Properties of weakly 2-absorbing primal ideals

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Abstract. Let $R$ be a commutative ring with unity ($1 \neq 0$). In this paper we introduce the concept of weakly 2-absorbing primal ideal which is a generalization of weakly primal ideal. Let $I$ be a proper ideal of $R$. An element $a \in R$ is defined to be a weakly 2-absorbing prime to $I$ if for any $r, s, t \in R$ with $0 \neq rsta \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$. An element $a \in R$ is not a weakly 2-absorbing prime to $I$ if there exist $r, s, t \in R$, with $0 \neq rsta \in I$, such that $rs, rt, st \in R \setminus I$. We denote by $\nu_0(I)$ the set of all elements in $R$ that are not weakly 2-absorbing prime to $I$. We define a proper ideal $I$ of $R$ to be a weakly 2-absorbing primal if the set $\nu_0(I) \cup \{0\}$ forms an ideal of $R$. Many results concerning weakly 2-absorbing primal ideals and examples of weakly 2-absorbing primal ideals are given.

Keywords: weakly 2-absorbing ideal, weakly 2-absorbing primal ideal.

1. Introduction

We study in this paper weakly 2-absorbing primal ideals in commutative rings with unity, which are generalization of weakly primal ideals. Ebrahimi Atani and Ahmad Darani gave a generalization of weakly primal ideals (see [3]). Recall from [3] that if $R$ is a commutative ring with unity and $I$ is a proper ideal of $R$, then $a \in R$ is a weakly prime to $I$ if $0 \neq ra \in I$, for some $r \in R$, then $r \in I$. Also recall from [3] that $a \in R$ is not a weakly prime to $I$ if there exists $r \in R \setminus I$ such that $0 \neq ra \in I$. In [3] Ebrahimi Atani defined $S_0(I)$ by the set of all elements $a$ in $R$ that are not weakly prime to $I$. In [3] Ebrahimi Atani defined $I$ to be a weakly primal ideal in $R$ if $S_0(I) \cup \{0\}$ forms an ideal in $R$.

The idea of a weakly 2-absorbing primal ideal of a given ring $R$ which is a generalization of a weakly primal ideal of $R$ is come from the concept of a weakly 2-absorbing ideal of $R$ which is a generalization of a weakly prime ideal of $R$. Recall that weakly 2-absorbing ideals, which are generalizations of weakly prime ideals, were introduced in [5]. Let $I$ be a proper ideal of $R$, in [5] A. Badawi and A. Y. Darani defined $I$ to be a weakly 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

In this paper, we define $a \in R$ to be a weakly 2-absorbing prime to a proper ideal $I$ of $R$, if $0 \neq rsta \in I$, where $r, s, t \in R$, then $rs \in I$ or $rt \in I$ or $st \in I$. Also we define $a \in R$ to be not a weakly 2-absorbing prime to a proper ideal $I$ of
Let $R$ be a commutative ring with unity and let $I$ be a proper ideal of $R$. An element $a \in R$ is a weakly 2-absorbing prime to $I$ if for any $r, s, t \in R$ with $0 \neq rsta \in I$, then $rs$ or $rt$ or $st$ is in $I$. An element $a \in R$ is not a weakly 2-absorbing prime to $I$ if there exist $r, s, t \in R$, with $0 \neq rsta \in I$, such that $rs$, $rt$ and $st$ are in $R \setminus I$. We denote by $\nu_0(I)$ the set of all elements in $R$ that are not weakly 2-absorbing prime to $I$.

It is clear that every weakly primal ideal of a ring $R$ is weakly 2-absorbing primal ideal of $R$. If $R = \mathbb{Z}_{16}$ and $I = \{0, 8\}$, then one can easily see that $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$ since $0 \neq 2.2.2 \in I$ and $4 \notin I$, so $I$ is a weakly 2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$. Also one can easily see that $S_0(I) \cup \{0\} = 2\mathbb{Z}_{16} \neq \nu_0(I) \cup \{0\}$. Therefore, $I = \{0, 8\}$ is a weakly primal and weakly 2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $S_0(I) \neq \nu_0(I)$. Also if $I = \{0\}$ is an ideal in $\mathbb{Z}_6$, then, from the definition of weakly 2-absorbing primal ideals, $I$ is a weakly 2-absorbing primal ideal of $\mathbb{Z}_6$ with $\nu_0(I) \cup \{0\} = \{0\}$, but easy computations implies that $I$ is not a 2-absorbing primal ideal of $\mathbb{Z}_6$. The following are two examples of nonzero weakly 2-absorbing primal ideals that are not weakly primal ideals.

**Example 2.1.** (1) Let $R = \mathbb{Z}$ and let $I = 30\mathbb{Z}$. Then $I$ is a weakly 2-absorbing primal ideal of $\mathbb{Z}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}$, since $(2)(3)(5) = 30 \in I$ and $(2)(3) = 6 \notin I$, $(2)(5) = 10 \notin I$ and $(3)(5) = 15 \notin I$. On the other hand $I$ is not a weakly primal ideal in $\mathbb{Z}$, because $2,3 \in S_0(I)$ but $1 \notin S_0(I)$. Note that if $1 \in S_0(I)$, then there exists $r \notin I$ with $1.r = r \in I$, a contradiction.

