Properties of weakly 2-absorbing primal ideals

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Abstract. Let R be a commutative ring with unity $(1 \neq 0)$. In this paper we introduce the concept of weakly 2-absorbing primal ideal which is a generalization of weakly primal ideal. Let I be a proper ideal of R. An element $a \in R$ is defined to be a weakly 2-absorbing prime to I if for any $r, s, t \in R$ with $0 \neq rsta \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$. An element $a \in R$ is not a weakly 2-absorbing prime to I if there exist $r, s, t \in R$, with $0 \neq rsta \in I$, such that $rs, rt, st \in R \setminus I$. We denote by $\nu_0(I)$ the set of all elements in R that are not weakly 2-absorbing prime to I. We define a proper ideal I of R to be a weakly 2-absorbing primal if the set $\nu_0(I) \cup \{0\}$ forms an ideal of R. Many results concerning weakly 2-absorbing primal ideals and examples of weakly 2-absorbing primal ideals are given.
Keywords: weakly 2-absorbing ideal, weakly 2-absorbing primal ideal.

1. Introduction

We study in this paper weakly 2-absorbing primal ideals in commutative rings with unity, which are generalization of weakly primal ideals. Ebrahimi Atani and Ahmad Darani gave a generalization of weakly primal ideals (see [3]). Recall from [3] that if R is a commutative ring with unity and I is a proper ideal of R, then $a \in R$ is a weakly prime to I if $0 \neq ra \in I$, for some $r \in R$, then $r \in I$. Also recall from [3] that $a \in R$ is not a weakly prime to I if there exists $r \in R \setminus I$ such that $0 \neq ra \in I$. In [3] Ebrahimi Atani defined $S_0(I)$ by the set of all elements a in R that are not weakly prime to I. In [3] Ebrahimi Atani defined I to be a weakly primal ideal in R if $S_0(I) \cup \{0\}$ forms an ideal in R.

The idea of a weakly 2-absorbing primal ideal of a given ring R which is a generalization of a weakly primal ideal of R is come from the concept of a weakly 2-absorbing ideal of R which is a generalization of a weakly prime ideal of R. Recall that weakly 2-absorbing ideals, which are generalizations of weakly prime ideals, were introduced in [5]. Let I be a proper ideal of R, in [5] A. Badawi and A. Y. Darani defined I to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

In this paper, we define $a \in R$ to be a weakly 2-absorbing prime to a proper ideal I of R, if $0 \neq rsta \in I$, where $r, s, t \in R$, then $rs \in I$ or $rt \in I$ or $st \in I$. Also we define $a \in R$ to be not a weakly 2-absorbing prime to a proper ideal I of

R, if there exist $r, s, t \in R$ with rs, rt and st are in $R \setminus I$ such that $0 \neq rsta \in I$. Let I be a proper ideal of R, and let $\nu_0(I)$ be the set of all $a \in R$ such that a is not a weakly 2-absorbing prime to I. In this paper I is defined to be a weakly 2-absorbing primal ideal of R if $\nu_0(I) \cup \{0\}$ forms an ideal in R. Let I be a proper ideal of R, in [9] we define $a \in R$ to be a 2-absorbing prime to I if for any $r, s, t \in R$ such that $rsta \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$. In [9] we define I to be a 2-absorbing primal ideal of R if $\nu(I)$ forms an ideal of R, where $\nu(I) = \{a \in R \ : \ a \text{ is not } 2-\text{absorbing prime to } I\}$. In this paper some basic properties of weakly 2-absorbing primal ideals are studied and classified, and some examples are given. Also the relations between 2-absorbing primal ideals and weakly 2-absorbing primal ideals are studied.

2. Basic properties of weakly 2-absorbing primal ideals

Let R be a commutative ring with unity and let I be a proper ideal of R. An element $a \in R$ is a weakly 2-absorbing prime to I if for any $r, s, t \in R$ with $0 \neq rsta \in I$, then rs or rt or st is in I. An element $a \in R$ is not a weakly 2-absorbing prime to I if there exist $r, s, t \in R$, with $0 \neq rsta \in I$, such that rs, rt and st are in $R \setminus I$. We denote by $\nu_0(I)$ the set of all elements in R that are not weakly 2-absorbing prime to I.

It is clear that every weakly primal ideal of a ring R is weakly 2-absorbing primal ideal of R. If $R = \mathbb{Z}_{16}$ and $I = \{0, 8\}$, then one can easily see that $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$ since $0 \neq 2.2.2 \in I$ and $4 \notin I$, so I is a weakly 2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$. Also one can easily see that $S_0(I) \cup \{0\} = 2\mathbb{Z}_{16} \neq \nu_0(I) \cup \{0\}$. Therefore, $I = \{0, 8\}$ is a weakly primal and weakly 2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $S_0(I) \neq \nu_0(I)$. Also if $I = \{0\}$ is an ideal in $\mathbb{Z}_6$, then, from the definition of weakly 2-absorbing primal ideals, I is a weakly 2-absorbing primal ideal of $\mathbb{Z}_6$ with $\nu_0(I) \cup \{0\} = \{0\}$, but easy computations implies that I is not a 2-absorbing primal ideal of $\mathbb{Z}_6$. The following are two examples of nonzero weakly 2-absorbing primal ideals that are not weakly primal ideals.

