On supra $\widehat{\omega}_p$-Lindelöf spaces

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Abstract. We submit in this research a new kind of supra compact and supra Lindelöf spaces via new kind of open sets in supra topological space, also, to introduce new type of supra continuous functions. Our results supported by examples, remarks and some facts.

Keywords: supra $\widehat{\omega}_p$-open set, supra $\widehat{\omega}_p$-limit point supra $\widehat{\omega}_p$-compact space, supra $\widehat{\omega}_p$-Lindelöf space, supra $\widehat{\omega}_p$-continuous function.

1. Introduction

Mashhour was the first researcher whose defined the supra topological space in (1983) and he introduced the notion of supra continuous functions [1], in which every topological space is supra topological space [2]. After him many researchers dealt with this space such as in [3]. In [4] Sayed, submitted the definition of supra closure and supra interior for any set in a supra space. Also, he posed the notion of supra pre-open set in [5] and he defined it as (any set $\mathfrak{L}$

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of a supra space is named supra pre-open in case $\mathfrak{L} \subseteq$ su.int (su.cl ($\mathfrak{L}$)). In [6], Humadi and Ali defined new set in supra spaces named it supra $\widehat{\omega}$-open set [6], where they defined it as (Any set $\mathfrak{L}$ in a supra space is named supra $\widehat{\omega}$-open set whenever for each $x \in \mathfrak{L}$ we can fined supra open set $\mathfrak{g}$ in X with $x \in \mathfrak{g}$ and $\mathfrak{g} - \mathfrak{L}$ is a countable set). In the other hand, Al-Shami in [7] defined the supra compact space and supra Lindelöf space. And in [8] the researchers defined supra $\widehat{\omega}$-compact space. In this our research we introduce the definition of new set named it supra $\widehat{\omega}$-pre-open set (shortly *su.* $\widehat{\omega}_p$-open) also we defined su. $\widehat{\omega}_p$-compact and su. $\widehat{\omega}_p$-Lindelöf spaces as well as, we defined su. ${}^*\widehat{\omega}_p$-continuous function in the same manner of definition for su.${}^*\widehat{\omega}$-continuous function in [9], [10].

2. Supra $\widehat{\omega}_p$-compact and supra $\widehat{\omega}_p$-Lindelöf spaces

We will define the notions of su. $\widehat{\omega}_p$-compact and su. $\widehat{\omega}_p$-Lindelöf spaces. Also we will show the relation between them. We will use the abbreviation "su" instead of the word "supra".

Definition 2.1. *A subset $\mathfrak{L}$ of a su. space is said to be su. $\widehat{\omega}_p$-open set if for each $x \in \mathfrak{L}$ we can fined a su. pre-open set $\mathfrak{g}$ in X with $x \in \mathfrak{g}$ and $\mathfrak{g} - \mathfrak{L}$ is a countable set.*

Remark 2.2. Each su. open set is su. $\widehat{\omega}_p$-open, this remark is irreversible as in the next example.

Example 2.3. Take indiscrete su. space ($\mathcal{R}$,μ_{ind}), for any point x in $\mathcal{R} - 0$, we can find a unique su. pre-open set $\mathcal{R}$ containing x and $\mathcal{R} - (\mathcal{R} - 0)$ is finite, so it is countable. This implies that $\mathcal{R} - 0$ is su. $\widehat{\omega}_p$-open but not su. open set.

Example 2.4. The set Z^+ in the indiscrete su. space (Z, μ_{ind}) is su. $\widehat{\omega}_p$-open set.

Lemma 2.5. *Suppose (X, μ) is a su. space. Then the union of any family of su. $\widehat{\omega}_p$-open sets is su. $\widehat{\omega}_p$-open.*

Proof. Give $\{U_\beta : \beta \in \Lambda\}$ as a family of su. $\widehat{\omega}_p$-open sets in X and $x \in \bigcup_{\beta\in\Lambda} U_\beta$. Then $x \in U_\alpha$ for some $\alpha \in \Lambda$. Hence there is su. pre-open subset $\mathcal{H}$ of X with $x \in \mathcal{H}$ in which $\mathcal{H} - U_\alpha$ is countable. We know that $\mathcal{H} - \bigcup_{\beta\in\Lambda} U_\beta \subseteq \mathcal{H} - U_\alpha$, then $\mathcal{H} - \bigcup_{\beta\in\Lambda} U_\beta$ is countable. Therefore, $\bigcup_{\beta\in\Lambda} U_\beta$ is su. $\widehat{\omega}_p$-open. □