(2) Let $R = \mathbb{Z}[x,y,z]$ and let $I = xyzR$. Then $I$ is a proper ideal of $R$ and since $0 \neq xyz \in I$ with $xy$, $xz$, and $yz$ are in $R \setminus I$ we get that $\nu_0(I) \cup \{0\} = R$. Therefore, $I$ is a weakly 2-absorbing primal ideal of $R$.

On the other hand, since $0 \neq xyz \in I$ and $yz \in R \setminus I$ we get that $x \in S_0(I)$. Similarly, $y \in S_0(I)$. Now we show that $x + y$ can’t be in $S_0(I)$. If there exists $f(x,y,z) \in \mathbb{Z}[x,y,z]$ with $0 \neq (x+y)f(x,y,z) \in I$, then $xyz$ divides $(x+y)f(x,y,z)$ and since $x$ divides $xyz$ we get that $x$ divides $(x+y)f(x,y,z)$ but $x$ does not divide $x+y$, so $x$ must divide $f(x,y,z)$. Similarly, $y$ divides...
Let $f(x, y, z)$ and $z$ divides $f(x, y, z)$. Therefore, $xyz$ divides $f(x, y, z)$ which implies that $f(x, y, z) \in I$, so $x + y \not\in S_0(I)$ and hence $I$ is not a weakly primal ideal of $R$.

Now, we have the following easy result.

**Lemma 2.1.** Every weakly primal ideal of $R$ is a weakly 2-absorbing primal.

**Theorem 2.1.** Let $I$ be a proper ideal of $R$ such that $I$ is a weakly 2-absorbing primal ideal of $R$ with $v_0(I) \cup \{0\} \neq R$. Then $v_0(I) \cup \{0\}$ is a weakly prime ideal of $R$.

**Proof of Theorem 2.1.** Let $a, b \in R$ such that $0 \neq ab \in v_0(I)$. Then there exist $r, s, t \in R$ with $0 \neq rst(ab) \in I$ such that $rs, rt, st \in R \setminus I$. Assume that $a \not\in v_0(I)$. We must show that $b \in v_0(I)$. Since $0 \neq r(sb)ta \in I$ and $a \not\in v_0(I)$, $0 \neq rsb \in I$ or $0 \neq rt \in I$ or $0 \neq sbl \in I$. But $rt \in R \setminus I$ implies $rsb \in I$ or $sbt \in I$. If $rsb \in I$, then $b \in v_0(I)$ since $r, s, rs \in R \setminus I$. Similarly, if $sbt \in I$, then $b \in v_0(I)$. Therefore, $v_0(I) \cup \{0\}$ is a weakly prime ideal of $R$.

For example $I = 4\mathbb{Z}$ is a proper ideal of $\mathbb{Z}$ with $v_0(I) \cup \{0\} = 2\mathbb{Z}$. So $I$ is a weakly 2-absorbing primal ideal of $\mathbb{Z}$ with $v_0(I) \cup \{0\} = 2\mathbb{Z}$ is a weakly prime ideal of $\mathbb{Z}$. But if $I = 6\mathbb{Z}$, then $I$ is not a weakly 2-absorbing primal ideal of $\mathbb{Z}$, since $0 \neq (2)(3) \in I$ and $2, 3 \not\in I$ so $2, 3 \in v_0(I)$, therefore, if $v_0(I) \cup \{0\}$ is an ideal of $\mathbb{Z}$, then $1 \in v_0(I)$ which implies that there exist $r, s, t \in \mathbb{Z} \setminus 6\mathbb{Z}$ such that $0 \neq rst \in 6\mathbb{Z}$ with $rs, rt, st \not\in 6\mathbb{Z}$, but since $6$ divides $rst$ we have that $2$ must divide $r$ or $s$ or $t$ and $3$ must divide $r$ or $s$ or $t$ so $6$ must divide $rs$ or $st$ or $rt$, a contradiction.

**Definition 2.1.** Let $R$ be a commutative ring with unity $(1 \neq 0)$ and let $I$ be a proper ideal of $R$ such that $I$ is a weakly 2-absorbing primal ideal of $R$. Let $r, s, t \in R$, then $(r, s, t)$ is called a triple zero of $I$ if $rst = 0$ with $rs, rt, st \in R \setminus I$.

The following five results on weakly 2-absorbing primal ideals over $R$ and the results on weakly 2-absorbing ideals of $R$ are the same and proved in [5] with another approach.

**Theorem 2.2.** Let $R$ be a commutative ring with unity $(1 \neq 0)$ and let $I$ be a proper ideal of $R$. Suppose that $I$ is a weakly 2-absorbing primal ideal of $R$ with $1 \not\in v_0(I)$. If $(r, s, t)$ is a triple zero of $I$, then:

1. $rsI = rtI = stI = 0$;
2. $rI^2 = sI^2 = tI^2 = 0$.

