Inverse one-dimensional wave equation problem under upper-base as additional information

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Abstract. Final displacement measurement 'upper-base' is used in this research for finding force in non-homogeneous hyperbolic equation. Same as [7, 8] finite difference method (FDM) with separation variable are applied, but different additional condition is investigated. That means additional data were given by the left end displacement measurement in [7], and the right end displacement measurement in [8].

Furthermore, the wave equation with force split in two parts, part one is direct homogenous problem and numerically using FDM and second one is inverse non-homogeneous ill-posed problem, solved separation variable has been applied. Sequentially, for solving inverse force problem require initial and the boundary condition with additional data, in this study extra data is upper-base. Moreover, second part is unstable due to ill-posness of the problem where a small error in addition data causes major error in out force data [6]. A stable solution for direct problem and reasonable for inverse problem has been obtained as shown on figures and presented numerical result. Compare with the [7, 8] different shapes of figures and different condition numbers are provided.

Keywords: final displacement measurement, direct problem, finite difference method, separation variables method, regularization, inverse force problem.

1. Introduction

The target of this research is to solve inverse force problem with initial, boundary and additional condition, which is different to the one had used in [7, 8]. Furthermore, dealing with end displacement boundary in [7, 8] and upper based in this research are considered. An application of wave equation in science especially physics and engineering is dominant [2, 3, 6]. Added small amount of error to the extra data 'noisy data' to see instability after that using zeroth order Tikhonov regularization in order to stabilize the solution. For choosing regularization parameter the minimum error is explored. Numerically finite difference method [7, 8, 9] is used instead boundary element method (BEM) [1, 4, 6] with separation variable and different condition are studied.

The innovation of this study is to develop difference additional data and to see how far is working compare to the [4, 6, 7, 8] for that reason same numerically example have been applied.

The mathematical formulation is in Section 2, numerical results and discussed are in Section 3 and conclusions are in Section 4.

2. Mathematical formulation

The governing equation for a boundary $[0, L]$ with force $f(x)$ is given by the hyperbolic equation [4, 5, 6, 7, 8]

$$u_{tt}(x,t) = \nabla^2 u(x,t) + f(x), \quad (x,t) \in (0,L) \times (0,T), \quad T > 0 \tag{1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x), \quad x \in [0,L], \tag{2}$$

$$u_x(0,t) = q_0(t), \quad u(L,t) = p_L(t), \quad t \in (0,T), \tag{3}$$

$$u(0,t) = p_0(t), \quad u_x(L,t) = q_L(t), \quad t \in (0,T), \tag{4}$$

$$u(x,T) = u_T(x), \quad x \in (0,L), \quad T > 0, \tag{5}$$

where (5) is called the final displacement measurement, when deal with inverse problem part for reaching force $f(x)$ is used. In addition, $\{u(x,t), u_0, v_0\}$ variables represent {displacement,initial displacement,velocity}, respectively.

One of the equation (3) or (4) using with (1) and (2) in case solving hyperbolic equation.

Split equation (1) into $u = v + w$. In direct problem: $v_{tt} = v_{xx}$ with initial condition (2) and boundary condition (3) (or (4)), and inverse problem: $w_{tt} = w_{xx} + f(x)$ with initial condition $u(x,0) = u_t(x,0) = 0$, boundary condition $u_x(0,t) = u(L,t) = 0$ (or $u(0,t) = u_x(L,t) = 0$) and $w(x,T) = u_T(x) - v_T(x)$.

2.1 Direct problem

v in part one is well-posed direct problem without force/source ($v_{tt} = v_{xx}$). Numerically FDM applying to get $v(x,T) = v_T(x)$ (i.e $v_T(x_i)$), with initial condition (2) and boundary condition (3) (or (4))

$$v_{i,j+1} = r^2 v_{i+1,j} + 2(1-r^2)v_{i,j} + r^2 v_{i-1,j} - v_{i,j-1}, \quad i = \overline{1,(M-1)}, \quad j = \overline{1,(N-1)}, \tag{6}$$

$$v_{i,1} = \frac{1}{2}r^2 u_0(x_{i+1}) + (1-r^2)u_0(x_i) + \frac{1}{2}r^2 u_0(x_{i-1}) + (\Delta t)v_0(x_i), \quad j = 0 i = \overline{1,(M-1)}. \tag{7}$$

$$v_{i,0} = u_0(x_i), \quad i = \overline{0,M}, \frac{v_{i,1} - v_{i,-1}}{2\Delta t} = v_0(x_i), i = \overline{1,(M-1)}, \tag{8}$$