Example 2.1. (1) Let $R = \mathbb{Z}$ and let $I = 30\mathbb{Z}$. Then I is a weakly 2-absorbing primal ideal of $\mathbb{Z}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}$, since $(2)(3)(5) = 30 \in I$ and $(2)(3) = 6 \notin I$, $(2)(5) = 10 \notin I$ and $(3)(5) = 15 \notin I$. On the other hand I is not a weakly primal ideal in $\mathbb{Z}$, because $2, 3 \in S_0(I)$ but $1 \notin S_0(I)$. Note that if $1 \in S_0(I)$, then there exists $r \notin I$ with $1.r = r \in I$, a contradiction.

(2) Let $R = \mathbb{Z}[x, y, z]$ and let $I = xyzR$. Then I is a proper ideal of R and since $0 \neq xyz \in I$ with xy, xz, and yz are in $R \setminus I$ we get that $\nu_0(I) \cup \{0\} = R$. Therefore, I is a weakly 2-absorbing primal ideal of R.

On the other hand, since $0 \neq xyz \in I$ and $yz \in R \setminus I$ we get that $x \in S_0(I)$. Similarly, $y \in S_0(I)$. Now we show that $x + y$ can't be in $S_0(I)$. If there exists $f(x, y, z) \in \mathbb{Z}[x, y, z]$ with $0 \neq (x + y)f(x, y, z) \in I$, then xyz divides $(x + y)f(x, y, z)$ and since x divides xyz we get that x divides $(x + y)f(x, y, z)$ but x does not divide $x + y$, so x must divide $f(x, y, z)$. Similarly, y divides

$f(x,y,z)$ and z divides $f(x,y,z)$. Therefore, xyz divides $f(x,y,z)$ which implies that $f(x,y,z) \in I$, so $x+y \notin S_0(I)$ and hence I is not a weakly primal ideal of R.

Now, we have the following easy result.

Lemma 2.1. *Every weakly primal ideal of R is a weakly 2-absorbing primal.*

Theorem 2.1. *Let I be a proper ideal of R such that I is a weakly 2-absorbing primal ideal of R with $\nu_0(I) \cup \{0\} \neq R$. Then $\nu_0(I) \cup \{0\}$ is a weakly prime ideal of R.*

Proof of Theorem 2.1. *Let $a, b \in R$ such that $0 \neq ab \in \nu_0(I)$. Then there exist $r, s, t \in R$ with $0 \neq rst(ab) \in I$ such that $rs, rt, st \in R \setminus I$. Assume that $a \notin \nu_0(I)$. We must show that $b \in \nu_0(I)$. Since $0 \neq r(sb)ta \in I$ and $a \notin \nu_0(I)$, $0 \neq rsb \in I$ or $0 \neq rt \in I$ or $0 \neq sbt \in I$. But $rt \in R \setminus I$ implies $rsb \in I$ or $sbt \in I$. If $rsb \in I$, then $b \in \nu_0(I)$ since $r, s, rs \in R \setminus I$. Similarly, if $stb \in I$, then $b \in \nu_0(I)$. Therefore, $\nu_0(I) \cup \{0\}$ is a weakly prime ideal of R.* □

For example $I = 4\mathbb{Z}$ is a proper ideal of $\mathbb{Z}$ with $\nu_0(I) \cup \{0\} = 2\mathbb{Z}$. So I is a weakly 2-absorbing primal ideal of $\mathbb{Z}$ and $\nu_0(I) \cup \{0\} = 2\mathbb{Z}$ is a weakly prime ideal of $\mathbb{Z}$. But if $I = 6\mathbb{Z}$, then I is not a weakly 2-absorbing primal ideal of $\mathbb{Z}$, since $0 \neq (2)(3) \in I$ and $2, 3 \notin I$ so $2, 3 \in \nu_0(I)$, therefore, if $\nu_0(I) \cup \{0\}$ is an ideal of $\mathbb{Z}$, then $1 \in \nu_0(I)$ which implies that there exist $r, s, t \in \mathbb{Z} \setminus 6\mathbb{Z}$ such that $0 \neq rst \in 6\mathbb{Z}$ with $rs, rt, st \notin 6\mathbb{Z}$, but since 6 divides rst we have that 2 must divide r or s or t and 3 must divide r or s or t so 6 must divide rs or st or rt, a contradiction.

Definition 2.1. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R such that I is a weakly 2-absorbing primal ideal of R. Let $r, s, t \in R$, then (r, s, t) is called a triple zero of I if $rst = 0$ with $rs, rt, st \in R \setminus I$.*

The following five results on weakly 2-absorbing primal ideals over R and the results on weakly 2-absorbing ideals of R are the same and proved in [5] with another approach.

Theorem 2.2. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R. Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$. If (r, s, t) is a triple zero of I, then:*

(1) $rsI = rtI = stI = 0$;

(2) $rI^2 = sI^2 = tI^2 = 0$.

Proof of Theorem 2.2. (1) If $rsI \neq 0$, then there exists $a \in I$ such that $0 \neq rsa \in I$. So, $0 \neq rs(t+a) = rsa \in I$ with $rs, r(t+a), s(t+a) \in R \setminus I$ implies that $1 \in \nu_0(I)$, a contradiction. Therefore, $rsI = 0$. Similarly, $rtI = 0$ and $stI = 0$.