**Proof of Theorem 2.2.** (1) If $rsI \neq 0$, then there exists $a \in I$ such that $0 \neq rsa \in I$. So, $0 \neq rs(t + a) = rsa \in I$ with $rs, r(t + a), s(t + a) \in R \setminus I$ implies that $1 \not\in v_0(I)$, a contradiction. Therefore, $rsI = 0$. Similarly, $rtI = 0$ and $stI = 0$.

(2) Suppose $rtI \neq 0$, then there exist $a, b \in I$ such that $rab \neq 0$. So, $0 \neq (s + a)(t + b) = ab \in I$ with $r(s + a), r(t + b), (s + a)(t + b) \in R \setminus I$ implies...
that \(1 \in \nu_0(I)\), a contradiction. Therefore, \(rI^2 = 0\). Similarly, \(sI^2 = 0\) and \(tI^2 = 0\).

Let \(I\) be a proper ideal of \(R\) such that \(I\) is a weakly 2-absorbing primal ideal of \(R\) with \(1 \notin \nu_0(I)\). If \(I\) is a 2-absorbing primal ideal of \(R\) with \(\nu(I) = R\) then, by using Theorem 2.2, we have the following result.

**Theorem 2.3.** Let \(R\) be a commutative ring with unity \((1 \neq 0)\) and let \(I\) be a proper ideal of \(R\). Suppose that \(I\) is a weakly 2-absorbing primal ideal of \(R\) with \(1 \notin \nu_0(I)\) such that \(\nu(I) = R\), then \(I^3 = 0\).

**Proof of Theorem 2.3.** Since \(\nu(I) = R\) we have that \(1 \in \nu(I)\), and hence there exist \(r, s, t \in R\) with \(rst = 0\) such that \(rs, rt, st \in R \setminus I\). Thus, \((r, s, t)\) is a triple zero of \(I\), since if \(0 \neq rst \in I\), then \(1 \in \nu_0(I)\), a contradiction. Suppose that \(I^3 \neq 0\), then there exist \(a, b, c \in I\) such that \(0 \neq abc \in I\). So by Theorem 2.2 we have \(0 \neq (r+a)(s+b)(t+c) = abc \in I\) and since \(1 \notin \nu_0(I)\) we must have that \((r+a)(s+b) \in I\) or \((r+a)(t+c) \in I\) or \((s+b)(t+c) \in I\) and hence we have either \(rs \in I\) or \(rt \in I\) or \(st \in I\), a contradiction. Hence \(I^3 = 0\).

Let \(R\) be a commutative ring with unity \((1 \neq 0)\). Then \(\text{Nil}(R)\) denotes the ideal of nilpotent elements of \(R\).

**Corollary 2.1.** Let \(R\) be a commutative ring with unity \((1 \neq 0)\) and let \(I\) be a proper ideal of \(R\). Suppose that \(I\) is a weakly 2-absorbing primal ideal of \(R\) with \(1 \notin \nu_0(I)\). If \(\nu(I) = R\), then \(I \subseteq \text{Nil}(R)\).

**Theorem 2.4.** Let \(I\) be a proper ideal of \(R\). Suppose that \(I\) is a weakly 2-absorbing primal ideal of \(R\) with \(1 \notin \nu_0(I)\) such that \(\nu(I) = R\). Then:

1. If \(a \in \text{Nil}(R)\), then either \(a^2 \in I\) or \(a^2I = aI^2 = 0\);
2. \((\text{Nil}(R))^2I^2 = 0\).

**Proof of Theorem 2.4.** (1) Let \(a \in \text{Nil}(R)\). First, we show that if \(a^2I \neq 0\), then \(a^2 \in I\). Now assume that \(a^2I \neq 0\). Let \(i \in I\) such that \(0 \neq a^2i \in I\) and suppose that \(n\) is the smallest positive integer such that \(a^n = 0\). Then \(n \geq 3\) and we have \(0 \neq a^2(i + a^{n-2}) \in I\), since \(1 \notin \nu_0(I)\) we must have \(a^2 \in I\) or \(a^{n-1} \in I\). If \(a^2 \in I\), then we are done. If \(a^{n-1} \in I\), then \(0 \neq a^2a^{n-3} \in I\) again since \(1 \notin \nu_0(I)\) we have \(a^{n-2} \in I\). Continuing this procedure, then we arrive at \(a^2 \in I\). Therefore for each \(a \in \text{Nil}(R)\) we have either \(a^2 \in I\) or \(a^2I = 0\). Now, assume that \(b^2 \notin I\) for some \(b \in \text{Nil}(R)\), then \(b^2I = 0\). We show that \(bI^2 = 0\). If \(bI^2 \neq 0\), then there exist \(i_1, i_2 \in I\) such that \(bi_1i_2 \neq 0\). Let \(m\) be the smallest positive integer such that \(b^m = 0\), then \(m \geq 3\) since \(b^2 \notin I\). Hence \(0 \neq b(b+ i_1)(b^{m-2} + i_2) = bi_1i_2 \in I\) and since \(1 \notin \nu_0(I)\) we have \(b(b+ i_1) \in I\) which implies that \(b^2 \in I\) (a contradiction) or \(b(b^{m-2} + i_2) \in I\) which implies that \(b^{m-1} \in I\) (a contradiction). Therefore, \(bI^2 = 0\).