Definition 2.6. *A su. $\widehat{\omega}_p$-open cover for any set $\mathcal{M}$ in a su. space X is a family $\{\mathfrak{g}_\alpha : \alpha \in \Lambda\}$ of su. $\widehat{\omega}_p$-open sets in in which $\mathcal{M} \subseteq \cup\{\mathfrak{g}_\alpha : \alpha \in \Lambda\}$.*

Definition 2.7. *A su. space X is named su. $\widehat{\omega}_p$-compact (su. $\widehat{\omega}_p$-Lindelöf) whenever each su. $\widehat{\omega}_p$-open cover of X owns a finite (countable) su. sub cover.*

Remark 2.8. If X is su. $\widehat{\omega}_p$-compact (su. $\widehat{\omega}_p$-Lindelöf) space, so it is su. compact (su. Lindelöf) but the convers is not true.

Example 2.9. The su. space $(\mathcal{R}, \mu_{ind})$ is su. compact (su. Lindelöf) but not su. $\widehat{\omega}_p$-compact, since $\{x\}_{x\in\mathcal{R}}$ is su. $\widehat{\omega}_p$-open cover to $\mathcal{R}$ which has no finite (countable) su. sub cover, where is su. $\widehat{\omega}_p$-open (Since $\mu Int(\mu cl\{x\}) = \mathcal{R}$, so $\{x\} \subseteq \mu Int(\mu cl\{x\}) = \mathcal{R}$).

Proposition 2.10. *If X is finite su. space, then it is su. $\widehat{\omega}_p$-compact.*

Proof. Since X is finite, so any su. $\widehat{\omega}_p$-open cover to it is finite. Also X is countable then any su. $\widehat{\omega}_p$-open cover to it is countable. □

Proposition 2.11. *Every su. $\widehat{\omega}_p$-compact space is su. $\widehat{\omega}_p$-Lindelöf.*

Proof. Give $\mathcal{C} = \{G_\alpha, \alpha \in \Lambda\}$ as a su. $\widehat{\omega}_p$-open cover to a su. $\widehat{\omega}_p$-compact space X, so $\mathcal{C}$ owns a finite sub cover to X, then it is countable, therefore X is su. $\widehat{\omega}_p$-Lindelöf. □

Example 2.12. Suppose $\mu_{\mathcal{R}} = \{\phi, \mathcal{R}, G_{\mathrm{j}} \subseteq \mathcal{R} : G_{\mathrm{j}} = \{\mathrm{j}, \mathrm{j}+1, \mathrm{j}+2, ... : \mathrm{j} \in N\}\}$ is su. topology on $\mathcal{R}$, so $(\mathcal{R}, \mu_{\mathcal{R}})$ is su. compact space, but not su. $\widehat{\omega}_p$-compact, since $\mathcal{C} = \{Q^c \cup \{x\} : x \in Q\}$ is su. $\widehat{\omega}_p$-open cover to $\mathcal{R}$. But it has no finite su. sub cover.

Remark 2.13. In case X is su. $\widehat{\omega}_p$-compact space, hence it is su. $\widehat{\omega}_p$-Lindelöf, this remark is irreversible as in above example.

Proposition 2.14. *Each su. $\widehat{\omega}_p$-closed subset of su. $\widehat{\omega}_p$-compact space is su. $\widehat{\omega}_p$-compact.*

Proof. Take $\mathfrak{L}$ as a su. $\widehat{\omega}_p$-closed subset of a su. $\widehat{\omega}_p$-compact space X and $\mathcal{C} = \{V_\alpha : \alpha \in \Lambda\}$ be su. $\widehat{\omega}_p$-open cover to $\mathfrak{L}$, thus $\mathfrak{L} \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$. But $\mathfrak{L}$ is a su. $\widehat{\omega}_p$-closed, so $\mathfrak{L}^c$ is $\widehat{\omega}_p$-open set. Therefore, $X \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha \cup \mathfrak{L}^c$ which is $\widehat{\omega}_p$-compact, so $X \subseteq \bigcup_{i=1}^{n} V_{\alpha_i} \cup \mathfrak{L}^c$, then $\mathfrak{L} \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$ which implies $\mathfrak{L}$ is su. $\widehat{\omega}_p$-compact. The case of su. $\widehat{\omega}_p$-Lindelöf, the proof is similar. □