(2) Suppose $rI^2 \neq 0$, then there exist $a, b \in I$ such that $rab \neq 0$. So, $0 \neq r(s+a)(t+b) = ab \in I$ with $r(s+a), r(t+b), (s+a)(t+b) \in R \setminus I$ implies

that $1 \in \nu_0(I)$, a contradiction. Therefore, $rI^2 = 0$. Similarly, $sI^2 = 0$ and $tI^2 = 0$. □

Let I be a proper ideal of R such that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$. If I is a 2-absorbing primal ideal of R with $\nu(I) = R$. Then, by using Theorem 2.2, we have the following result.

Theorem 2.3. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R. Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$ such that $\nu(I) = R$, then $I^3 = 0$.*

Proof of Theorem 2.3. Since $\nu(I) = R$ we have that $1 \in \nu(I)$, and hence there exist $r, s, t \in R$ with $rst = 0$ such that $rs, rt, st \in R \setminus I$. Thus, (r, s, t) is a triple zero of I, since if $0 \neq rst \in I$, then $1 \in \nu_0(I)$, a contradiction. Suppose that $I^3 \neq 0$, then there exist $a, b, c \in I$ such that $0 \neq abc \in I$. So by Theorem 2.2 we have $0 \neq (r+a)(s+b)(t+c) = abc \in I$ and since $1 \notin \nu_0(I)$ we must have that $(r+a)(s+b) \in I$ or $(r+a)(t+c) \in I$ or $(s+b)(t+c) \in I$ and hence we have either $rs \in I$ or $rt \in I$ or $st \in I$, a contradiction. Hence $I^3 = 0$. □

Let R be a commutative ring with unity ($1 \neq 0$). Then $\mathrm{Nil}(R)$ denotes the ideal of nilpotent elements of R.

Corollary 2.1. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R. Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$. If $\nu(I) = R$, then $I \subseteq \mathrm{Nil}(R)$.*

Theorem 2.4. *Let I be a proper ideal of R. Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$ such that $\nu(I) = R$. Then:*

(1) *If $a \in \mathrm{Nil}(R)$, then either $a^2 \in I$ or $a^2 I = aI^2 = 0$;*

(2) $(\mathrm{Nil}(R))^2 I^2 = 0$.

Proof of Theorem 2.4. (1) Let $a \in \mathrm{Nil}(R)$. First, we show that if $a^2 I \neq 0$, then $a^2 \in I$. Now assume that $a^2 I \neq 0$. Let $i \in I$ such that $0 \neq a^2 i \in I$ and suppose that n is the smallest positive integer such that $a^n = 0$. Then $n \geq 3$ and we have $0 \neq a^2(i + a^{n-2}) \in I$, since $1 \notin \nu_0(I)$ we must have $a^2 \in I$ or $a^{n-1} \in I$. If $a^2 \in I$, then we are done. If $a^{n-1} \in I$, then $0 \neq a^2 a^{n-3} \in I$ again since $1 \notin \nu_0(I)$ we have $a^{n-2} \in I$. Continuing this procedure, then we arrive at $a^2 \in I$. Therefore for each $a \in \mathrm{Nil}(R)$ we have either $a^2 \in I$ or $a^2 I = 0$. Now, assume that $b^2 \notin I$ for some $b \in \mathrm{Nil}(R)$, then $b^2 I = 0$. We show that $bI^2 = 0$. If $bI^2 \neq 0$, then there exist $i_1, i_2 \in I$ such that $bi_1 i_2 \neq 0$. Let m be the smallest positive integer such that $b^m = 0$, then $m \geq 3$ since $b^2 \notin I$. Hence $0 \neq b(b + i_1)(b^{m-2} + i_2) = bi_1 i_2 \in I$ and since $1 \notin \nu_0(I)$ we have $b(b + i_1) \in I$ which implies that $b^2 \in I$ (a contradiction) or $b(b^{m-2} + i_2) \in I$ which implies that $b^{m-1} \in I$ (a contradiction). Therefore, $bI^2 = 0$.

(2) Let $r, s \in \mathrm{Nil}(R)$. If $r^2 \notin I$ or $s^2 \notin I$, then, by(1), $(rs)I^2 = 0$. Therefore we may assume that $r^2 \in I$ and $s^2 \in I$. So, $rs(r + s) \in I$. If $(r, s, r + s)$ is a triple zero of I, then, by Theorem 2.2(1), $(rs)I = 0$ and hence $(rs)I^2 = 0$.