$$-\frac{\partial v}{\partial x}(0,t_j) = -\frac{4v_{1,j} - v_{2,j} - 3v_{0,j}}{2\Delta x}, j = \overline{1,N}, v_{M,j} = P_L(t_j), j = \overline{0,N}, \tag{9}$$

$$v_{0,j} = P_0(t_j), j = \overline{0,N}, \frac{\partial v}{\partial x}(L,t_j) = \frac{3v_{M,j} - 4v_{M-1,j} + v_{M-2,j}}{2\Delta x}, j = \overline{1,N}, \tag{10}$$

where $v_{i,j} := v(x_i, t_j)$, $x_i = i\Delta x$, $t_j = j\Delta t$ and $r = c\Delta t/\Delta x$, for $i = \overline{0, M}$, $j = \overline{0, N}$, in addition $\Delta x = L/M$ and $\Delta t = T/N$ such that divide the domain $(0, L) \times (0, T)$ into M and N[9, 4, 5, 6, 7, 8].

After that using $v_T(x)$ in the inverse problem part to get (w, f).

2.2 Inverse problem

Separation variable has studied to solve $w_{tt} = w_{xx} + f(x)$ with 'zero' initial condition $u(x, 0) = u_t(x, 0) = 0$ and 'zero' boundary condition $u_x(0, t) = u(L, t) = 0$ get equations (11) and (12), or with 'zero' boundary condition $u(0, t) = u_x(L, t) = 0$ reach equations (13) and (14) [1, 9, 4, 5, 6, 7, 8], as following:

$$w_K(x, t; \underline{b}) = \frac{\sqrt{2}}{c^2} \sum_{k=1}^{K} \frac{b_k}{\lambda_k^2} (1 - \cos(c\lambda_k t)) \cos(\lambda_k x), \tag{11}$$

$$f_K(x) = \sqrt{2} \sum_{k=1}^{K} b_k \cos(\lambda_k x), \quad x \in (0, L), \tag{12}$$

$$w_K(x, t; \underline{b}) = \frac{\sqrt{2}}{c^2} \sum_{k=1}^{K} \frac{b_k}{\lambda_k^2} (1 - \cos(c\lambda_k t)) \sin(\lambda_k x), \tag{13}$$

$$f_K(x) = \sqrt{2} \sum_{k=1}^{K} b_k \sin(\lambda_k x), \quad x \in (0, L), \tag{14}$$

where K is a truncation number and $\lambda_k = \frac{(k-\frac{1}{2})\pi}{L}$ for $k = \overline{1, K}$. As it is clear from equations (11)-(14) $\underline{b} = (b_k)_{k=\overline{1,K}}$ is unknown, which can calculate from $w(x, T) = u_T(x) - v_T(x)$, i.e

$$\begin{aligned} w(x, T; \underline{b}) &= u_T(x) - v_T(x) =: g(x)|h(x) \\ &= \frac{\sqrt{2}}{c^2} \sum_{k=1}^{K} \frac{b_k}{\lambda_k^2} (1 - \cos(c\lambda_k T)) \cos(\lambda_k x)| \sin(\lambda_k x), \; x \in [0, L], \end{aligned} \tag{15}$$

the left side of the equation (15) is known and b is unknown on the right side. Moreover, the least squares solution is given $\underline{b}_\lambda = (Q^{\text{tr}} Q)^{-1} Q^{\text{tr}} \underline{g|h}$ where the superscript 'tr' denotes the transpose and $Q_{mk} = \frac{\sqrt{2}(1-\cos(c\lambda_k T)) \cos(\lambda_k x)| \sin(\lambda_k x)}{c^2 \lambda_k^2}$.

The existence and uniqueness had provided in [1, 4, 6], and for stability adding some noise to the $u_T(x)$ as

$$u_T^\epsilon(x_m) = u_T(x_m) + \epsilon, \quad m = \overline{1, M}. \tag{16}$$

For knowing the value of ϵ: studying a Gaussian normal distribution as mean zero and standard deviation σ are given by the following [4, 5, 6, 7] and $p\%$ represents the percentage of noise

$$\sigma = p\% \times \max_{x \in [0, L]} |u_T(x)|. \tag{17}$$

The noisy data (16) also causes noise in $g|h$ term as

$$g^{\epsilon}(x_m)|h^{\epsilon}(x_m) = u^{\epsilon}_T(x_m) - v_T(x_m) = g(x_m)|h(x_m) + \epsilon, \quad m = \overline{1, M}. \tag{18}$$