(2) Let \(r, s \in \text{Nil}(R)\). If \(r^2 \notin I\) or \(s^2 \notin I\), then, by (1), \((rs)^2 = 0\). Therefore we may assume that \(r^2 \in I\) and \(s^2 \in I\). So, \((rs)(r + s) \in I\). If \((r, s, r + s)\) is a triple zero of \(I\), then, by Theorem 2.2(1), \((rs)I = 0\) and hence \((rs)^2 = 0\).
If \( 0 \neq rs(r + s) \in I \), then \( rs \in I \) since \( 1 \notin \nu_0(I) \). So, by Theorem 2.3, 
\((rs)I^2 \subseteq I^2 = 0\).

\[ \text{Corollary 2.2.} \] Let \( R \) be a commutative ring with unity \((1 \neq 0)\) and let \( A, B, C \) be proper ideals of \( R \). Suppose that \( A, B, C \) are weakly 2-absorbing primal ideals of \( R \) with \( 1 \notin \nu_0(A) \cup \nu_0(B) \cup \nu_0(C) \) such that \( \nu(A) = \nu(B) = \nu(C) = R \). Then, 
\( A^2BC = AB^2C = ABC^2 = A^2B^2 = A^2C^2 = B^2C^2 = 0. \)

\[ \text{Proof of Corollary 2.2.} \] By Corollary 2.1, \( B \subseteq \text{Nil}(R) \) and \( C \subseteq \text{Nil}(R) \), so \( A^2BC \subseteq A^2(\text{Nil}(R))^2 \) and by Theorem 2.4(2), \( A^2(\text{Nil}(R))^2 = 0 \). Similarly, \( AB^2C = ABC^2 = 0 \). Also, by Corollary 2.1, \( A^2B^2 \subseteq (\text{Nil}(R))^2B^2 = 0 \). Similarly, \( A^2C^2 = B^2C^2 = 0. \)

In the next result we give a condition on a weakly 2-absorbing primal ideal of \( R \) to be a 2-absorbing primal ideal of \( R \).

**Theorem 2.5.** Let \( R \) be a commutative ring with unity \((1 \neq 0)\) and let \( I \) be a proper ideal of \( R \). If \( I \) is a weakly 2-absorbing primal ideal of \( R \) with \( I^2 \neq 0 \), then \( I \) is a 2-absorbing primal ideal of \( R \).

**Proof of Theorem 2.5.** If \( 1 \in \nu(I) \), then \( \nu(I) = R \) which implies that \( I \) is a 2-absorbing primal ideal of \( R \). Therefore we may assume that \( 1 \notin \nu(I) \). To prove that \( I \) is a 2-absorbing primal ideal of \( R \) we must show that \( \nu(I) = \nu_0(I) \cup \{0\} \). It is clear that \( \nu_0(I) \cup \{0\} \subseteq \nu(I) \). Conversely, let \( a \in \nu(I) \), then there exist \( r, s, t \in R \) with \( rs, rt, st \in R \setminus I \) such that \((rst)a \in I \). If \((rst)a \neq 0\), then \( a \in \nu_0(I) \). So we may assume that \( rsta = 0 \). If \((rst)I \neq 0 \), then there exists \( c \in I \) such that \((rst)c \neq 0 \). Therefore, \( 0 \neq (rst)(a + c) \in I \) which implies that \( a + c \in \nu_0(I) \) and hence \( a \in \nu_0(I) \), since \( c \in \nu_0(I) \). Therefore we may assume that \((rst)I = 0 \). If \((rst)I \in I \), then \( 1 \in \nu(I) \) which is a contradiction. Therefore we may assume that \((rst)I \notin I \). Since \( I^2 \neq 0 \), there exist \( x, y \in I \) such that \( xy \neq 0 \). Hence, \((a + y)(trs + x) = 0 + ax + 0 + xy \in I \). If \( 0 \neq ax + xy \in I \) and since \( trs + x \notin I \), then \( a + y \in \nu_0(I) \) which implies that \( a \in \nu_0(I) \), since \( y \in \nu_0(I) \). But, if \( ax + xy = 0 \), then \( 0 \neq ax \in I \) which implies that \( 0 \neq a(x + trs) = ax \in I \), so \( a \in \nu_0(I) \). Thus, \( \nu(I) = \nu_0(I) \cup \{0\} \).

We have to remark that if a proper ideal \( I \) of \( R \) is a weakly 2-absorbing 
primal ideal of \( R \) with \( I^2 \neq 0 \), and \( 1 \notin \nu(I) \), then \( \nu_0(I) \cup \{0\} \) is a prime ideal of \( R \) since by Theorem 2.5, \( \nu(I) = \nu_0(I) \cup \{0\} \).

We recall that if \( R \), and \( S \) are commutative rings with unity and \( P, Q \) are weakly prime ideals in \( R \), \( S \) (respectively), then \( P \times S \) and \( R \times Q \) are weakly prime ideals of \( R \times S \).

**Theorem 2.6.** Let \( R \times S \) be a commutative ring with unity, where \( R, S \) are commutative rings with units, let \( I \) be a proper ideal of \( R \) with \( I \times S \notin \text{Nil}(R \times S) \). Then, the following statements are equivalent.