If $0 \neq rs(r+s) \in I$, then $rs \in I$ since $1 \notin \nu_0(I)$. So, by Theorem 2.3, $(rs)I^2 \subseteq I^3 = 0$. □

Corollary 2.2. *Let R be a commutative ring with unity ($1 \neq 0$) and let A, B, C be proper ideals of R. Suppose that A, B, C are weakly 2-absorbing primal ideals of R with $1 \notin \nu_0(A) \cup \nu_0(B) \cup \nu_0(C)$ such that $\nu(A) = \nu(B) = \nu(C) = R$. Then, $A^2BC = AB^2C = ABC^2 = A^2B^2 = A^2C^2 = B^2C^2 = 0$.*

Proof of Corollary 2.2. By Corollary 2.1, $B \subseteq \text{Nil}(R)$ and $C \subseteq \text{Nil}(R)$, so $A^2BC \subseteq A^2(\text{Nil}(R))^2$ and by Theorem 2.4(2), $A^2(\text{Nil}(R))^2 = 0$. Similarly, $AB^2C = ABC^2 = 0$. Also, by Corollary 2.1, $A^2B^2 \subseteq (\text{Nil}(R))^2B^2 = 0$. Similarly, $A^2C^2 = B^2C^2 = 0$. □

In the next result we give a condition on a weakly 2-absorbing primal ideal of R to be a 2-absorbing primal ideal of R.

Theorem 2.5. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R. If I is a weakly 2-absorbing primal ideal of R with $I^2 \neq 0$, then I is a 2-absorbing primal ideal of R.*

Proof of Theorem 2.5. If $1 \in \nu(I)$, then $\nu(I) = R$ which implies that I is a 2-absorbing primal ideal of R. Therefore we may assume that $1 \notin \nu(I)$. To prove that I is a 2-absorbing primal ideal of R we must show that $\nu(I) = \nu_0(I) \cup \{0\}$. It is clear that $\nu_0(I) \cup \{0\} \subseteq \nu(I)$. Conversely, let $a \in \nu(I)$, then there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I$ such that $(rst)a \in I$. If $(rst)a \neq 0$, then $a \in \nu_0(I)$. So we may assume that $rsta = 0$. If $(rst)I \neq 0$, then there exists $c \in I$ such that $rstc \neq 0$. Therefore, $0 \neq (rst)(a+c) \in I$ which implies that $a + c \in \nu_0(I)$ and hence $a \in \nu_0(I)$, since $c \in \nu_0(I)$. Therefore we may assume that $(rst)I = 0$. If $rst \in I$, then $1 \in \nu(I)$ which is a contradiction. Therefore we may assume that $rst \notin I$. Since $I^2 \neq 0$, there exist $x, y \in I$ such that $xy \neq 0$. Hence, $(a+y)(trs+x) = 0 + ax + 0 + xy \in I$. If $0 \neq ax + xy \in I$ and since $trs + x \notin I$, then $a + y \in \nu_0(I)$ which implies that $a \in \nu_0(I)$, since $y \in \nu_0(I)$. But, if $ax + xy = 0$, then $0 \neq ax \in I$ which implies that $0 \neq a(x + trs) = ax \in I$, so $a \in \nu_0(I)$. Thus, $\nu(I) = \nu_0(I) \cup \{0\}$. □

We have to remark that if a proper ideal I of R is a weakly 2-absorbing primal ideal of R with $I^2 \neq 0$, and $1 \notin \nu(I)$, then $\nu_0(I) \cup \{0\}$ is a prime ideal of R since by Theorem 2.5, $\nu(I) = \nu_0(I) \cup \{0\}$.

We recall that if R and S are are commutative rings with unity and P, Q are weakly prime ideals in R, S (respectively), then $P \times S$ and $R \times Q$ are weakly prime ideals of $R \times S$.

Theorem 2.6. *Let $R \times S$ be a commutative ring with unity, where R, S are commutative rings with unities, let I be a proper ideal of R with $I \times S \nsubseteq \text{Nil}(R \times S)$. Then, the following statements are equivalent.*

(1) *$I \times S$ is a weakly 2-absorbing primal ideal of $R \times S$;*
(2) *$I \times S$ is a 2-absorbing primal ideal of $R \times S$;*
(3) *I is a 2-absorbing primal ideal of R.*

Proof of Theorem 2.6. $(1 \to 2)$. Since $I \times S \nsubseteq \mathrm{Nil}(R \times S)$, by Corollary 2.1 we have that $\nu(I \times S) \neq R \times S$. To prove that $I \times S$ is a 2-absorbing primal ideal of $R \times S$ we must show that $\nu(I) = \nu_0(I) \cup I$. Let $a \in \nu(I)$ and let $(rst)a = 0$ for some $r, s, t \in R$ with $rs, rt, st \in R \setminus I$. Since $1 \notin \nu(I)$ and $rs \notin I$ we must have $rta \in I$ or $sta \in I$. If $rta \in I$, then $ra \in I$ or $ta \in I$ since $1 \notin \nu(I)$ and $rt \notin I$. If $ra \in I$, then $a \in I$ since $r \notin I$ and $1 \notin \nu(I)$, similarly, if $ta \in I$, then $a \in I$. Similarly, if $sta \in I$, then $a \in I$. Therefore, $\nu(I) = \nu_0(I) \cup I$ and hence $\nu(I \times S) = \nu(I) \times S$ is an ideal of $R \times S$ which implies that $I \times S$ is a 2-absorbing primal ideal of $R \times S$.

$(2 \to 3)$. Since $\nu(I \times S) = \nu(I) \times S$ is a prime ideal of $R \times S$ we have that $\nu(I)$ is a prime ideal of R, so I is a 2-absorbing primal ideal of R.