Proposition 2.15. *A su. space X is su. $\widehat{\omega}_p$-compact if and only if each family of su. $\widehat{\omega}_p$-closed sets in X, satisfies the finite intersection property (shortly FIP), has itself a non-null intersection necessity.*

Proof. Necessity give $\mathcal{C} = \{V_\alpha : \alpha \in \Lambda\}$ as a family of su. $\widehat{\omega}_p$-closed sets in X and has FIP, suppose $\bigcap_{\alpha\in\Lambda} V_\alpha = \phi$, then $\bigcup_{\alpha\in\Lambda} V_\alpha^c = X$. But X is su. $\widehat{\omega}_p$-compact, thus $X = \bigcup_{i=1}^{n} V_{\alpha_i}^c$, so $\bigcap_{i=1}^{n} V_{\alpha_i} = \phi$ which this contradicts with fact that $\mathcal{C}$ has FIP. Therefore, $\mathcal{C}$ owns a non-null intersection. Sufficiency: Give $\mathcal{M} = \{\mathcal{K}_i : i \in I\}$ as a su. $\widehat{\omega}_p$-open cover to X and it owns no finite su. sub cover. Thus $X - \bigcup_{i=1}^{n} \mathcal{K}_i \neq \phi$, so $\bigcap_{i=1}^{n} \mathcal{K}_i^c \neq \phi$, which implies the collection $\{\mathcal{K}_i^c : i \in I\}$ of su. $\widehat{\omega}_p$-closed subsets of X which has FIP, then $\bigcap_{i\in I} \mathcal{K}_i^c \neq \phi$ by hypothesis. So $\bigcup_{i\in I} \mathcal{K}_i \neq X$ which is a contradiction. So $\mathcal{M}$ owns a finite sub cover to X. □

Proposition 2.16. *If $\mathcal{H}$ is a su. $\widehat{\omega}_p$-compact set in a su. space X and $\mathcal{K}$ is su. $\widehat{\omega}_p$-closed set in X, then $\mathcal{H}\cap\mathcal{K}$ is su. $\widehat{\omega}_p$-compact.*

Proof. If $\mathcal{C}=\{U_\alpha:\alpha\in\Lambda\}$ is a su. $\widehat{\omega}_p$-open cover to $\mathcal{H}\cap\mathcal{K}$, then $\mathcal{H}\subseteq(\bigcup_{\alpha\in\Lambda}U_\alpha)\cup\mathcal{K}^c$. Since $\mathcal{H}$ is su. $\widehat{\omega}_p$-compact, then $\mathcal{H}\subseteq(\bigcup_{i=1}^{n}U_{\alpha_i})\cup\mathcal{K}^c$, which implies $\mathcal{H}\cap\mathcal{K}\subseteq(\bigcup_{i=1}^{n}U_{\alpha_i})\cup\mathcal{K}^c\cap\mathcal{K}$. Hence, $\mathcal{H}\cap\mathcal{K}$ is su. ω_p-compact. □

Corollary 2.17. *If $\mathcal{H}$ is a su. $\widehat{\omega}_p$-Lindelöf and $\mathcal{K}$ is a su. $\widehat{\omega}_p$-closed set in a su. space X, thus $\mathcal{H}\cap\mathcal{K}$ is su. $\widehat{\omega}_p$-Lindelöf.*

Proof. Is similar to Proposition 2.16 □

Definition 2.18. *A function h from a su. space X into a su. space Y is named su.$^*\widehat{\omega}_p$-continuous whenever $h^{-1}(\mathfrak{L})$ is su. $\widehat{\omega}_p$-open in X for any su. open set $\mathfrak{L}$ in Y.*

Proposition 2.19. *If a function h from a su. space X into a su. space Y is su.$^*\widehat{\omega}_p$-continuous and $\mathfrak{L}$ is su. $\widehat{\omega}_p$-compact set in X, then $h(\mathfrak{L})$ is su. compact.*