1. \( I \times S \) is a weakly 2-absorbing primal ideal of \( R \times S \);
2. \( I \times S \) is a 2-absorbing primal ideal of \( R \times S \);
3. \( I \) is a 2-absorbing primal ideal of \( R \).
Proof of Theorem 2.6. (1 → 2). Since \( I \times S \not\subseteq \text{Nil}(R \times S) \), by Corollary 2.1 we have that \( \nu(I \times S) \neq R \times S \). To prove that \( I \times S \) is a 2-absorbing primal ideal of \( R \times S \) we must show that \( \nu(I) = \nu_0(I) \cup I \). Let \( a \in \nu(I) \) and let \((rst)a = 0 \) for some \( r, s, t \in R \) with \( rs, rt, st \in R \setminus I \). Since \( 1 \notin \nu(I) \) and \( rs \notin I \) we must have \( rta \in I \) or \( sta \in I \). If \( rta \in I \), then \( ra \in I \) or \( ta \in I \) since \( 1 \notin \nu(I) \) and \( rt \notin I \). If \( ra \in I \), then \( a \in I \) since \( r \notin I \) and \( 1 \notin \nu(I) \), similarly, if \( ta \in I \), then \( a \in I \). Similarly, if \( sta \in I \), then \( a \in I \). Therefore, \( \nu(I) = \nu_0(I) \cup I \) and hence \( \nu(I \times S) = \nu(I) \times S \) is an ideal of \( R \times S \) which implies that \( I \times S \) is a 2-absorbing primal ideal of \( R \times S \).

(2 → 3). Since \( \nu(I \times S) = \nu(I) \times S \) is a prime ideal of \( R \times S \) we have that \( \nu(I) \) is a prime ideal of \( R \), so \( I \) is a 2-absorbing primal ideal of \( R \).

(3 → 1). Since \( I \) is a 2-absorbing primal ideal of \( R \) we have that \( I \times S \) is a 2-absorbing primal ideal of \( R \times S \), hence \( I \times S \) is a weakly 2-absorbing primal ideal of \( R \times S \).

□

Theorem 2.7. Let \( R \times S \) be a commutative ring with unity, where \( R, S \) are commutative rings with unities, let \( I \) be a nonzero proper ideal of \( R \) and \( J \) a nonzero ideal of \( S \) with \( I \times J \not\subseteq \text{Nil}(R \times S) \). Then, the following statements are equivalent.

1. \( I \times J \) is a weakly 2-absorbing primal ideal of \( R \times S \);
2. \( J = S \) and \( I \) is a 2-absorbing primal ideal of \( R \);
3. \( I \times J \) is a 2-absorbing primal ideal of \( R \times S \).

Proof of Theorem 2.7. (1 → 2). Suppose \( I \times J \) is a weakly 2-absorbing primal ideal of \( R \times S \). Since \( I \times J \not\subseteq \text{Nil}(R \times S) \) and if \( J = S \), then \( I \) is a 2-absorbing primal ideal of \( R \) by Theorem 2.6. We show that the case \( J \neq S \) cannot be happened. Suppose \( J \neq S \), we show that \( J \) is a prime ideal in \( S \) and \( I \) is a prime ideal of \( R \). Since \( I \times J \not\subseteq \text{Nil}(R \times S) \), by Corollary 2.1 we have that \( \nu(I \times J) \neq R \times S \). Let \( a, b \in S \) such that \( ab \in J \) and let \( 0 \neq i \in I \). Then, \((i, 1)(1, a)(b, 1) = (i, ab) \neq (0, 0) \) in \( I \times J \) since \((1, ab) \notin I \times J \) and since \((1, 1) \notin \nu_0(I \times J) \) we must have \((i, a) \in I \times J \) or \((i, b) \in I \times J \) so \( a \in J \) or \( b \in J \). Thus \( J \) is a prime ideal of \( S \). Similarly, let \( c, d \in R \) such that \( cd \in I \), and let \( 0 \neq j \in J \). Then, \((c, 1)(d, 1)(1, j) = (cd, j) \neq (0, 0) \) in \( I \times J \), since \((cd, 1) \notin I \times J \) and since \((1, 1) \notin \nu_0(I \times J) \) we must have \((c, j) \in I \times J \) or \((d, j) \in I \times J \) so \( c \in I \) or \( d \in I \). Hence \( I \) is a prime ideal of \( R \). In this case we show that \((1, 1) \in \nu(I \times J) \). Now, \((1, 0)(0, 1) \in I \times J \) and \((1, 0) \notin I \times J \), \((0, 1) \notin I \times J \), so \((1, 0), (0, 1) \in \nu(I \times J) \). Therefore, if \( \nu(I \times J) \) is an ideal in \( R \times S \), then \((1, 1) = (1, 0) + (0, 1) \in \nu(I \times J) \), a contradiction to Corollary 2.1. Therefore the only case of part \( 2 \) is that \( J = S \) and \( I \) is a 2-absorbing primal ideal of \( R \).

