a_3\}$, $A_2 = A_2^C \cup A_2^N$, $A_2^C = \{a_2\}$ and $A_2^N = \{a_4\}$. Let $X = \{x_2, x_4, x_5\}$ and $\delta = 1$. According to Definition 10, we have that for any $k \in \{1, 2\}$, the relative neighborhood relation matrix can be calculate as follows,

$$M_{A_1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, M_{A_2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

and

$$R_{A_1}(X) = \{x_1, x_2, x_4, x_5\} = \{y_1, y_2, y_3, y_4\} - V^1,$$
$$R_{A_2}(X) = \{x_1, x_2, x_4, x_5, x_6\} = \{y_1, y_2, y_3, y_4, y_5\} = V^2.$$

then, we have $f(R_{A_1}(X)) = V^1$, $f(R_{A_2}(X)) = V^2$, $f^{-1}(V^1) = R_{A_1}(X)$, $f^{-1}(V^2) = R_{A_2}(X)$. $G^U(X) = [0, 1, 0, 1, 1, 0]^T$, $G^{V^1}(X) = [0, 1, 1, 1]^T$, $G^{V^2}(X) = [0, 1, 1, 1, 0]^T$, $F(G^U(X)) {=} G^{V^1}(X)$, $F^{-1}(G^{V^1}(X)) {=} G^U(X)$.

$F(G^U(X)) = G^{V^2}(X)$, $F^{-1}(G^{V^2}(X)) = G^U(X)$.

Definition 11. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information* *system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \ldots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix,*

$$\Lambda_{A_k} = diag[\frac{1}{\lambda^1_{A_k}}, \frac{1}{\lambda^2_{A_k}}, \ldots, \frac{1}{\lambda^{r_k}_{A_k}}],$$

where $\lambda^q_{A_k} = \sum_{i=1}^{r_k} mx^{A_k}_{ij}$ for any $i \in \{1, 2, \ldots, r_k\}$

Example 2 (Continuation of Example 1)**.** by Definition 11, for any $k \in \{1, 2, 3\}$, we can calculate the diagonal matrix of A_k as follows.

$$\begin{aligned} \Lambda_{A_1} &= diag[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], \\ \Lambda_{A_2} &= diag[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}]. \end{aligned} \tag{2}$$

Theorem 3. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \ldots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, then*

$$\Lambda_{A_k} = diag[\frac{1}{|n_{A_1^C \cup A_k^N}(y_1^k)|}, \frac{1}{|n_{A_1^C \cup A_k^N}(y_2^k)|}, \ldots, \frac{1}{|n_{A_1^C \cup A_k^N}(y_r^k)|}],$$

where $|n_{A_1^C \cup A_k^N}(y_s)|$ denote the cardinality of the neighborhood class of y_s with respect to granular structure A_k, $d \in 1, 2, \ldots, r_k$.

Definition 12. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \ldots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, then the column matrix $H_{A_k}(X)$ of the granular structure A_k can be calculated as follows*

$$H_{A_k}(X) = \Lambda_{A_k} \cdot (MX_{A_k} \cdot G^{V^k}(X)),$$

where " $\cdot$ " denote dot product of matrix.

Example 3 (Continuation of Example 1)**.** According to Definition 12, the column matrix $H_{A_1}(X)$ and $H_{A_2}(X)$ can be calculated as follows.

$$\begin{aligned} H_{A_1}(X) &= \Lambda_{A_1} \cdot (MX_{A_1} \cdot G^{V^1}(X)) \\ &= diag\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \cdot \left[\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1/2 \end{bmatrix} \end{aligned}$$

$$H_{A_2}(X) = \Lambda_{A_2} \cdot (MX_{A_2} \cdot G^{V^2}(X))$$

$$= diag\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\right] \left[\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1 \\ 1/2 \end{bmatrix}$$

Definition 13 ([4]). *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \ldots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, $H_{A_K}(X)$ is the column matrix of granular structure A_k, then two cut matrices of A_k can be denoted as follows,*

- $H^{[\alpha,\beta]}_{A_k}(X) = (h^{i\downarrow}_{A_k})_{r\times 1}$, *where*

$$h^{i\downarrow}_{A_k} = \begin{cases} 1, & \alpha \leq h^i_{A_k} \leq \beta \\ 0, & otherwise \end{cases}, \qquad i \in \{1, 2, \ldots, r_k\} \tag{3}$$

- $H^{(\alpha,\beta]}_{A_k}(X) = (h^{i\uparrow}_{A_k})_{r\times 1}$, *where*

$$h^{i\downarrow}_{A_k} = \begin{cases} 1, & \alpha < h^i_{A_k} \leq \beta \\ 0, & otherwise \end{cases}, \qquad i \in \{1, 2, \ldots, r_k\} \tag{4}$$

Example 4 (Continuation of Example 3). Assume that $\alpha = 1$ and $\beta = 1$. According to Definition 13. the cut matrix of H_{A_1} and H_{A_2} can be calculated as follows,

$$H^{[1,1]}_{A_1}(X) = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T,$$
$$H^{[1,1]}_{A_2}(X) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

Assume that $\alpha = 0$ and $\beta = 1$. the cut matrix of H_{A_1} and H_{A_2} can be calculated as follows,

$$H^{(0,1]}_{A_1}(X) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T,$$
$$H^{(0,1]}_{A_2}(X) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T.$$

Theorem 4. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \ldots, m\}$, δ is a nonnegative number and* *for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix. If $H_{A_K}(X)$ is the column matrix of granular structure A_k, the lower and upper approximations in ONMGRS are denoted by $\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)$ and $\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)$, respectively, the following results hold,*

1. *If $G^U(\underline{\sum_{k=1}^m \delta^O_{A_k}}(X))$ denotes the boolean column matrix of lower approximation in ONMGRS, then we have*

$$G^U(\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)) = \bigvee_{k=1}^m F^{-1}(H^{[1,1]}_{A_k}(X)),$$

2. *If $G^U(\overline{\sum_{k=1}^m \delta^O_{A_k}}(X))$ denotes the boolean column matrix of upper approximation in ONMGRS, then we have*

$$G^U(\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)) = \bigwedge_{k=1}^m F^{-1}(H^{(0,1]}_{A_k}(X)).$$

Proof. 1. $\forall s \in \{1, 2, \dots, r_k\}$, $g_s^{V_k}(\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)) = 1 \Leftrightarrow y_s \in \underline{\sum_{k=1}^m \delta^O_{A_k}}(X) \Leftrightarrow \exists k \in \{1, 2, \dots, m\}$, $n_{A_k^C \cup A_k^N}(y_s^{A_k}) \subseteq X$. Since $(y_t, y_s) \in R_k$ and $y_s \in X$, we have $mx_{st}^{A_k} = 1$ and $g_s^{V_k}(X) = 1$, thus we have $g_s^{V_k}(\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)) = 1 \Leftrightarrow \exists k \in \{1, 2, \dots, m\}$, s.t $h^{s\downarrow}_{A_k} = \frac{\sum_{t=1}^{r_k} mx_{st}^{A_k} \times g_t^{V_k}(X)}{\sum_{t=1}^{r_k} mx_{st}^{A_k}} = 1$ holds. Then we have $g_s^{V_k}(\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)) = 1 \Leftrightarrow \bigvee_{k=1}^m h^{s\downarrow}_{A_k} = 1$, $G^{V_k}(\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)) = \bigvee_{k=1}^m H^{[1,1]}_{A_k}(X)$, in other words, $G^U(\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)) = \bigvee_{k=1}^m F^{-1}(H^{[1,1]}_{A_k}(X))$.