As mentioned above this noise makes unstable for solution as appear in example section, then to overcome this obstacle zeroth-order Tikhonov regularization is applied to find the solutions (for more detail can see [4, 6, 7]) as:

$$\underline{b}_{\lambda} = (Q^{\text{tr}}Q + \lambda I)^{-1}Q^{\text{tr}}\underline{g}^{\epsilon}|\underline{h}^{\epsilon}. \tag{19}$$

Note that in equations (15),(18) and (19) the "—" means "or". Furthermore, this 'or' has been used when (3) or (4) was applied.

3. Numerical results and discussion

With an observation of understanding how changing additional condition is modified solution for the problem, the same example in [4, 6, 7, 8] is chosen with analytic solution for $(u(x,t), f(x))$

$$u(x,t) = \sin(\pi x) + t + \frac{t^2}{2}, \quad f(x) = 1 + \pi^2 \sin(\pi x), \quad x \in [0,1], \tag{20}$$

$$u(x,0) = u_0(x) = \sin(\pi x), \quad u_t(x,0) = v_0(x) = 1, \quad x \in [0,1], \tag{21}$$

$$u_x(0,t) = q_0(t) = \pi, \quad u(1,t) = p_L(t) = t + \frac{t^2}{2}, \quad t \in (0,1]. \tag{22}$$

Instead of (22) can be used boundary condition

$$u(0,t) = p_0(t) = t + \frac{t^2}{2}, \quad u_x(L,t) = q_L(t) = -\pi, \quad t \in (0,1]. \tag{23}$$

An additional overdeterminated boundary condition is upper-base

$$u(x, T = 1) = u_T(x) = \frac{3}{2} + \sin(\pi x), \quad x \in [0,1]. \tag{24}$$

First, find $v_T(x)$ by using finite difference method (FDM), the discrete form of v problem for more details see [5, 7, 8], as shown in Figure 1 $M = N \in \{20, 40, 80\}$ increase the result quite close to each other (this means solution good convergence), when using boundary condition (22) and (23) appear in Figure 1(a) and 1(b), respectively. After that, is obtained $\underline{b}$ from equation (13) after substituting $v_T(x)$ in it. Moreover, condition number for matrix Q 'cond(Q)' is calculating in Table 1, the left side of the Table 1 is getting with boundary condition (22), and the right side from boundary condition (23). However, when M and K are both increased, the condition number of Q increases too, they are not effected to the numerically result for $\underline{b}$. Figure 2 showing ill-condition for inverse problem, the figure comes from singular values for matrix Q.