(2 → 3). If \( J = S \) and \( I \) is a 2-absorbing primal ideal of \( R \), then \( I \times J \) is a 2-absorbing primal ideal of \( R \times S \) by Theorem 2.6(2).

(3 → 1). Clearly from Theorem 2.6

□

Theorem 2.8. Let \( R \times S \) be a commutative ring with unity, where each \( R \) and \( S \) are commutative rings with unities, let \( I = I_1 \times I_2 \) be a proper ideal of \( R \times S \).
If $I$ is a weakly 2-absorbing primal ideal of $R \times S$ with $\nu_0(I) \neq \emptyset$. Then, either $I = (0,0)$ or $I$ is a 2-absorbing primal ideal of $R \times S$.

**Proof of Theorem 2.8.** Suppose $I \neq (0,0)$. Since $\nu_0(I) \neq \emptyset$, by Theorem 2.1, $\nu_0(I) \cup \{(0,0)\}$ is a weakly prime ideal of $R \times S$. Therefore, $\nu_0(I) \cup \{(0,0)\}$ has one of the following cases:

1. $\nu_0(I) \cup \{(0,0)\} = P_1 \times P_2$, where each $P_i = \{0\}$. In this case $\nu_0(I) \cup \{(0,0)\} = (0,0)$, a contradiction.

2. $\nu_0(I) \cup \{(0,0)\} = P_1 \times S$, where $P_1$ is a weakly 2-absorbing prime ideal of $R$ and $S \neq 0$. $P_1$ is a 2-absorbing prime ideal of $R \times S$ in $\nu_0(I)$ or $\nu_0(I)$ is a prime ideal of $R \times S$. Now, we show that $I_2 = S$. Since $S \neq 0$, there exists $(a,b) \neq (0,0) \in I$ and $(0,0) \neq (a,b) = (1,1)(a,1)$. If $(a,1) \notin I$, then $(1,b) \in \nu_0(I)$ and hence $1 \in P_1$, a contradiction. Therefore, $(a,1) \in I$ which implies that $1 \in I_2$, hence $I_2 = S$. Now, we show that $I_1$ is a 2-absorbing primal ideal of $R$ with $\nu(I_1) = P_1$. Let $a_1 \in P_1$, then $(a_1,0) \in \nu_0(I)$. If $a_1 = 0$, then $a_1 \in I_1 \subseteq \nu_0(I_1)$, so we may assume that $a_1 \neq 0$. Thus, there exist $(r_1, r_2), (s_1, s_2), (t_1, t_2) \in R \times S$ with $(r_1 s_1, r_2 s_2)(t_1 t_2) \in R \times S \setminus I = (R \setminus I_1) \times S$ such that $(0,0) \neq (r_1 s_1 t_1, r_2 s_2 t_2)(a_1,0) \in I$, so $0 \neq r_1 s_1 t_1 a_1 \in I_1$ with $r_1 s_1, r_2 s_2, t_1 t_2 \notin I_1$ which implies that $a_1$ is not a weakly 2-absorbing prime to $I_1$, hence $a_1$ is not a 2-absorbing prime to $I_1$ and so $a_1 \in \nu(I_1)$. Conversely, let $a_1 \in \nu(I_1)$. Then, there exist $r,s,t \in R$ with $rs, rt, st \in R \setminus I_1$ such that $rs t a_1 \in I_1$. Since $S \neq 0$, we have $(0,0) \neq (rs t a_1,1) = (r,1)(s,1)(t,1)(a_1,1) \in I_1 \times S$ with $(rs,1), (rt,1), (st,1) \in (R \setminus I_1) \times S = (R \times S) \setminus I$ which implies that $(a_1,1)$ is not a weakly 2-absorbing prime to $I$, hence $(a_1,1) \in \nu_0(I) \cup \{(0,0)\} = P_1 \times S$, so $a_1 \in P_1$. Therefore, $\nu(I_1) = P_1$. Thus, $\nu(I) = P_1 \times S$. Hence $I = I_1 \times S$ is a 2-absorbing primal ideal of $R \times S$.

3. $\nu_0(I) \cup \{(0,0)\} = R \times P_2$, where $P_2$ is a weakly 2-absorbing primal ideal in $S$. By using the same approach in part 2 we conclude that $I = R \times I_2$ and $\nu(I_2) = P_2$, and so $\nu(I) = R \times P_2$. Hence $I = R \times I_2$ is a 2-absorbing primal ideal of $R \times S$. □

3. More properties of weakly 2-absorbing primal ideals

For a commutative ring $R$, let $J(R)$ denotes the intersection of all maximal ideals of $R$.

**Lemma 3.1.** Let $R$ be a commutative ring and $a, b \in J(R)$. Then the ideal $I = abR$, where $1 \notin \nu_0(I)$, is a weakly 2-absorbing primal ideal of $R$ if and only if $ab = 0$.