2. The proof is similar to that of 1. □

Theorem 5. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix. If $H_{A_K}(X)$ is the column matrix of granular structure A_k, the lower and upper approximations in ONMGRS are denoted by $\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)$ and $\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)$, respectively, the following results hold,*

1. *If $G^U(\underline{\sum_{k=1}^m \delta^P_{A_k}}(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have*

$$G^U(\underline{\sum_{k=1}^m \delta^P_{A_k}}(X)) = \bigwedge_{k=1}^m F^{-1}(H^{[1,1]}_{A_k}(X)),$$

2. *If* $G^U(\overline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X))$ *denotes the boolean column matrix of upper approximation in ONMGRS, then we have*

$$G^U(\overline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)) = \bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{(0,1]}(X)).$$

Proof. The proof is similar to that of Theorem 4. □

Example 5 (Continuation of Example 3)**.** The boolean column matrix of lower and upper approximations in PNMGRS can be computed as follows,

$$\begin{aligned}
G^U(\underline{\sum_{k=1}^{m}\delta_{A_k}^{O}}(X)) &= \bigvee_{k=1}^{2} F^{-1}(H_{A_k}^{[1,1]}(X)) \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^T.
\end{aligned}$$

$$\begin{aligned}
G^U(\overline{\sum_{k=1}^{m}\delta_{A_k}^{O}}(X)) &= \bigwedge_{k=1}^{2} F^{-1}(H_{A_k}^{(0,1]}(X)) \\
&= \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^T \wedge \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^T.
\end{aligned}$$

The boolean column matrix of lower and upper approximations in PNMGRS can be computed as follows,

$$\begin{aligned}
G^U(\underline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)) &= \bigwedge_{k=1}^{2} F^{-1}(H_{A_k}^{[1,1]}(X)) \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \wedge \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\end{aligned}$$

$$\begin{aligned}
G^U(\overline{\sum_{k=1}^{m}\delta_{A_k}^{P}}(X)) &= \bigvee_{k=1}^{2} F^{-1}(H_{A_k}^{(0,1]}(X)) \\
&= \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T.
\end{aligned}$$

Algorithm 1 is a relative matrix based algorithm for computing approximations in NMGRS. If $N_X = \max_{k=1}^{m}\{|R_{A_k}(X)|\}$, and then he total time complexity of the algorithm is $O(|AT|N_X^2)$. Steps 4-9 are to calculate $R_k(X)$ whose time complexity is $O(|AT||X|| \sim X|))$, Steps 12-17 are to calculate MX_{A_k} whose time complexity is $O(|AT|N_X^2))$. Steps 18-22 are to calculate H_{A_k} whose time complexity is $O(|AT|N_X^2))$, 24-27 are to compute the approximations of NMGRS whose time complexity is $O(|AT||U|)$.

Algorithm 1 Relative relation matrix based algorithm for computing approximations in NMGRS

Require: (1)A neighborhood information system $NIS = (U, AT, \delta)$, the granular structure is $A_1, A_2, \ldots, A_m$. (2)A target concept $X \subseteq U$

Ensure: Approximations in NMGRS.

$m \longleftarrow |AT^t|$
$n \longleftarrow |U|$
for $k = 1 \rightarrow m$ **do**
  for $i = 1 \rightarrow n$ **do**
    for $j = 1 \rightarrow n$ **do**
      if $g_i^U(X) = 1 \wedge g_j^U(n_{A_1^C \cup A_1^N}(x_i)) = 1$ **then** $g_i^U(R_k(X)) = 1$
      end if
    end for
  end for
  $V \longleftarrow f^{-1}(R_k(X))$
  $r \longleftarrow |V^k|$
  for $i = 1 \rightarrow r_k$ **do**
    for $j = 1 \rightarrow r_k$ **do**
      if $(y_i, y_j) \in R_k = 1$ **then** $mx_{ij}^{A_k} = 1$
      end if
    end for
  end for
  for $i = 1 \rightarrow r_k$ **do**
    $g_i^{V^k}(\Lambda_{A_k}) \longleftarrow \sum_{j=1}^{r_k} mx_{ij}^{A_k}$
  end for
  $G^{V^k}(X) \longleftarrow F^{-1}(G^U(X))$
  $H_{A_k} \longleftarrow diag(\Lambda_{A_k}) \cdot (MX_{A_k} \cdot G^{V^k}(X))$
end for
$G^U(\underline{\sum_{k=1}^m \delta_{A_k}^O}(X)) \longleftarrow \bigvee_{i=1}^m F^{-1}(H_(A_k)^{[1,1]})$
$G^U(\overline{\sum_{k=1}^m \delta_{A_k}^O}(X)) \longleftarrow \bigwedge_{i=1}^m F^{-1}(H_(A_k)^{(0,1]})$
$G^U(\underline{\sum_{k=1}^m \delta_{A_k}^P}(X)) \longleftarrow \bigwedge_{i=1}^m F^{-1}(H_(A_k)^{[1,1]})$
$G^U(\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)) \longleftarrow \bigvee_{i=1}^m F^{-1}(H_(A_k)^{(0,1]})$
Return $\underline{\sum_{k=1}^m \delta_{A_k}^O}(X)$, $\overline{\sum_{k=1}^m \delta_{A_k}^O}(X)$, $\underline{\sum_{k=1}^m \delta_{A_k}^P}(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)$.