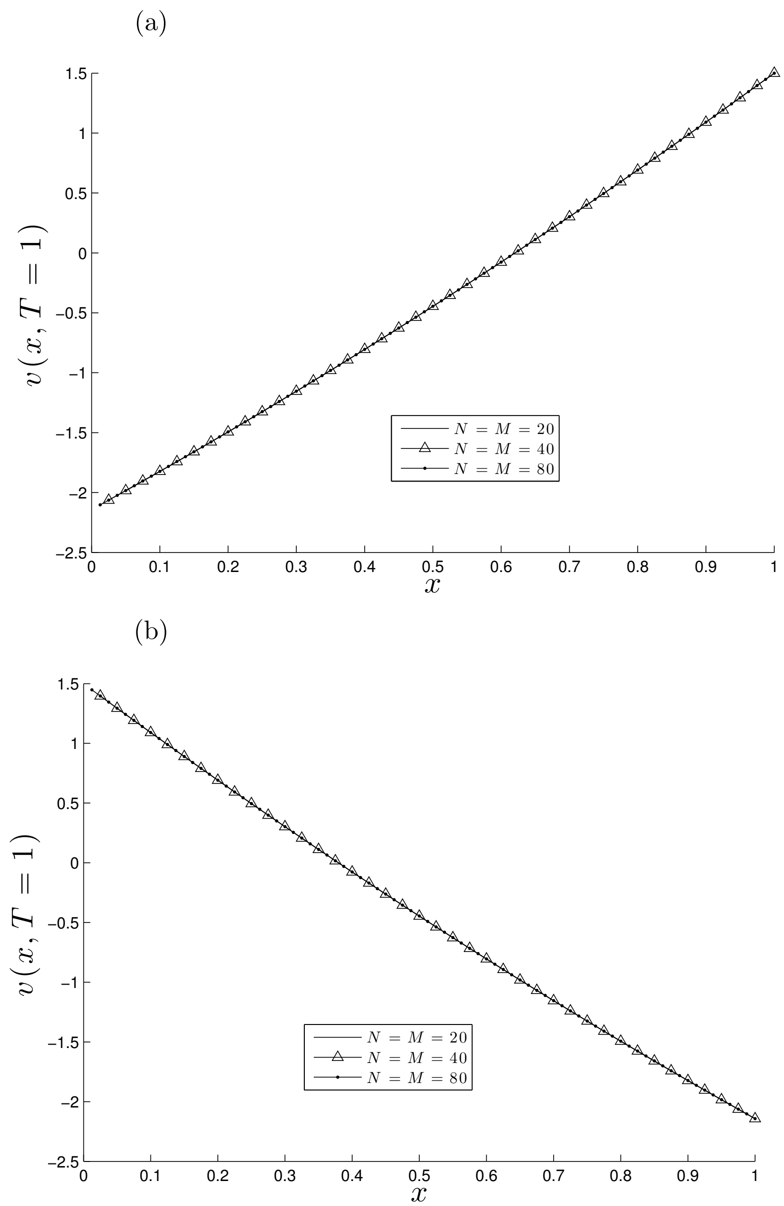


Figure 1: The numerical values of $v_T(x)$ using the boundary condition (22) are shown in (a), whereas the corresponding $v_T(x)$ values when boundary condition (23) is applied, are shown in (b), obtained using the FDM (see [5, 7, 8]) with $M = N \in \{20, 40, 80\}$.

Next step is finding a numerical solution for $\underline{b}$ from equation (15) and for $f(x)$ then compare them with exact one.

In Figure 4(a) the exact solution is $b_k = \sqrt{2}\int_0^1 f(x)\cos\left(\left(k - \frac{1}{2}\right)\pi x\right)dx$. In Figure 4(b) is $b_k = \sqrt{2}\int_0^1 f(x)\sin\left(\left(k - \frac{1}{2}\right)\pi x\right)dx$ (see [4, 6, 7, 8]). From these

Table 1: Condition number of matrix Q.

BC (22) K	$M = 20$	$M = 40$	$M = 80$
5	6666.95	6583.35	6566.18
10	139798.90	131715.67	130607.63
20	$1.28E + 21$	2432128.57	2328937.11
BC (23) K	$M = 20$	$M = 40$	$M = 80$
5	6621.60	6577.98	6565.51
10	132709.88	131061.46	130531.14
20	2380475.12	2337248.57	2320963.73

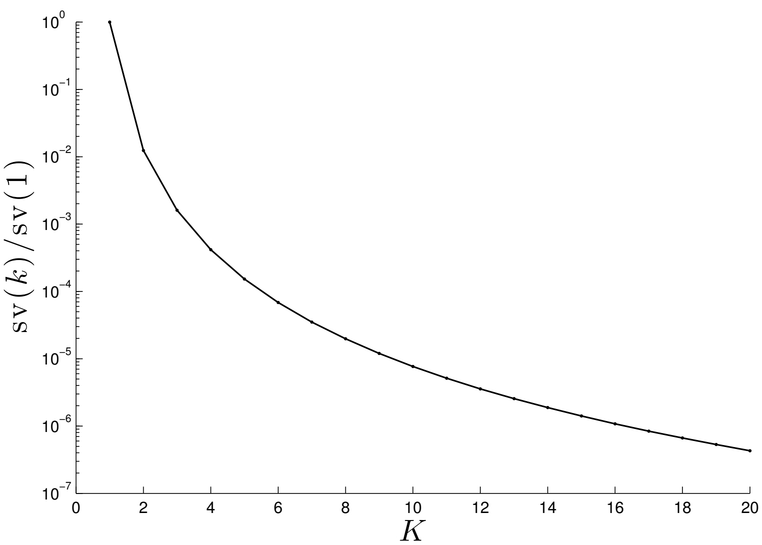


Figure 2: $sv(k)/sv(1)$ normalised singular values for $k = \overline{1, 20}$ and $M = 80$

figures fixed $K = 20, N = M = 80$, whereas different shapes obtain where boundary condition has changed, but still can observe a good approach exist between exact and numerical one.