$(3 \to 1)$. Since I is a 2-absorbing primal ideal of R we have that $I \times S$ is a 2-absorbing primal ideal of $R \times S$, hence $I \times S$ is a weakly 2-absorbing primal ideal of $R \times S$. □

Theorem 2.7. *Let $R \times S$ be a commutative ring with unity, where R, S are commutative rings with unities, let I be a nonzero proper ideal of R and J a nonzero ideal of S with $I \times J \nsubseteq \mathrm{Nil}(R \times S)$. Then, the following statements are equivalent.*

(1) *$I \times J$ is a weakly 2-absorbing primal ideal of $R \times S$;*

(2) *$J = S$ and I is a 2-absorbing primal ideal of R;*

(3) *$I \times J$ is a 2-absorbing primal ideal of $R \times S$.*

Proof of Theorem 2.7. $(1 \to 2)$. Suppose $I \times J$ is a weakly 2-absorbing primal ideal of $R \times S$. Since $I \times J \nsubseteq \mathrm{Nil}(R \times S)$ and if $J = S$, then I is a 2-absorbing primal ideal of R by Theorem 2.6. We show that the case $J \neq S$ can not be happened. Suppose $J \neq S$, we show that J is a prime ideal in S and I is a prime ideal of R. Since $I \times J \nsubseteq \mathrm{Nil}(R \times S)$, by Corollary 2.1 we have that $\nu(I \times J) \neq R \times S$. Let $a, b \in S$ such that $ab \in J$ and let $0 \neq i \in I$. Then, $(i, 1)(1, a)(1, b) = (i, ab) \neq (0, 0) \in I \times J$ since $(1, ab) \notin I \times J$ and since $(1, 1) \notin \nu_0(I \times J)$ we must have $(i, a) \in I \times J$ or $(i, b) \in I \times J$ so $a \in J$ or $b \in J$. Thus J is a prime ideal of S. Similarly, let $c, d \in R$ such that $cd \in I$, and let $0 \neq j \in J$. Then, $(c, 1)(d, 1)(1, j) = (cd, j) \neq (0, 0) \in I \times J$, since $(cd, 1) \notin I \times J$ and since $(1, 1) \notin \nu_0(I \times J)$ we must have $(c, j) \in I \times J$ or $(d, j) \in I \times J$ so $c \in I$ or $d \in I$. Hence I is a prime ideal of R. In this case we show that $(1, 1) \in \nu(I \times J)$. Now, $(1, 0)(0, 1) \in I \times J$ and $(1, 0) \notin I \times J$, $(0, 1) \notin I \times J$, so $(1, 0), (0, 1) \in \nu(I \times J)$. Therefore, if $\nu(I \times J)$ is an ideal in $R \times S$, then $(1, 1) = (1, 0) + (0, 1) \in \nu(I \times J)$, a contradiction to Corollary 2.1. Therefore the only case of part (2) is that $J = S$ and I is a 2-absorbing primal ideal of R.

$(2 \to 3)$. If $J = S$ and I is a 2-absorbing primal ideal of R, then $I \times J$ is a 2-absorbing primal ideal of $R \times S$ by Theorem 2.6(2).

$(3 \to 1)$. Clearly from Theorem 2.6 □

Theorem 2.8. *Let $R \times S$ be a commutative ring with unity, where each R and S are commutative rings with unities, let $I = I_1 \times I_2$ be a proper ideal of $R \times S$.*

If I is a weakly 2-absorbing primal ideal of $R \times S$ with $\nu_0(I) \neq \phi$. Then, either $I = (0,0)$ or I is a 2-absorbing primal ideal of $R \times S$.

Proof of Theorem 2.8. Suppose $I \neq (0,0)$. Since $\nu_0(I) \neq \phi$, by Theorem 2.1, $\nu_0(I) \cup \{(0,0)\}$ is a weakly prime ideal of $R \times S$. Therefore, $\nu_0(I) \cup \{(0,0)\}$ has one of the following cases:

(1) $\nu_0(I) \cup \{(0,0)\} = P_1 \times P_2$, where each $P_i = \{0\}$. In this case $\nu_0(I) \cup \{(0,0)\} = (0,0)$, a contradiction.