2. Relative relation matrix based dynamic approaches for updating approximations in NMGRS while neighborhood classes are decreasing and increasing

2.1 Relative relation matrix based approaches for updating approximations while neighborhood classes are decreasing

In this subsection, we present relative relation matrix based dynamic approaches for updating approximations in NMGRS while neighborhood classes are decreasing. We denote $NIS^t = (U, AT^t, \Delta)$ be an neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be an neighborhood information system at time $t+1$, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \delta)$ be a relative neighborhood information system at time $t+1$ where δ and Δ are two nonnegative numbers and $\delta \leq \Delta$. For all $A_k^t \in AT^t (k \leq m)$, exists $A_k^{t+1} \in AT^{t+1}$, s.t. $(A_k^C)^t \subseteq (A_k^C)^{t+1}$, $(A_k^N)^t \subseteq (A_k^N)^{t+1}$. Also, for all $x \in U$, we denote neighborhood class of x at time t by $n^t_{A_k^C \cup A_k^N}(x)$. Denote neighborhood class of x at time $t+1$ by $n^{t+1}_{A_k^C \cup A_k^N}(x_i)$. For all $X \in U$, we denote approximations in PNMGRS by $\underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X)$ and $\overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X)$ at time t. Denote approximations in PNMGRS $\underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X)$ and $\overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X)$ at time $t+1$. Denote approximations in ONMGRS by $\underline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X)$ and $\overline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X)$ at time t. Denote approximations in ONMGRS by $\underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$ and $\overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$ at time $t+1$.

Theorem 6 ([32])**.** *Let $NIS^t = (U, AT^t, \Delta)$ be an neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t+1$. For any $X \subseteq U$, the following result holds,*

$$n^{t+1}_{A_k^C \cup A_k^N}(x) \subseteq n^t_{A_k^C \cup A_k^N}(x).$$

Theorem 7 ([32])**.** *Let $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t+1$. For any $X \subseteq U$, the following results hold,*
(1) $\underline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X) \subseteq \underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$;
(2) $\overline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X) \supseteq \overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$.

Theorem 8 ([32])**.** *Let $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t+1$. For any $X \subseteq U$, the following results hold,*
(1) $\underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) \subseteq \underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X)$;
(2) $\overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) \supseteq \overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X)$.

Theorem 9. *Let $NIS^t = (U, AT^t, \Delta)$ be an neighborhood information system at time t, $RNIS^t = (VT^t, AT^t, \Delta)$ be a relative neighborhood information system* *at time t; $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be an neighborhood information system at time t+1, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \delta)$ be a relative neighborhood information system at time $t+1$. For any $X \in U$, the upper approximations in NMGRS at time $t+1$ are denoted by $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X))$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{P}(X)$, respectively. If $BN^P(X) = \overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)$, the following results hold,*

1. *If $G^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have*

$$G^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X)) = G^U(\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{O}(X)) \wedge \sim (\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))),$$

2. *If $G^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{P}(X))$ denote the boolean column matrix of upper approximation in ONMGRS, then we have*

$$G^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{P}(X)) = G^U(\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)) \wedge \sim (\bigwedge_{k=1}^{m} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))).$$

Proof. 1. $\forall i \in \{1, 2, \ldots, n\}, g_i^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X)) = 1 \Leftrightarrow \forall k \in \{1, 2, \ldots, m\}$, $n_{A_k^C \cup A_k^N}(x_i) \subseteq X \wedge \forall x_j \in BN^P(X), \exists k \in \{1, 2, \ldots, m\}, n_{A_k^C \cup A_k^N}(x_j) \nsubseteq X \Leftrightarrow g_i^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X)) = 1 \wedge g_j^U(\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) = 0$.

2. The proof is similar to that of 1.

□

Theorem 10. *Let $NIS^t = (U, AT^t, \Delta)$ be an neighborhood information system at time t, $RNIS^t = (VT^t, AT^t, \Delta)$ be a relative neighborhood information system at time t; $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be an neighborhood information system at time t+1, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \delta)$ be a relative neighborhood information system at time t+1. For any $X \in U$, the lower approximations in NMGRS at time $t+1$ are denoted by $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X)$ and $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{P}(X)$, respectively. If $BN^P(X) = \overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)$, the following results hold,*

1. *If $G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have*

$$G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^{O}(X)) = G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t}}^{O}(X)) \vee (\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[1,1]}(BN^P(X)))),$$

2. *If* $G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X))$ *denote the boolean column matrix of lower approximation in ONMGRS, then we have*

$$G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)) = G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t}}^P(X)) \vee (\bigwedge_{k=1}^{m} F^{-1}(H_{A_k}^{[1,1]}(BN^P(X)))).$$

Proof. The proof is similar to that of Theorem 9. □

Example 6. (Continuation of Example 1)Supposed that $AT^t = \{A_1^t, A_2^t\}$, $A_1^t = (A_1^C)^t \cup (A_1^N)^t$, $(A_1^C)^t = \{a_1\}$ and $(A_1^N)^t = \emptyset$, $A_2^t = (A_2^C)^t \cup (A_2^N)^t$, $(A_2^C)^t = \emptyset$ and $(A_2^N)^t = \{a_3\}$, $AT^{t+1} = \{A_1^{t+1}, A_2^{t+1}\}$, $A_1^{t+1} = (A_1^C)^{t+1} \cup (A_1^N)^{t+1}$, $(A_1^C)^{t+1} = \{a_1\}$ and $(A_1^N)^{t+1} = \{a_4\}$, $A_2^{t+1} = (A_2^C)^{t+1} \cup (A_2^N)^{t+1}$, $(A_2^C)^{t+1} = \{a_1\}$ and $(A_2^N)^{t+1} = \{a_3\}$. Let and $\Delta = 0.5, \delta = 0.5$. According to Definition 10, we have that for any $k \in \{1, 2\}$, the relative neighborhood relation matrix can be calculate as follows. According to Theorem 9 and Theorem 10, $BN^P(X) = \{x_1, x_2, x_3, x_5\}$, $R_{A_2}(BN^P(X)) = \{x_1, x_2, x_3, x_5\}$, $R_{A_1}(BN^P(X)) = U$, the column matrix $H_{A_1}(BN^P(X))$ and $H_{A_2}(BN^P(X))$ can be calculated as follows.

$$\begin{aligned}
&H_{A_1}(BN^P(X)) = \Lambda_{A_1} \cdot (M_{A_1} \cdot G^{V^1}(X)) \\
&= diag\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \cdot \left[\begin{bmatrix} 1&0&0&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&0&1 \\ 0&1&0&1&0&0 \\ 1&0&0&0&1&0 \\ 0&0&1&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 0\\1\\0\\1\\1\\0 \end{bmatrix}\right] = \begin{bmatrix} 1/2\\1\\0\\1\\1/2\\0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&H_{A_2}(BN^P(X)) = \Lambda_{A_2} \cdot (M_{A_2} \cdot G^{V^2}(X)) \\
&= diag\left[\frac{1}{2}, 1, 1, \frac{1}{2}\right] \left[\begin{bmatrix} 1&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 0\\1\\0\\1 \end{bmatrix}\right] = \begin{bmatrix} 1/2\\1/2\\0\\1/2 \end{bmatrix}.
\end{aligned}$$