The value of force $f(x)$ has obtained after substituted numerical values b in equation (12)—$_{\text{or}}$(14) where $k \in \{5, 10, 20\}$ and $N = M = 80$. Furthermore, Figure 6 shows comparative between numerical (12)—$_{\text{or}}$ (14) and exact values (20) of $f(x)$, where applying boundary condition (22) for finding force appear in Figure 6(a), similarly, for boundary condion (23) shown in Figure 6(b). Although, there is unstable from left of the Figure 6(a) and right Figure 6(b), still get quite good approximation where K increase between exact and numerical values. Add noise to the input data (16) to see how far out put data "force" stay stable which clear in Figure 8 oscillations appear high as K increase and amount of noise $p\% = 1\%$. In Figure 8(a) right side and Figure 8(b) left side are more unstable. Consequently, boundary conditions (22) and (23) have researched for achieving Figures 8(a) and 8(b), respectively.

Final step, for stable solution zeroth-order Tikhonov regularization has been applied (19). In order to find good parameter, several parameters λ are tested such as

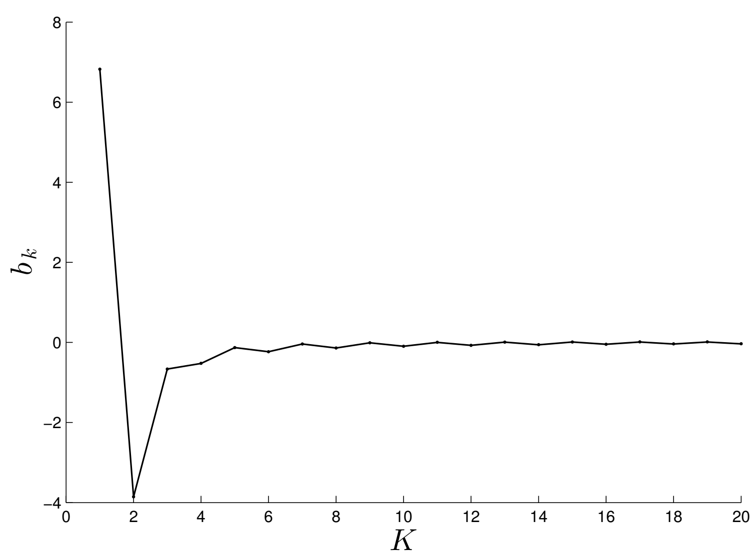
$$\lambda \in \{10^{-4}, 5 \times 10^{-4}, 10^{-3}, 5 \times 10^{-3}, ..., 10^{0}\}$$

and minimum norm error

$$||f_{\text{numerical}} - f_{\text{exact}}|| = \sqrt{\sum_{n=1}^{N} (f_{\text{numerical}}(t_n) - f_{\text{exact}}(t_n))^2}$$

[4, 6, 7, 8] have applied to choose the best one, as shown in Figure 10. In both Figures 10(a) and 10(b) receive $\lambda = 10^{-3}$ best approach $f(x)$ reach with exact solution which is showing in Figure 12. This result is derived from the fact that there are two figures in Figures 10 and 12, using boundary condition (22)$|_{\text{or}}$ (23).

(a)



(a)