(2) $\nu_0(I) \cup \{(0,0)\} = P_1 \times S$, where P_1 is a weakly prime ideal in R and since $S \neq 0$, P_1 is a prime ideal of R, so $\nu_0(I) \cup \{(0,0)\}$ is a prime ideal of $R \times S$. Now, we show that $I_2 = S$. Since $S \neq 0$, there exists $(a,b) \neq (0,0) \in I$ and so $(0,0) \neq (a,b) = (1,1)(a,1)(1,b)$. If $(a,1) \notin I$, then $(1,b) \in \nu_0(I)$ and hence $1 \in P_1$, a contradiction. Therefore, $(a,1) \in I$ which implies that $1 \in I_2$, hence $I_2 = S$. Now, we show that I_1 is a 2-absorbing primal ideal of R with $\nu(I_1) = P_1$. Let $a_1 \in P_1$, then $(a_1, 0) \in \nu_0(I)$. If $a_1 = 0$, then $a_1 \in I_1 \subseteq \nu(I_1)$, so we may assume that $a_1 \neq 0$. Thus, there exist $(r_1,r_2),(s_1,s_2),(t_1,t_2) \in R \times S$ with $(r_1s_1, r_2s_2), (r_1t_1, r_2t_2), (t_1s_1, t_2s_2) \in R \times S \setminus I = (R \setminus I_1) \times S$ such that $(0,0) \neq (r_1s_1t_1, r_2s_2t_2)(a_1, 0) \in I$, so $0 \neq r_1s_1t_1a_1 \in I_1$ with $r_1s_1, r_1t_1, s_1t_1 \notin I_1$ which implies that a_1 is not a weakly 2-absorbing prime to I_1, hence a_1 is not a 2-absorbing prime to I_1 and so $a_1 \in \nu(I_1)$. Conversely, let $a_1 \in \nu(I_1)$. Then, there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I_1$ such that $rsta_1 \in I_1$. Since $S \neq 0$, we have $(0,0) \neq (rsta_1, 1) = (r,1)(s,1)(t,1)(a_1,1) \in I_1 \times S$ with $(rs,1), (rt,1), (st,1) \in (R \setminus I_1) \times S = (R \times S) \setminus I$ which implies that $(a_1, 1)$ is not a weakly 2-absorbing prime to I, hence $(a_1,1) \in \nu_0(I) \cup \{(0,0)\} = P_1 \times S$, so $a_1 \in P_1$. Therefore, $\nu(I_1) = P_1$. Thus, $\nu(I) = P_1 \times S$. Hence $I = I_1 \times S$ is a 2-absorbing primal ideal of $R \times S$.

(3) $\nu_0(I) \cup \{(0,0)\} = R \times P_2$, where P_2 is a weakly prime ideal in S. By using the same approach in part 2 we conclude that $I = R \times I_2$ and $\nu(I_2) = P_2$, and so $\nu(I) = R \times P_2$. Hence $I = R \times I_2$ is a 2-absorbing primal ideal of $R \times S$. □

3. More properties of weakly 2-absorbing primal ideals

For a commutative ring R, let $\mathcal{J}(R)$ denotes the intersection of all maximal ideals of R.

Lemma 3.1. *Let R be a commutative ring and $a, b \in \mathcal{J}(R)$. Then the ideal $I = abR$, where $1 \notin \nu_0(I)$, is a weakly 2-absorbing primal ideal of R if and only if $ab = 0$.*

Proof of Lemma 3.1. If $ab = 0$, then $I = 0$ is a weakly 2-absorbing primal ideal of R by definition. If $0 \neq ab \in I$ with $a, b \notin I$, then $1 \in \nu_0(I)$, contradiction. Therefore, a or b is in $I = abR$. If $a \in I$, then $a = abk$ for some $k \in R$. So, $a(1 - bk) = 0$ and since $bk \in \mathcal{J}(R)$ we have that $1 - bk$ is a unit in R. Thus, $a(1 - bk) = 0$ implies that $a = 0$ and hence $ab = 0$, a contradiction. Therefore, $I = 0$. □

We recall that R is defined to be quasi-local ring if R has a unique maximal ideal. If (R, M) is a quasi-local ring, where M is the unique maximal ideal of R, then we have the following two results about a weakly 2-absorbing primal ideal I of R with $1 \notin \nu_0(I)$.

Theorem 3.1. *Let (R, M) be a quasi-local ring with $\nu_0(I) \neq R$, for all proper ideals I of R. Then every proper ideal of R is a weakly 2-absorbing primal if and only if $M^2 = 0$.*

Proof of Theorem 3.1. Let $a, b \in M$, then $I = abR$ is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$, hence, by Lemma 3.1, $M^2 = 0$. Conversely, suppose $M^2 = 0$. Let I be a proper ideal of R and let $a \in \nu_0(I)$. If a is a unit in R, then $1 \in \nu_0(I)$, a contradiction. So we may assume that a is not a unit in R. Let $r, s, t, \in R$ with $0 \neq rsta \in I$ such that $rs, rt, st \notin I$. If $0 \neq rst \in I$, then $M^2 = 0$ and $a \in M$ implies $rst \notin M$. So rst is a unit in R which implies that $a \in I$. So $\nu_0(I) \cup \{0\} = I$ which implies that I is weakly 2-absorbing primal ideal of R. □

Corollary 3.1. *Let (R, M) be a quasi-local ring with $M^2 = 0$ and with $\nu_0(I) \neq R$, for all proper ideals I of R. Then every proper ideal of R is a 2-absorbing primal ideal of R.*

Proof of Corollary 3.1. Let I be a proper ideal of R, then, by Theorem 3.1, I is a weakly 2-absorbing primal ideal of R since $M^2 = 0$. We show that $\nu(I)$ is an ideal in R. Let a, b be nonzero elements in $\nu(I)$. Then, there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I$ such that $rsta \in I$. If $0 \neq rsta \in I$, then, by Theorem 3.1, $a \in I \subseteq M$. Since $rs \notin I$ we have r or s is a unit in R. Therefore, if $rsta = 0$, then $(st)a = 0$ or $(rt)a = 0$. Say $(st)a = 0$ again since $st \notin I$ we have s or t is a unit in R which implies that $sa = 0$ or $ta = 0$. Say $ta = 0$, hence t is not a unit in R, since $0 \neq a \in I$. Therefore if $ta = 0 \in I \subseteq M$ and a is not a unit in R (if a is a unit in R, then $t = 0$ a contradiction), then a must be in M since M is a prime ideal. Similarly, $b \in M$, so $a + b \in M$. If $t(a + b) \neq 0$, then t is a unit in R since $a + b \in M$, a contradiction. Therefore, $t(a + b) = 0 \in I$ which implies that $a + b \in \nu(I)$ since $t \notin I$. Hence $\nu(I)$ is an ideal of R. □

In the next result we give the condition on a proper ideal I of R such that I/J is a weakly 2-absorbing primal ideal of R/J where J is a proper ideal of R contained in I.