According to Theorem 9 and Theorem 10 we have

$$\begin{aligned}
G^U(\overline{\sum_{k=1}^{2} \delta_{A_k}^{t+1}}^O(X)) &= G^U(\overline{\sum_{k=1}^{2} \Delta_{A_k}^{t}}^O(X)) \wedge \sim (\bigvee_{k=1}^{2} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) \\
&= \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T \wedge \sim \\
&\left[\begin{bmatrix} 0&0&1&0&0&1 \end{bmatrix}^T \vee \begin{bmatrix} 0&0&1&0&0&0 \end{bmatrix}^T\right] \\
&= \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T \wedge \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T \\
&= \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T.
\end{aligned}$$

$$
\begin{aligned}
G^U(\overline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{P}(X)) &= G^U(\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{P}(X)) \wedge \sim (\bigwedge_{k=1}^{2} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T \wedge \sim \\
&\Big[\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T \wedge \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T\Big] \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T \wedge \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^T .
\end{aligned}
$$

$$
\begin{aligned}
G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{O}(X)) &= G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)) \vee (\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[1,1]}(BN^P(X)))) \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \\
&\Big[\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T\Big] \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T .
\end{aligned}
$$

$$
\begin{aligned}
G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{P}(X)) &= G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{P}(X)) \vee (\bigwedge_{k=1}^{m} F^{-1}(H_{A_k}^{[1,1]}(BN^P(X)))) \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \\
&\Big[\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \wedge \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T\Big] \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T .
\end{aligned}
$$

2.2 Matrix-based approaches for updating approximations while neighborhood classes are increasing

In this subsection, we present matrix based dynamic approaches for updating approximations in neighborhood multigranulation rough set while neighborhood classes are increasing. We denote $NIS^t = (U, AT^t, \delta)$ be an neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be an neighborhood information system at time $t+1$,$RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \Delta)$ be an relative neighborhood information system at time $t+1$ where δ and Δ are two nonnegative numbers and $\delta \leq \Delta$. For all $A_k^t \in AT^t (k \leq m)$, exists $A_k^{t+1} \in AT^{t+1}$, s.t. $(A_k^C)^{t+1} \subseteq (A_k^C)^t$ and $(A_k^N)^{t+1} \subseteq (A_k^N)^t$. Also, for all $x \in U$, we denote neighborhood class of x at time t by $n^t_{A_k^C \cup A_k^N}(x)$. Denote neighborhood class of x at time $t+1$ by $n^{t+1}_{A_k^C \cup A_k^N}(x)$. For all $X \in U$, we denote approximations in PNMGRS by $\underline{\sum_{k=1}^m \delta^t_{A_k}}^P(X)$ and $\overline{\sum_{k=1}^m \delta^t_{A_k}}^P(X)$ at time t. Denote approximations in PNMGRS $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)$ at time $t+1$. Denote approximations in ONMGRS by $\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)$ and $\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)$ at time t. Denote approximations in ONMGRS by $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)$ and $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)$ at time $t+1$.

Theorem 11 ([32]). *Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t+1$, . For any $X \subseteq U$, the following result holds,*

$$n^{t+1}_{A_k^C \cup A_k^N}(x) \supseteq n^{t}_{A_k^C \cup A_k^N}(x).$$

Theorem 12 ([32]). *Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t+1$, . For any $X \subseteq U$, the following results hold:*
(1) $\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X) \supseteq \underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)$;
(2) $\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X) \subseteq \overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)$.

Theorem 13 ([32]). *Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t+1$, . For any $X \subseteq U$, the following results hold:*
(1) $\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X) \supseteq \underline{\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}}^{P}(X)$;
(2) $\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X) \subseteq \overline{\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}}^{P}(X)$.

Theorem 14. *Let $NIS^t = (U, AT^t, \delta)$ be an neighborhood information system at time t, $RNIS^t = (VT^t, AT^t, \delta)$ be an relative neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be an neighborhood information system at time $t+1$,$RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \Delta)$ be an relative neighborhood information system at time $t+1$. For all $X \in U$, the upper approximations in NMGRS are denoted by $\overline{\sum_{k=1}^{m}\delta_{A_k}^{O}}(X)$ and $\overline{\sum_{k=1}^{m}\delta_{A_k}^{O}}(X)$, respectively. If $BN^O(X) = \underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X) \cup (U - \overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)$, the following results hold,*

1. *If $G^U(\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X))$ denotes the boolean column matrix of upper approximation in PNMGRS, then we have*

$$G^U(\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)) = \overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X) \vee (\bigwedge_{k=1}^{m} F^{-1}(H_{A_k}^{(0,1]}(BN^P(X)))),$$

2. *If $G^U(\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{P}(X))$ denotes the boolean column matrix of upper approximation in ONMGRS, then we have*

$$G^U(\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)) = \overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X) \vee (\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{(0,1]}(BN^P(X)))).$$

Proof. 1. $\forall i \in \{1,2,\ldots,n\}, g_i^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)) = 1 \Leftrightarrow \forall k \in \{1,2,\ldots,m\}$, $n_{A_k^C \cup A_k^N}(x_i) \subseteq X \wedge \forall x_j \in BN^O(X), \exists k \in \{1,2,\ldots,m\}, n_{A_k^C \cup A_k^N}(x_j) \subseteq X \Leftrightarrow g_i^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)) = 1 \vee g_j^U(\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[0,1]}(BN^O(X)))) = 1$.

2. The proof is similar to that of 1. □

Definition 14. *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1,2,\ldots,m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1,2,\ldots,m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, then the cut matrix of A_k can be denoted as $H_{A_k}^{[\alpha,\beta)}(X) = (h_{A_k}^{i\downarrow})_{r\times 1}$, where*

$$h_{A_k}^{i\downarrow} = \begin{cases} 1, & \alpha \le h_{A_k}^i < \beta \\ 0, & otherwise \end{cases}, \qquad i \in \{1,2,\ldots,r\}.$$

Theorem 15. *Let $NIS^t = (U, AT^t, \delta)$ be an neighborhood information system at time t, $RNIS^t = (VT^t, AT^t, \delta)$ be an relative neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be an neighborhood information system at time $t+1$,$RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \Delta)$ be an relative neighborhood information system at time $t+1$. For all $X \in U$, the lower approximations in NMGRS are denoted by $\underline{\sum_{k=1}^{m} \delta_{A_k}^P}(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^P}(X)$, respectively. If $BN^O(X) = \underline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X) \cup (U - \overline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X)$, the following results hold,*

1. *If $G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have*

$$G^U(\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^O(X)) = \underline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X) \wedge \sim (\bigwedge_{k=1}^{m} F^{-1}(H_{A_k}^{[0,1)}(BN^P(X)))),$$