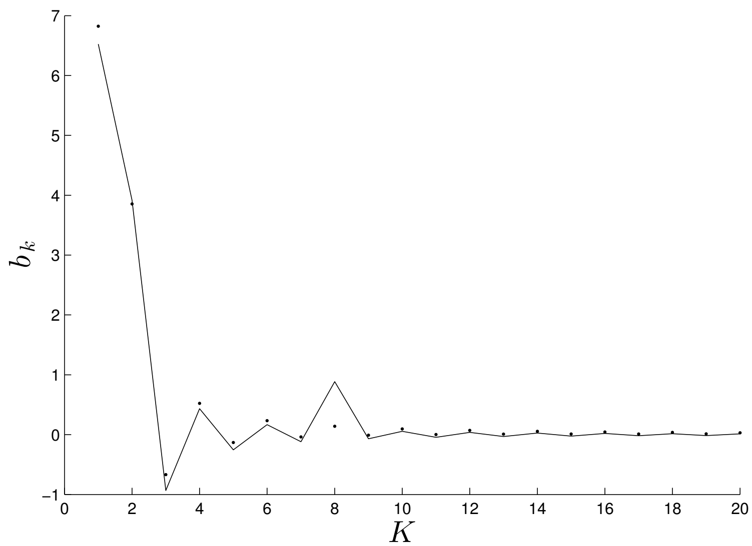
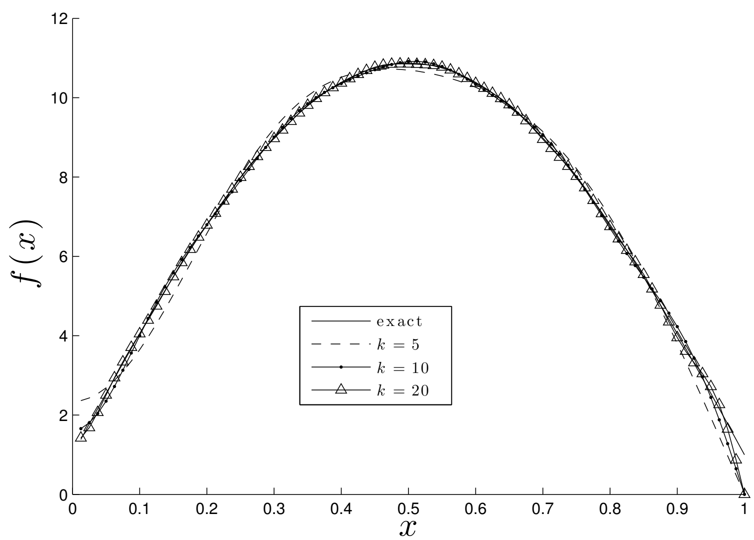


Figure 4: The numerical solution $(\ldots)$ for $\underline{b}$ for $K = 20, N = 80$, obtained from equation (15) in comparison with the exact solution (a) $b_k = \sqrt{2}\int_0^1 f(x)\cos\left(\left(k-\frac{1}{2}\right)\pi x\right)dx$ (-) ,(b) $b_k = \sqrt{2}\int_0^1 f(x)\sin\left(\left(k-\frac{1}{2}\right)\pi x\right)dx$ (-)

(a)



(a)

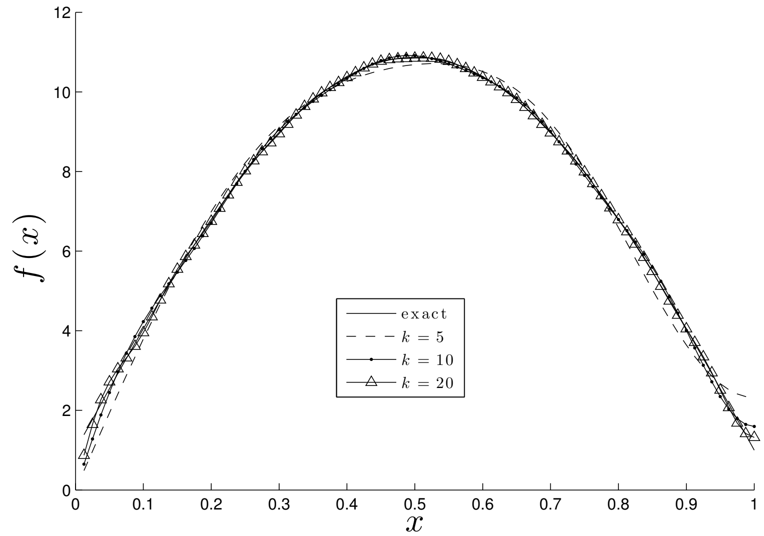
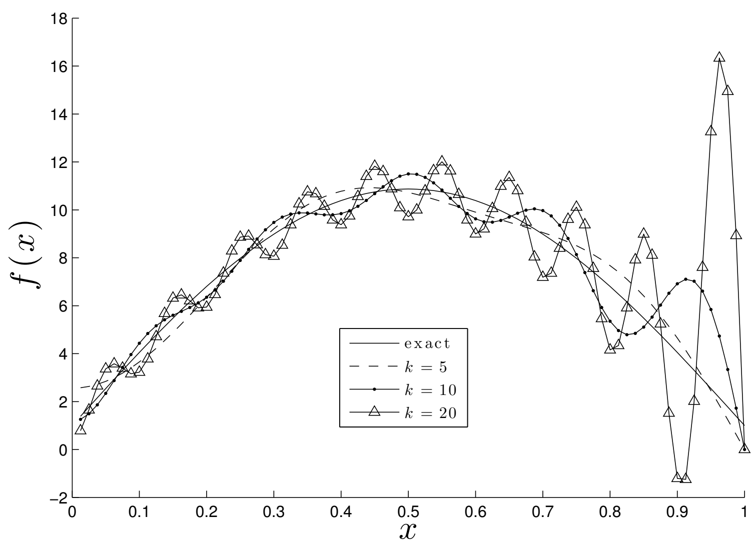


Figure 6: The exact solution (20) for $f(x)$ in comparison with the numerical solution (12)—or(14) (a) with condition (22) (b) with boundary condition (23) are used for various $K \in \{5, 10, 20\}$.