**Proof of Lemma 3.1.** If $ab = 0$, then $I = 0$ is a weakly 2-absorbing primal ideal of $R$ by definition. If $0 \neq ab \in I$ with $a, b \notin I$, then $1 \in \nu_0(I)$, contradiction. Therefore, $a$ or $b$ is in $I = abR$. If $a \in I$, then $a = abk$ for some $k \in R$. So, $a(1 - bk) = 0$ and since $bk \in J(R)$ we have that $1 - bk$ is a unit in $R$. Thus, $a(1 - bk) = 0$ implies that $a = 0$ and hence $ab = 0$, a contradiction. Therefore, $I = 0$. □
We recall that \( R \) is defined to be quasi-local ring if \( R \) has a unique maximal ideal. If \((R, M)\) is a quasi-local ring, where \( M \) is the unique maximal ideal of \( R \), then we have the following two results about a weakly 2-absorbing primal ideal \( I \) of \( R \) with \( 1 \not\in \nu_0(I) \).

**Theorem 3.1.** Let \((R, M)\) be a quasi-local ring with \( \nu_0(I) \neq R \), for all proper ideals \( I \) of \( R \). Then every proper ideal of \( R \) is a weakly 2-absorbing primal if and only if \( M^2 = 0 \).

**Proof of Theorem 3.1.** Let \( a, b \in M \), then \( I = abR \) is a weakly 2-absorbing primal ideal of \( R \) with \( 1 \not\in \nu_0(I) \), hence, by Lemma 3.1, \( M^2 = 0 \). Conversely, suppose \( M^2 = 0 \). Let \( I \) be a proper ideal of \( R \) and let \( a \in \nu_0(I) \). If \( a \) is a unit in \( R \), then \( 1 \in \nu_0(I) \), a contradiction. So we may assume that \( a \) is not a unit in \( R \). Let \( r, s, t, \in R \) with \( 0 \neq rst \in I \) such that \( rs, rt, st \not\in I \). If \( 0 \neq rst \in I \), then \( M^2 = 0 \) and \( a \in M \) implies \( rst \not\in M \). So \( rst \) is a unit in \( R \) which implies that \( a \in I \). So \( \nu_0(I) \cup \{0\} = I \) which implies that \( I \) is weakly 2-absorbing primal ideal of \( R \).

**Corollary 3.1.** Let \((R, M)\) be a quasi-local ring with \( M^2 = 0 \) and with \( \nu_0(I) \neq R \), for all proper ideals \( I \) of \( R \). Then every proper ideal of \( R \) is a 2-absorbing primal ideal of \( R \).

**Proof of Corollary 3.1.** Let \( I \) be a proper ideal of \( R \), then, by Theorem 3.1, \( I \) is a weakly 2-absorbing primal ideal of \( R \) since \( M^2 = 0 \). We show that \( \nu(I) \) is an ideal in \( R \). Let \( a, b \) be nonzero elements in \( \nu(I) \). Then, there exist \( r, s, t, \in R \) with \( rs, rt, st \in R \setminus I \) such that \( rsta \in I \). If \( 0 \neq rsta \in I \), then, by Theorem 3.1, \( a \in I \subseteq M \). Since \( rs \not\in I \) we have \( r \) or \( s \) is a unit in \( R \). Therefore, if \( rsta = 0 \), then \( (st)a = 0 \) or \( (rt)a = 0 \). Say \( (st)a = 0 \) again since \( st \not\in I \) we have \( s \) or \( t \) is a unit in \( R \) which implies that \( sa = 0 \) or \( ta = 0 \). Say \( ta = 0 \), hence \( t \) is not a unit in \( R \), since \( 0 \neq a \in I \). Therefore if \( ta = 0 \in I \subseteq M \) and \( a \) is not a unit in \( R \) (if \( a \) is a unit in \( R \), then \( t = 0 \) a contradiction), then \( a \) must be in \( M \) since \( M \) is a prime ideal. Similarly, \( b \in M \), so \( a + b \in M \). If \( t(a + b) \neq 0 \), then \( t \) is a unit in \( R \) since \( a + b \in M \), a contradiction. Therefore, \( t(a + b) = 0 \in I \) which implies that \( a + b \in \nu(I) \) since \( t \not\in I \). Hence \( \nu(I) \) is an ideal of \( R \).

In the next result we give the condition on a proper ideal \( I \) of \( R \) such that \( I/J \) is a weakly 2-absorbing primal ideal of \( R/J \) where \( J \) is a proper ideal of \( R \) contained in \( I \).

**Theorem 3.2.** Let \( I, J \) be proper ideals of \( R \) with \( J \subseteq I \). If \( I \) is a weakly 2-absorbing primal ideal of \( R \) with \( \nu_0(J) \subseteq I \). Then \( I/J \) is a weakly 2-absorbing primal ideal of \( R/J \).

**Proof of Theorem 3.2.** To prove that \( I/J \) is a weakly 2-absorbing primal ideal of \( R/J \), we must show that \( \nu_0(I/J) \cup \{0\} = [\nu_0(I) \cup J]/J \). Let \( a + J \in \nu_0(I/J) \), then there exist \( r + J, s + J, t + J \in R/J \) with \( 0 \neq rsta + J \in I/J \) such that \( rs + J, rt + J, st + J \not\in I/J \). So \( 0 \neq rsta \in I \), since \( rsta \not\in J \), with
Theorem 3.3. \( rs, rt, st \notin I \) hence \( a \in \nu_0(I) \), therefore, \( a + J \in [\nu_0(I) \cup J]/J \). Conversely, let \( 0 \neq a + J \in [\nu_0(I) \cup J]/J \), then \( a \in \nu_0(I) - J \). If \( a \in I \), then \( a + J \in \nu_0(I)/J \).