2. *If $G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have*

$$G^U(\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^P(X)) = \underline{\sum_{k=1}^{m} \delta_{A_k}^t}^P(X) \wedge \sim (\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[0,1)}(BN^P(X)))).$$

Proof. The proof is similar to that of Theorem 9. □

Example 7. (Continuation of Example 1)Supposed that $AT^t = \{A_1^t, A_2^t\}, A_1^t = (A_1^C)^t \cup (A_1^N)^t$, $(A_1^C)^t = \{a_1, a_2\}$ and $(A_1^N)^t = \emptyset$, $A_2^t = (A_2^C)^t \cup (A_2^N)^t$, $(A_2^C)^t = \emptyset$ and $(A_2^N)^t = \{a_3, a_4\}$, $AT^{t+1} = \{A_1^{t+1}, A_2^{t+1}\}, A_1^{t+1} = (A_1^C)^{t+1} \cup (A_1^N)^{t+1}$,

$(A_1^C)^{t+1} = \{a_2\}$ and $(A_1^N)^{t+1} = \emptyset$, $A_2^{t+1} = (A_2^C)^{t+1} \cup (A_2^N)^{t+1}$, $(A_2^C)^{t+1} = \emptyset$ and $(A_2^N)^{t+1} = \{a_4\}$. Let $\delta = 0.5, \Delta = 0.5$. According to Definition 10, we have that for any $k \in \{1,2\}$, the relative neighborhood relation matrix can be calculate as follows,According to Theorem 14 and Theorem 15, $BN^O(X) = \{x_2, x_3, x_4, x_6\}$, $R_{A_2}(BN^P(X)) = U$, $R_{A_1}(BN^P(X)) = U$, the column matrix $H_{A_1}(BN^P(X))$ and $H_{A_2}(BN^P(X))$ can be calculated as follows.

$$H_{A_1}(BN^P(X)) = \Lambda_{A_1} \cdot (M_{A_1} \cdot G^{V^1}(X))$$
$$= diag\left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right] \cdot \left[\begin{bmatrix} 1&0&0&1&0&0\\ 0&1&0&0&0&1\\ 0&0&1&0&1&0\\ 1&0&0&1&0&0\\ 0&0&1&0&1&0\\ 0&1&0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 0\\1\\0\\1\\1\\0 \end{bmatrix}\right] = \begin{bmatrix} 1/2\\1/2\\1/2\\1/2\\1/2\\1/2 \end{bmatrix},$$

$$H_{A_2}(BN^P(X)) = \Lambda_{A_2} \cdot (M_{A_2} \cdot G^{V^2}(X))$$
$$= diag\left[\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{2}\right] \left[\begin{bmatrix} 1&1&0&1&1&0\\ 1&1&0&1&1&0\\ 0&0&1&0&0&1\\ 1&1&0&1&1&0\\ 1&1&0&1&1&0\\ 0&0&1&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 0\\1\\0\\1\\1\\0 \end{bmatrix}\right] = \begin{bmatrix} 3/4\\3/4\\0\\3/4\\3/4\\0 \end{bmatrix}.$$

According to Theorem 9) and Theorem 10) we have

$$\begin{aligned} G^U(\overline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{O}(X)) &= G^U(\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{O}(X)) \vee (\bigwedge_{k=1}^{2} F^{-1}(H_{A_k}^{(0,1]}(BN^P(X)))) \\ &= \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T \vee \\ &\left[\begin{bmatrix} 1&1&1&1&1&1 \end{bmatrix}^T \wedge \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T\right] \\ &= \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T \wedge \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T \\ &= \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T. \end{aligned}$$

$$\begin{aligned} G^U(\overline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{P}(X)) &= G^U(\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{P}(X)) \wedge \sim (\bigwedge_{k=1}^{2} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) \\ &= \begin{bmatrix} 1&1&1&1&1&0 \end{bmatrix}^T \vee \\ &\left[\begin{bmatrix} 1&1&1&1&1&1 \end{bmatrix}^T \vee \begin{bmatrix} 1&1&0&1&1&0 \end{bmatrix}^T\right] \\ &= \begin{bmatrix} 1&1&1&1&1&0 \end{bmatrix}^T \wedge \begin{bmatrix} 1&1&1&1&1&1 \end{bmatrix}^T \\ &= \begin{bmatrix} 1&1&1&1&1&0 \end{bmatrix}^T. \end{aligned}$$

$$
\begin{aligned}
G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^O(X)) &= G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^O(X))\wedge\sim(\bigwedge_{k=1}^{m}F^{-1}(H_{A_k}^{[0,1)}(BN^P(X))))\\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \wedge \sim\\
&\Big[\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \vee \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T\Big]\\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \wedge \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T\\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\end{aligned}
$$

$$
\begin{aligned}
G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^P(X)) &= G^U(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^P(X))\vee(\bigvee_{k=1}^{m}F^{-1}(H_{A_k}^{[0,1)}(BN^P(X))))\\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \wedge \sim\\
&\Big[\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \vee \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T\Big]\\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \wedge \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T\\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\end{aligned}
$$

3. Matrix-based dynamic algorithms for updating approximations while neighborhood classes decreasing and increasing

Based on Theorem 9 and 10, we propose a relative matrix based algorithm for updating approximations in NMGRS while neighborhood classes are decreasing. Algorithm 2 is a relative relation matrix based algorithm for updating the approximations of NMGRS while neighborhood classes are decreasing. Suppose that $N_{BN^P(X)} = \max_{k=1}^{|AT^t|}\left\{|R_{A_k}(BN^P(X))|\right\}$, the total time complexity of Algorithm 2 is $O(|AT^t|N^2_{BN^P(X)})$. Steps 4-12 are to calculate $R_{A_k}(BN^P(X))$ ($k \in \{1, 2, \ldots, m\}$) whose time complexity is $O(|AT^t||X|| \sim X|)$. Steps 13-20 are to compute MX_{A_k} ($k \in \{1, 2, \ldots, m\}$) whose time complexity is $O(|AT^t|R(BN^P(X))^2)$. Steps 21-25 are to compute $H_{A_k}(BN^P(X))$ whose time complexity is $O(|AT^t|N^2_{BN^P(X)})$. Steps 27-29 are to compute $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^P(X)$ whose time complexity is $O(|AT^t||U|)$. steps 24-27 are to update the approximations of NMGRS while neighborhood classes are decreasing whose time complexity is $O(|AT^t||U|)$.

Since the time complexity of Algorithm 1 is $O(|AT^t|N_X^2)$, and in general, $O(|AT^t|N^2_{BN^P(X)}) \leq O(|AT^t|N_X^2)$ is not hold, Algorithm 3 was proposed to make sure the total time complexity is no more than $O(|AT^t|N_X^2)$, in other words, when $N_{BN^P(X)} \leq N_X$ we call Algorithm 2, otherwise, we call Algorithm 1.