(a)



(a)

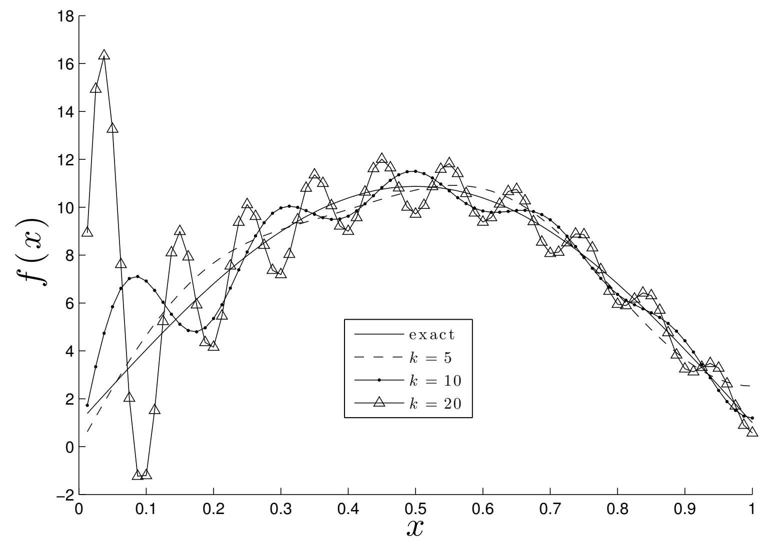
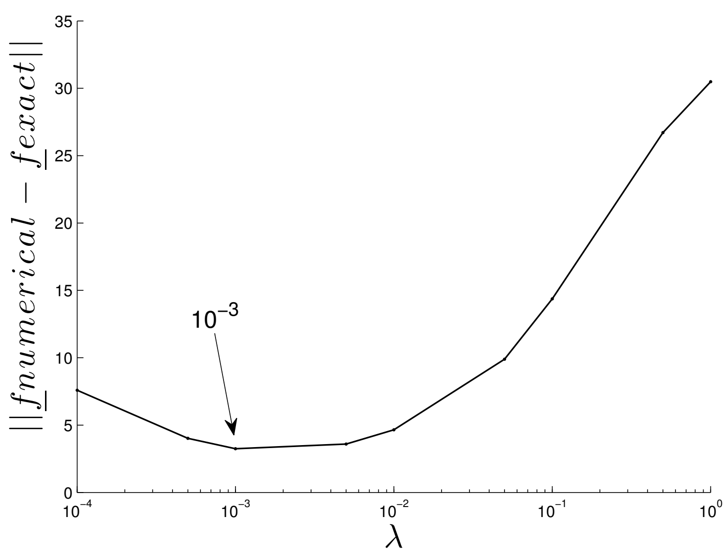


Figure 8: The exact solution (20) for $f(x)$ in comparison with the numerical solution (12)—$_{\text{or}}$(14), for $p\% = 1\%$ and $K \in \{5, 10, 20\}$ using (a) boundary condition (22) (b) boundary condition (23), in noisy data.

(a)



(a)