Theorem 3.2. *Let I, J be proper ideals of R with $J \subseteq I$. If I is a weakly 2-absorbing primal ideal of R with $\nu_0(J) \subseteq I$. Then I/J is a weakly 2-absorbing primal ideal of R/J.*

Proof of Theorem 3.2. To prove that I/J is a weakly 2-absorbing primal ideal of R/J, we must show that $\nu_0(I/J) \cup \{0\} = [\nu_0(I) \cup J]/J$. Let $a + J \in \nu_0(I/J)$, then there exist $r + J, s + J, t + J \in R/J$ with $0 \neq rsta + J \in I/J$ such that $rs + J, rt + J, st + J \notin I/J$. So $0 \neq rsta \in I$, since $rsta \notin J$, with

$rs, rt, st \notin I$ hence $a \in \nu_0(I)$, therefore, $a + J \in [\nu_0(I) \cup J]/J$. Conversely, let $0 \neq a + J \in [\nu_0(I) \cup J]/J$, then $a \in \nu_0(I) - J$. If $a \in I$, then $a + J \in \nu_0(I/J)$. So we may assume that $a \notin I$. Then, there exist $r, s, t \in R$ with $0 \neq rsta \in I$ such that $rs, rt, st \notin I$. If $0 \neq rsta \in J$, then $a \in \nu_0(J)$, a contradiction, since $\nu_0(J) \subseteq I$ and $a \notin I$. Therefore, $r + J, s + J, t + J \in R/J$ with $0 \neq rsta + J = (rst + J)(a + J) \in I/J$ such that $rs + J, rt + J, st + J \notin I/J$, so $a + J \in \nu_0(I/J)$. Hence $\nu_0(I/J) \cup \{0\} = [\nu_0(I) \cup J]/J$ which implies that I/J is a weakly 2-absorbing primal ideal of R/J. □

Corollary 3.2. *Let R_0 be a subring of R with unity. If I is a weakly 2-absorbing primal ideal of R, then $I \cap R_0$ is a weakly 2-absorbing primal ideal of R_0.*

Proof of Corollary 3.2. The easy proof is left to the reader. □

Let R be a commutative ring with unity $(1 \neq 0)$ and let S be a multiplicatively closed proper subset of R with $1 \in S$. We recall that if R is a commutative ring with unity, then $R_S = \{\frac{a}{s} \ : \ a \in R \ , \ s \in S\}$ is a commutative ring with unity. Also if I is an ideal in R, then I_S is an ideal of R_S, where $I_S = \{\frac{a}{s} \ : \ a \in I \ , \ s \in S\}$. Moreover, if J is an ideal of R_S, then $J \cap R$ is an ideal R.

Consider the canonical homomorphism $\rho : R \to R_S$ which is defined by $r \mapsto \frac{r}{1}$, for all $r \in R$. Then ρ is a homogenous homomorphism of degree 0. Note that if P is a prime ideal of R with $P \cap S = \emptyset$, then $\rho^{-1}(P_S) = P$.

Theorem 3.3. *Let I be a weakly 2-absorbing-primal ideal of R with $\nu_0(I) \cap S = \emptyset$. If $0 \neq \frac{a}{s} \in I_S$, then $a \in I$.*

Proof of Theorem 3.3. Let $0 \neq \frac{a}{s} \in I_S$, then $\frac{a}{s} = \frac{b}{t}$ for some $b \in I$. So there exists $u \in S$ such that $uta = usb \in I$, since $b \in I$. If $uta = 0$, then $\frac{a}{s} = 0$, a contradiction, so $0 \neq uta \in I$. If $a \notin I$, then ut is not a weakly 2-absorbing prime to I which implies that $ut \in \nu_0(I)$, a contradiction. Therefore, $a \in I$. □

We recall that $a \in R$ is a regular element in R, if $(0 \ : \ a) = 0$. In the rest of this section we assume that S is multiplicatively closed subset of R with $1 \in S$ such that all elements of S are regular.

Suppose that J is a 2-absorbing-primal ideal of R_S, we define the set of all elements in R_S that are not weakly 2-absorbing prime to J by $\nu_0(J)$. Let I be a proper ideal of R, under the condition $\nu_0(I) \cap S = \emptyset$ we have the following two results.