Algorithm 2 Matrix-based algorithm for computing approximations in NM-GRS

Require (1)Let $NIS^t = (U, AT^t, \Delta)(2)NIS^{t+1} = (U, AT^{t+1}, \delta)$(3)A target concept $X \subseteq U$ (4)$\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{O}(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{O}(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)$.(4)neighborhood classes $n_{A_k^C \cup A_k^N}^{t+1}(x), \forall x \in U, \forall k \in \{1, 2, \ldots, m\}$.

Ensure: $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{O}(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{P}(X)$.

$m \longleftarrow |AT^t|$
$n \longleftarrow |U|$
$BN^P(X) \longleftarrow \overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)$
for $k = 1 \to m$ **do**
  for $i = 1 \to n$ **do**
    for $j = 1 \to n$ **do**
      if $g_i^U(BN^P(X)) = 1 \wedge g_j^U(n_{A_1^C \cup A_1^N}(x_i)) = 1$ **then**
$g_i^U(R_k(BN^P(X))) = 1$
      end if
    end for
  end for
  $V \longleftarrow f^{-1}(R_k(BN^P(X)))$
  $r_k \longleftarrow |V^k|$
  for $i = 1 \to r_k$ **do**
    for $j = 1 \to r_k$ **do**
      if $(y_i, y_j) \in R_k = 1$ **then** $m_{ij}^{A_k} = 1$
      end if
    end for
  end for
  for $i = 1 \to r_k$ **do**
    $g_i^{V^k}(\Lambda_{A_k}) \longleftarrow \sum_{j=1}^{r} m_{ij}^{A_k}$
  end for
  $G^{V^k}(BN^P(X)) \longleftarrow F^{-1}(G^U(X))$
  $H_{A_k}(BN^P(X)) \longleftarrow diag(\Lambda_{A_k})) \cdot (M_{A_k} \cdot G^{V^k}(X))$
end for
$G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{O}}(X)) \longleftarrow G^U(\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{O}(X)) \vee (\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[1,1]}(BN^P(X))))$
$G^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{O}}(X)) \longleftarrow G^U(\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{O}(X)) \wedge \sim (\bigvee_{k=1}^{m} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X))))$
$G^U(\underline{\sum_{k=1}^{m} \delta_{A_k}^{P}}(X)) \longleftarrow G^U(\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)) \vee (\bigwedge_{k=1}^{m} F^{-1}(H_{A_k}^{[1,1]}(BN^P(X))))$
$G^U(\overline{\sum_{k=1}^{m} \delta_{A_k}^{P}}(X)) \longleftarrow G^U(\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t}}^{P}(X)) \wedge \sim (\bigwedge_{k=1}^{m} F^{-1}(H_{A_k}^{[0,0]}(BN^P(X))))$
Return $\overline{\sum_{k=1}^{m} \delta_{A_k}^{O}}(X)$, $\overline{\sum_{k=1}^{m} \delta_{A_k}^{P}}(X)$, $\underline{\sum_{k=1}^{m} \delta_{A_k}^{O}}(X)$ and $\underline{\sum_{k=1}^{m} \delta_{A_k}^{P}}(X)$

Algorithm 3 Ensure total time complexity of updating approximations in NMGRS while increasing granular structures no more than $O(|AT^t|R(X)^2)$

Require (1)Let $NIS^t = (U, AT^t, \Delta)$ (2)$NIS^{t+1} = (U, AT^{t+1}, \delta)$(3) A target concept $X \subseteq U$ (4)$\underline{\sum_{k=1}^m \Delta_{A_k}^t}^O(X)$, $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^O(X)$, $\underline{\sum_{k=1}^m \Delta_{A_k}^t}^P(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^P(X)$.(4)neighborhood classes $n^{t+1}_{A_k^C \cup A_k^N}(x), \forall x \in U, \forall k \in \{1, 2, \ldots, m\}$.

Ensure: $\underline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$.

if $N_{BN^P(X)} \leq N_X$ **then** Call Algorithm 3
end if
if $N_X \leq N_{BN^P(X)}$ **then** Call Algorithm 2
end if
Return $\underline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$.

Based on Theorem 14 and Theorem 15, we proposed a relative matrix based algorithm for updating approximations in NMGRS while neighborhood classes are increasing. Algorithm 3 is a relative matrix based algorithm for updating the approximations of NMGRS while neighborhood classes are increasing. Suppose that $N_{BN^O(X)} = \max_{k=1}^{|AT^t|} \{|R_{A_k}(BN^O(X))|\}$, the total time complexity of Algorithm 2 is $O(|AT^t|R(X)^2)$. Steps 5-12 are to calculate $R_{A_k}(BN^O(X))$ whose time complexity is $O(|X|| \sim X|)$. Steps 13-20 are to compute $MX^{A_k}(k \in \{1, 2, \ldots, m\})$ whose time complexity is $O(|AT^t|N^2_{BN^O(X)})$. Steps 21-25 are to compute $H_{A_k}(BN^O(X))$ whose time complexity is $O(|AT^t|N^2_{BN^O(X)})$.

Steps 27-29 are to update approximations in NMGTS whose time complexity is $O(|AT^t||U|)$. Since the time complexity of Algorithm 1 is $O(|AT^t|N_X^2)$, and In general, $O(|AT^t|N^2_{BN^O(X)}) \leq O(|AT^t|N_X^2)$ is not hold, Algorithm 5 was proposed to make sure the total time complexity is no more than $O(|AT^t|N_X^2)$, in other words, when $N_{BN^O(X)} \leq N_X$ we call Algorithm 4, otherwise, we call Algorithm 1.

4. Experimental evaluations

In this section, several experiments were conducted to evaluate effectiveness and efficiency of the algorithms that we devised. In order to verify the effectiveness and efficiency of Algorithm 3 and Algorithm 5, six data sets have been chosen from UCI machine learning repository to perform tests in the experiments. The details of data sets is listed in Table 2. All the experiments were carried out

Algorithm 4 Matrix-based algorithm for computing approximations in NM-GRS

Require (1)Let $NIS^t = (U, AT^t, \delta)(2)NIS^{t+1} = (U, AT^{t+1}, \Delta)$(3)A target concept $X \subseteq U$ (4)$\underline{\sum_{k=1}^m \delta^t_{A_k}}^O(X)$, $\overline{\sum_{k=1}^m \delta^t_{A_k}}^O(X)$, $\underline{\sum_{k=1}^m \delta^t_{A_k}}^P(X)$ and $\overline{\sum_{k=1}^m \delta^t_{A_k}}^P(X)$.(4)neighborhood classes $n^{t+1}_{A_k^C \cup A_k^N}(x), \forall x \in U, \forall k \in \{1, 2, \ldots, m\}$.