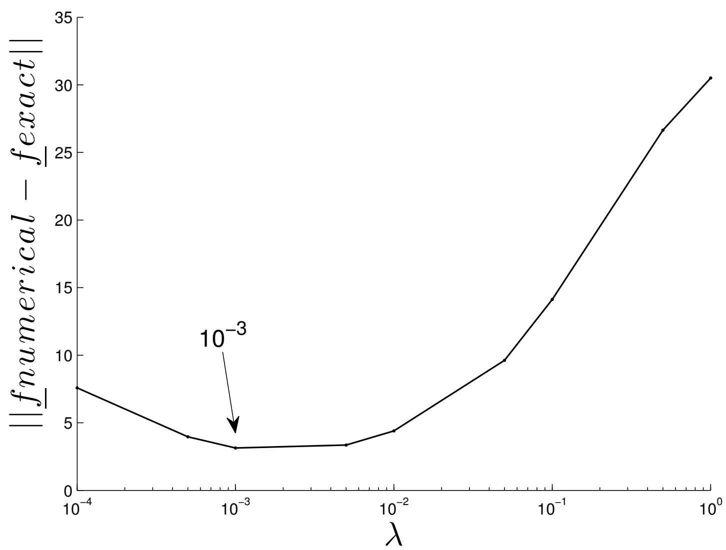
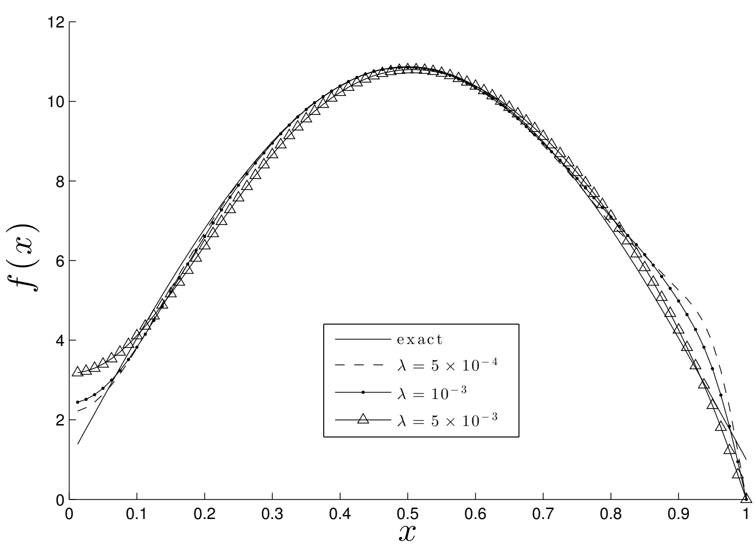


Figure 10: The accuracy error $||f_{numerical} - f_{exact}||$ after applying Tikhonov regularization techniqe, as a function of λ, for $K = 20$ and $p\% = 1\%$ noise, boundary condition are applied (a) (22) (b) (23).

(a)



(a)

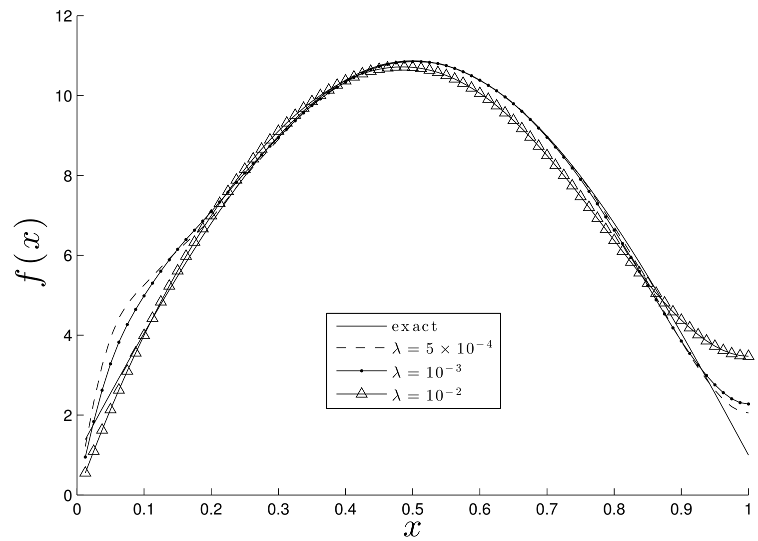


Figure 12: The exact solution (20) for $f(x)$ in comparison with the numerical solution (12)—$_{\text{or}}$(14), for $K = 20$, $p\% = 1\%$ noise, and regularization parameters (a) $\lambda \in \{5 \times 10^{-4}, 10^{-3}, 5 \times 10^{-3}\}$ boundary condition are applied (22) (b) $\lambda \in \{5 \times 10^{-4}, 10^{-3}, 10^{-2}\}$ boundary condition (23).

4. Conclusions

Applying upper-base as additional data has been studying in this research. The non-homogeneous hyperbolic problem has solved by splitting in two parts, part of direct problem and part of inverse problem. Furthermore, the direct problem is well-posed and inverse problem is ill-posed which is unstable small error in upper-base $u_T(x)$ case error in force $f(x)$. The difference from this work in comparison with the other two parts [7, 8] is additional data depended on x "space" whereas the other on t "time". As I said in [7, 8], zeroth-order Tikhonov regularization and minimum norm errors have been used to regularized the solution which is clear from the presented figures.

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