So we may assume that \( a \notin I \). Then, there exist \( r, s, t \in R \) with \( 0 \neq rsta \in I \) such that \( rs, rt, st \notin I \). If \( 0 \neq rsta \in J \), then \( a \in \nu_0(J) \), a contradiction, since \( \nu_0(J) \subseteq I \) and \( a \notin I \). Therefore, \( r + J, s + J, t + J \in R/J \) with \( 0 \neq rsta + J = (rsta + J)(a + J) \in I/J \) such that \( rsta + J, rt + J, st + J \notin I/J \), so \( a + J \in \nu_0(I)/J \). Hence \( \nu_0(I)/J \cup \{0\} = [\nu_0(I) \cup J]/J \) which implies that \( I/J \) is a weakly 2-absorbing primal ideal of \( R/J \).

Corollary 3.2. Let \( R_0 \) be a subring of \( R \) with unity. If \( I \) is a weakly 2-absorbing primal ideal of \( R \), then \( I \cap R_0 \) is a weakly 2-absorbing primal ideal of \( R_0 \).

Proof of Corollary 3.2. The easy proof is left to the reader.

Let \( R \) be a commutative ring with unity \( (1 \neq 0) \) and let \( S \) be a multiplicatively closed proper subset of \( R \) with \( 1 \in S \). We recall that if \( R \) is a commutative ring with unity, then \( R_S = \left\{ \frac{a}{s} : a \in R, s \in S \right\} \) is a commutative ring with unity. Also if \( I \) is an ideal in \( R \), then \( I_S \) is an ideal of \( R_S \), where \( I_S = \left\{ \frac{a}{s} : a \in I, s \in S \right\} \). Moreover, if \( J \) is an ideal of \( R_S \), then \( J \cap R \) is an ideal of \( R \).

Consider the canonical homomorphism \( \rho : R \rightarrow R_S \) which is defined by \( r \mapsto \frac{r}{1} \), for all \( r \in R \). Then \( \rho \) is a homogenous homomorphism of degree 0. Note that if \( P \) is a prime ideal of \( R \) with \( P \cap S = \emptyset \), then \( \rho^{-1}(P_S) = P \).

Theorem 3.3. Let \( I \) be a weakly 2-absorbing-primal ideal of \( R \) with \( \nu_0(I) \cap S = \emptyset \). If \( 0 \neq \frac{a}{s} \in I_S \), then \( a \in I \).

Proof of Theorem 3.3. Let \( 0 \neq \frac{a}{s} \in I_S \), then \( \frac{a}{s} = \frac{b}{t} \) for some \( b \in I \). So there exists \( u \in S \) such that \( uta = usb \in I \), since \( b \in I \). If \( uta = 0 \), then \( \frac{a}{s} = 0 \), a contradiction, so \( 0 \neq uta \in I \). If \( a \notin I \), then \( ut \) is not a weakly 2-absorbing prime to \( I \) which implies that \( ut \in \nu_0(I) \), a contradiction. Therefore, \( a \in I \).

We recall that \( a \in R \) is a regular element in \( R \), if \( (0 : a) = 0 \). In the rest of this section we assume that \( S \) is multiplicatively closed subset of \( R \) with \( 1 \in S \) such that all elements of \( S \) are regular.

Suppose that \( J \) is a 2-absorbing-primal ideal of \( R_S \), we define the set of all elements in \( R_S \) that are not weakly 2-absorbing prime to \( J \) by \( \nu_0(J) \). Let \( I \) be a proper ideal of \( R \), under the condition \( \nu_0(I) \cap S = \emptyset \) we have the following two results.

Lemma 3.2. let \( I \) be a weakly 2-absorbing-primal ideal of \( R \) with \( \nu_0(I) \cap S = \emptyset \). Then:

(1) \( I_S \) is a 2-absorbing-primal ideal of \( R_S \);
(2) \( I = I_S \cap R \).

Proof of Lemma 3.2. (1) It is enough to show that \( \nu_0(I_S) = (\nu_0(I))_S \). Let \( \frac{a}{u} \in \nu_0(I_S) \), then there exist \( \frac{r}{t_1}, \frac{s}{t_2}, \frac{t}{t_3} \in R_S \) with \( 0 \neq (\frac{r}{t_1} \frac{s}{t_2} \frac{t}{t_3}) (\frac{a}{u}) \in I_S \) such that...
Theorem 3.4. There is a one-to-one correspondence between weakly 2-absorbing primal ideals of $R$ with $\nu_0(I) \cap S = \phi$ and weakly 2-absorbing primal ideals of $R_S$.

Proof of Theorem 3.4. This follows directly from Lemma 3.2 and Proposition 3.1 and the fact that there is a one-to-one correspondence between weakly prime ideals $P$ of $R$ with $P \cap S = \phi$ and weakly prime ideals of $R_S$. 

References


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