Ensure: $\underline{\sum_{k=1}^m \Delta^{t+1}_{A_k}}^O(X)$, $\overline{\sum_{k=1}^m \Delta^{t+1}_{A_k}}^O(X)$, $\underline{\sum_{k=1}^m \Delta^{t+1}_{A_k}}^P(X)$ and $\overline{\sum_{k=1}^m \Delta^{t+1}_{A_k}}^P(X)$.

$m \longleftarrow |AT^t|$

$n \longleftarrow |U|$

$BN^O(X) \longleftarrow \underline{\sum_{k=1}^m \delta^t_{A_k}}^O(X) \cup (U - \overline{\sum_{k=1}^m \delta^t_{A_k}}^O(X)$

for $k = 1 \rightarrow m$ **do**

  for $i = 1 \rightarrow n$ **do**

    for $j = 1 \rightarrow n$ **do**

      if $g_i^U(BN^O(X)) = 1 \wedge g_j^U(n_{A_1^C \cup A_1^N}(x_i)) = 1$ **then**

$g_i^U(R_k(BN^O(X))) = 1$

      end if

    end for

  end for

  $V^k \longleftarrow f^{-1}(R_k(BN^O(X)))$

  $r_k \longleftarrow |V^k|$

  for $i = 1 \rightarrow r_k$ **do**

    for $j = 1 \rightarrow r_k$ **do**

      if $(y_i, y_j) \in R_k = 1$ **then** $m_{ij}^{A_k} = 1$

      end if

    end for

  end for

  for $i = 1 \rightarrow r_k$ **do**

    $g_i^{V^k}(\Lambda_{A_k}) \longleftarrow \sum_{j=1}^r m_{ij}^{A_k})$

  end for

  $G^{V^k}(X) \longleftarrow F^{-1}(G^U(X))$

  $H_{A_k}(BN^O(X) \longleftarrow diag(\Lambda_{A_k}) \cdot (M_{A_k} \cdot G^{V^k}(X))$

end for

$G^U(\underline{\sum_{k=1}^m \Delta^O_{A_k}}(X)) \longleftarrow G^U(\underline{\sum_{k=1}^m \delta^t_{A_k}}^O(X)) \wedge \sim (\bigwedge_{k=1}^m F^{-1}(H^{(0,1)}_{A_k}(BN^O(X))))$

$G^U(\overline{\sum_{k=1}^m \Delta^O_{A_k}}(X)) \longleftarrow G^U(\overline{\sum_{k=1}^m \delta^t_{A_k}}^O(X)) \vee (\bigwedge_{k=1}^m F^{-1}(H^{(0,1]}_{A_k}(BN^O(X))))$

$G^U(\underline{\sum_{k=1}^m \Delta^P_{A_k}}(X)) \longleftarrow G^U(\underline{\sum_{k=1}^m \delta^t_{A_k}}^P(X)) \wedge \sim (\bigvee_{k=1}^m F^{-1}(H^{(0,1)}_{A_k}(BN^O(X))))$

$G^U(\overline{\sum_{k=1}^m \Delta^P_{A_k}}(X)) \longleftarrow G^U(\overline{\sum_{k=1}^m \delta^t_{A_k}}^P(X)) \vee (\bigvee_{k=1}^m F^{-1}(H^{(0,1]}_{A_k}(BN^O(X))))$

Return $\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)$, $\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)$, $\underline{\sum_{k=1}^m \delta^O_{A_k}}(X)$ and $\underline{\sum_{k=1}^m \delta^P_{A_k}}(X)$

Algorithm 5 Ensure total time complexity of updating approximations in NMGRS while increasing granular structures no more than $O(|AT^t|R(X)^2)$

Require (1)Let $NIS^t = (U, AT^t, \delta)(2)NIS^{t+1} = (U, AT^{t+1}, \Delta)$(3)A target concept $X \subseteq U$ (4)$\underline{\sum_{k=1}^{m} \delta_{A_k}^{t}}^{O}(X)$, $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t}}^{O}(X)$, $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t}}^{P}(X)$.(4)neighborhood classes $n^{t+1}_{A_k^C \cup A_k^N}(x), \forall x \in U, \forall k \in \{1, 2, \dots, m\}$.

Ensure: $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{O}(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{P}(X)$.

if $N_{BN^O(X)} \leq N_X$ **then** Call Algorithm 4
end if
if $N_X \leq N_{BN^O(X)}$ **then** Call Algorithm 1
end if
Return $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{O}(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^{P}(X)$.

Table 2: Details of data sets

No.	Data sets	Samples	Attributes
1	Cylinder Bands Data Set	541	40
2	Dermatology	366	20
3	Flags	194	30
4	LasVegas Trip Advisor Reviews Data set	504	20
5	Soybean Large	307	36
6	Student	396	33

on a personal computer with 64-bit windows 10, Inter(R) Core(TM) i7 6700HQ CPU @2.60GHz, and 8GB memory. The program lauguage is Matlab r2015b.

4.1 The comparison of computational time with different size of data sets

The computational time were compared between static and incremental algorithms in NMGRS when the size of data increases. While neighborhood classes are decreasing and increasing, first of all, we made $\delta = 1, \Delta = 1$ and we randomly chose the categorical and numeric attributes $\widehat{a_C}$ and $\widehat{a_N}$ in the data set and divided the rest of the categorical and numeric attributes into three parts randomly respectively to contribute to three granular structures. While neighborhood classes decreasing, we added the categorical and numeric attributes $\widehat{a_C}$ and $\widehat{a_N}$ into the three granular structures at the same time. While neighborhood classes increasing, we combined the categorical and numeric attributes $\widehat{a_C}$

and $\widehat{a_N}$ with each granular structure and deleted the categorical and numeric attributes $\widehat{a_C}$ and $\widehat{a_N}$ from the three granular structures at the same time. We randomly divided every data set U into ten subsets which is denoted by $\{U_1, U_2, \ldots, U_{10}\}$. Then U_1 was chosen as the first temporary data set. After that, some samples of temporary data set was randomly selected to contribute to the target concept X, the size of target concept X was about 0.2 times the size of the each temporary data set. We calculated the four approximations in NMGRS by static and incremental algorithms ten times and compared the averages. Then we made $U_1 \cup U_2$ the second temporary data set and repeat the whole process.

When the size of data sets increases, Experimental results of the static and the incremental while neighborhood classes are decreasing and increasing in NMGRS are shown in Figure 1 and Figure 2, by comparing the computational time of the static and the incremental algorithms in Figure 1 and Figure 2, we have that the computational time of the incremental algorithms is less than or equal to the static algorithm in most situations of these experiments. Sometimes the computational time of the incremental algorithms was a little more than the static algorithm while deleting attributes, which is due to additional computation in the incremental algorithms and it is within the expected range. According to Figure 1 and Figure 2 we can observe that the incremental algorithms for updating approximations while neighborhood classes are decreasing and increasing are more efficient than the static algorithm in most of the situations.