Lemma 3.2. *let I be a weakly 2-absorbing-primal ideal of R with $\nu_0(I) \cap S = \emptyset$. Then:*

(1) *I_S is a 2-absorbing-primal ideal of R_S;*
(2) *$I = I_S \cap R$.*

Proof of Lemma 3.2. (1) It is enough to show that $\nu_0(I_S) = (\nu_0(I))_S$. Let $\frac{a}{u} \in \nu_0(I_S)$, then there exist $\frac{r}{t_1}, \frac{s}{t_2}, \frac{v}{t_3} \in R_S$ with $0 \neq (\frac{r}{t_1}\frac{s}{t_2}\frac{v}{t_3})(\frac{a}{u}) \in I_S$ such that

$\frac{rs}{t_1t_2}, \frac{rv}{t_1t_3}, \frac{sv}{t_2t_3} \notin I_S$. Therefore, $rs, rv, sv \notin I$ with $0 \neq (rsva)w \in I$ for some $w \in S$, since $\frac{rsva}{ut_1t_2t_3} \neq 0$. So $wa \in \nu_0(I)$ and hence $\frac{a}{u} \in (\nu_0(I))_S$. Conversely, let $\frac{a}{u} \in (\nu_0(I))_S$, then there exists $w \in S$ such that $aw \in \nu_0(I)$. So, let $r, s, t \in R$ with $rs, rt, st \notin I$ such that $0 \neq (rst)wa \in I$. If $rsta \notin I$, then $w \in \nu_0(I)$, a contradiction. Therefore, $0 \neq rsta \in I$ and since S consists of regular elements we have that $0 \neq \frac{rsta}{u} \in I_S$. If $\frac{rs}{1} \in I_S$, then $0 \neq rsv \in I$ for some $v \in S$ and so $v \in \nu_0(I)$, a contradiction. Therefore, $\frac{rs}{1} \notin I_S$, similarly, $\frac{rt}{1} \notin I_S$ and $\frac{st}{1} \notin I_S$. Thus, $\frac{a}{u} \in \nu_0(I_S)$ and hence I_S is a 2-absorbing-primal ideal of R_S.

(2) It is clear that $I \subseteq I_S \cap R$. Conversely, let $\frac{a}{1} \in I_S \cap R$, then there exists $w \in S$ such that $aw \in I$. If $aw = 0$, then $a = 0 \in I$. If $0 \neq aw \in I$ and $a \notin I$, then $w \in \nu_0(I)$, a contradiction. Thus, $a \in I$. So, $I = I_S \cap R$. □

Proposition 3.1. *If J is a weakly 2-absorbing primal ideal of R_S with $\nu_0(J) \neq R_S$, then $J \cap R$ is a weakly 2-absorbing primal ideal of R with $\nu_0(J \cap R) \cap S = \phi$.*

Proof of Proposition 3.1. To prove that $J \cap R$ is a weakly 2-absorbing primal ideal of R we must show that $\nu_0(J \cap R) = \nu_0(J) \cap R$. Let $a \in \nu_0(J) \cap R$, then $\frac{a}{1} \in \nu_0(J)$. So there exist $\frac{r}{t_1}, \frac{s}{t_2}, \frac{u}{t_3} \in R_S$ with $0 \neq (\frac{r}{t_1}\frac{s}{t_2}\frac{u}{t_3})(\frac{a}{1}) \in I_S$ such that $\frac{rs}{t_1t_2}, \frac{ru}{t_1t_3}, \frac{su}{t_2t_3} \notin I_S$. Therefore, $0 \neq rsua \in J \cap R$ with $rs, ru, su \notin J \cap R$, hence $a \in \nu_0(J \cap R)$. Conversely, let $a \in \nu_0(J \cap R)$, then there exist $r, s, t \in R$ such that $rs, rt, st \notin J \cap R$ with $0 \neq (rst)a \in J \cap R$. Therefore, $\frac{rs}{1}, \frac{rt}{1}, \frac{st}{1} \notin J$ with $0 \neq \frac{(rst)a}{1} \in J$, since S consists of regular elements, thus, $\frac{a}{1} \in \nu_0(J)$. So $J \cap R$ is a weakly 2-absorbing primal ideal of R. Now, we show that $\nu_0(J \cap R) \cap S = \phi$. Let $w \in \nu_0(J \cap R) \cap S$, then there exist $r, s, t \in R$ such that $rs, rt, st \notin J \cap R$ with $0 \neq (rst)w \in J \cap R$. Therefore, $0 \neq \frac{rst}{1} \in J$ with $\frac{rs}{1}, \frac{rt}{1}, \frac{st}{1} \notin J$ implies that $1 \in \nu_0(J)$, a contradiction. So, $\nu_0(J \cap R) \cap S = \phi$. □

It is well known that if S is multiplicatively closed subset of R with $1 \in S$ such that S consists of regular elements, then there is a one-to-one correspondence between weakly prime ideals P of R with $P \cap S = \phi$ and weakly prime ideals of R_S. In the next result we also prove that there is a one-to-one correspondence between weakly 2-absorbing primal ideals I of R with $\nu_0(I) \cap S = \phi$ and weakly 2-absorbing primal ideals of R_S.

Theorem 3.4. *There is a one-to-one correspondence between weakly 2-absorbing primal ideals I of R with $\nu_0(I) \cap S = \phi$ and weakly 2-absorbing primal ideals of R_S.*

Proof of Theorem 3.4. This follows directly from Lemma 3.2 and Proposition 3.1 and the fact that there is a one-to-one correspondence between weakly prime ideals P of R with $P \cap S = \phi$ and weakly prime ideals of R_S. □

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