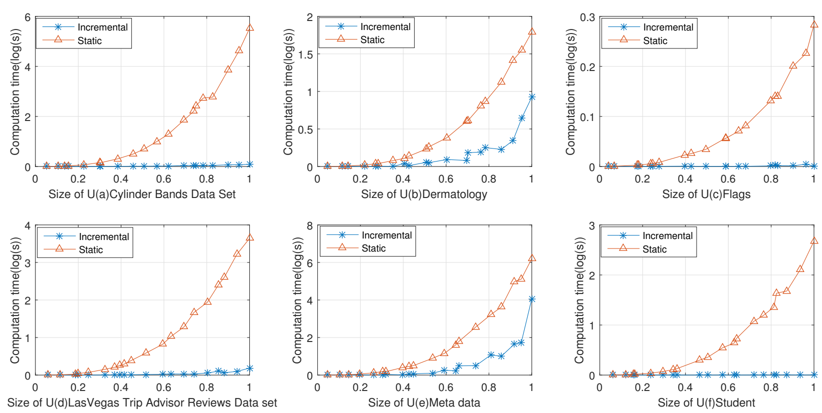

Figure 1: Computational time of Algorithm 3 when the size of U increasing gradually

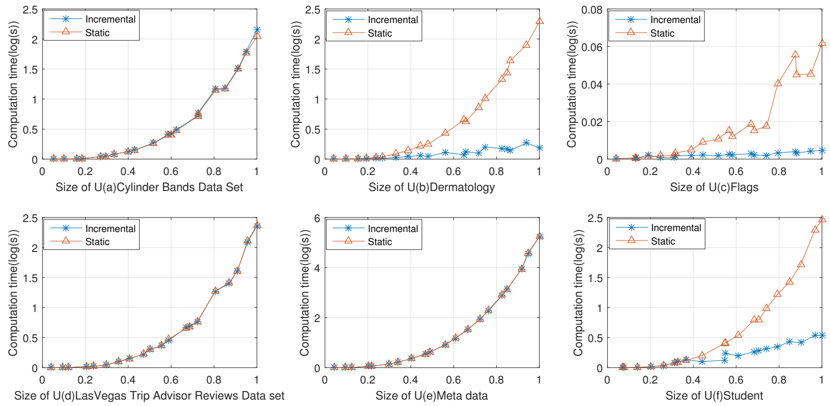

Figure 2: Computational time of Algorithm 5 when the size of U increasing gradually

4.2 The comparison of computational time with different size of target concept

The computational time are compared between the static and the incremental algorithms in NMGRS when the size of target concept increases. While neighborhood classes are decreasing and increasing, the process of setting parameters and constructing granular structures is similar to subsection 5.1. We randomly divided every data set U into ten subsets which is denoted by $\{U_1, U_2, \ldots, U_{10}\}$. And then U_1 was chosen as the first temporary target concept X. Finally, we calculated the four approximations in NMGRS by the static and the incremental algorithm ten times and compared the averages. We made $U_1 \cup U_2$ the second temporary target concept and repeat the whole process.

When the size of target concept increases, Experimental results of the static and the incremental while neighborhood classes decreasing and increasing in NMGRS are shown in Figure 3 and Figure 4, by comparing the computational time of the static and the incremental algorithms in Figure 3 and Figure 4, we can also get the same result that incremental methods are more efficient than static method. According to Figure 3 and Figure 4 we can observe that the incremental algorithms for updating approximations while neighborhood classes decreasing and increasing are more efficient than the static algorithm in most of the situations.

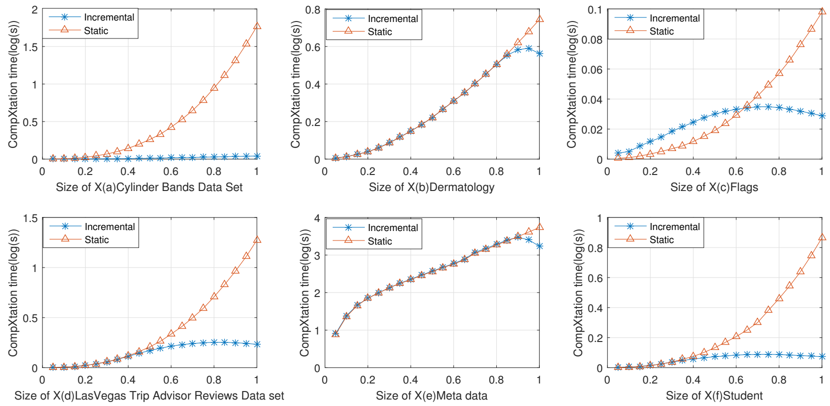

Figure 3: Computational time of Algorithm 3 when the size of X increasing gradually

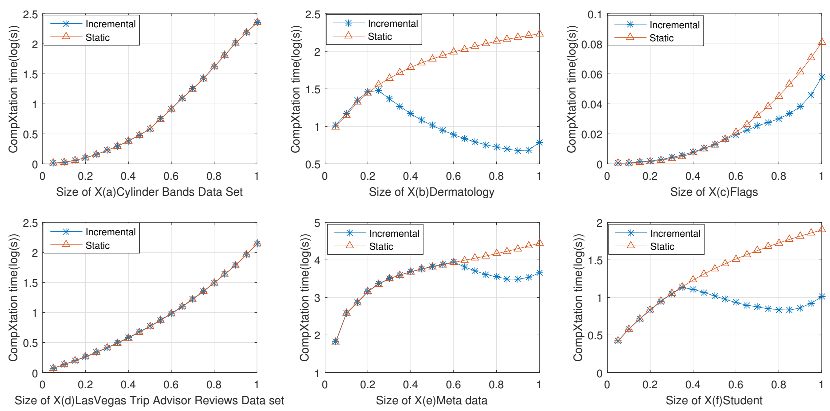

Figure 4: Computational time of Algorithm 5 when the size of X increasing gradually

5. Conclusion

Data sets in real life applications sometimes are complex and huge, i.e, it is difficult to use and granular structures often increase and decrease in some data sets. It is a significant issue to design algorithms to update approximation in NMGRS. In this paper, four algorithms have been proposed to ensure time complexity of incremental algorithms is less than static algorithm. Experimental results showed that the time complexity of the incremental algorithms is no more than the static algorithm in most of situations. Approximation computation is a basic process of attribute reduction, in the future, we will further investigate attribute reduction algorithm using approaches we proposed.

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