Proof. Give $\mathfrak{L}$ is a su. $\widehat{\omega}_p$-compact set in a su. space X and $\mathcal{C}=\{\bigcup_\alpha:\alpha\in\Lambda\}$ is a su. open cover for $h(\mathfrak{L})$, which means $h(\mathfrak{L})\subseteq(\bigcup_{\alpha\in\Lambda}U_\alpha)$, then $\mathfrak{L}\subseteq h^{-1}(f(\mathfrak{L}))\subseteq h^{-1}(\bigcup_{\alpha\in\Lambda}U_\alpha)$. But $\mathfrak{L}$ is su. $\widehat{\omega}_p$-compact and $h^{-1}(U_\alpha)$ is su. $\widehat{\omega}_p$-open set, since h is su. $^*\widehat{\omega}_p$-continuous, so $\mathfrak{L}\subseteq\bigcup_{i=1}^{n}h^{-1}(U_{\alpha_i})$, then $h(\mathfrak{L})\subseteq h(\bigcup_{i=1}^{n}h^{-1}(U_{\alpha_i}))\subseteq\bigcup_{i=1}^{n}h(h^{-1}(U_{\alpha_i}))\subseteq\bigcup_{i=1}^{n}U_{\alpha_i}$. Therefore, $h(\mathfrak{L})$ is su. compact. □

Proposition 2.20. *If f from a su. space X onto a su. space Y and if X is su. $\widehat{\omega}_p$-Lindelöf, then Y is a su. $\widehat{\omega}_p$-Lindelöf, whenever f is su. $^*\widehat{\omega}_p$-continuous.*

Proof. Is similar to above proposition 2.19 □

Definition 2.21. *A function h from a su. space X into a su. space Y is called su. $^*\widehat{\omega}_p^*(\widehat{\omega}_p^{**})$-continuous, whenever $h^{-1}(\mathfrak{L})$ is su. open (su. $\widehat{\omega}_p$-open)in X for any su. $\widehat{\omega}_p$-open set $\mathfrak{L}$ in Y.*

Proposition 2.22. *If h from a su. space X onto a su. space Y, and X is su. Lindelöf (su. $\widehat{\omega}_p$-Lindelöf), then Y is su. $\widehat{\omega}_p$-Lindelöf, whenever h is su.$^*\widehat{\omega}_p^*(\widehat{\omega}_p^{**})$-continuous.*

Proof. Let $\mathcal{C}=\{G_\alpha:\alpha\in\Lambda\}$ be a su. $\widehat{\omega}_p$-open cover to Y, thus $Y=\bigcup_{\alpha\in\Lambda}G_\alpha$. Thus $h^{-1}(Y)=h^{-1}(\bigcup_{\alpha\in\Lambda}G_\alpha)$, hence $X=\bigcup_{\alpha\in\Lambda}(h^{-1}(G_\alpha))$. But h is su.$^*\widehat{\omega}_p^*(\widehat{\omega}_p^{**})$-continuous, so $h^{-1}(G_\alpha)$ is su. $\widehat{\omega}_p$-open set in X for all, but X is su. Lindelöf (su. $\widehat{\omega}_p$-Lindelöf). Then $X\subseteq\bigcup_{i=1}^{n}(h^{-1}(G_{\alpha_i})$, so $Y=h(X)\subseteq h\bigcup_{i=1}^{n}(h^{-1}(G_{\alpha_i})=\bigcup_{i=1}^{n}(h(h^{-1}(G_{\alpha_i})))=\bigcup_{i=1}^{n}(G_{\alpha_i})$. Since h is onto, then Y is su. $\widehat{\omega}_p$-Lindelöf. □

Definition 2.23. *Any point $y\in X$ is named su. $\widehat{\omega}_p$-limit point to a subset $\mathfrak{L}$ of a su. space X, if every su. $\widehat{\omega}_p$-open set $\mathcal{K}$ where $y\in\mathcal{K}$ has non-empty intersection with $\mathfrak{L}$ whenever y has been select. The set of all su. $\widehat{\omega}_p$-limit points to $\mathfrak{L}$ is denoted by $\mu\widehat{\omega}_p-d(\mathfrak{L})$.*

Proposition 2.24. *Any set $\mathcal{D}$ of su. space X is su. $\widehat{\omega}_p$-closed if and only if su. $\mu\widehat{\omega}_p - d(\mathcal{D}) \subseteq \mathcal{D}$.*

Proof. Necessity: Let $\mathcal{D}$ be a su. $\widehat{\omega}_p$-closed set, and let $x \in \mu\widehat{\omega}_p - d(\mathcal{D})$. Suppose $x \notin \mathcal{D}$ then $x \in \mathcal{D}^c$ which is su. $\widehat{\omega}_p$-open set and $\mathcal{D} \cap \mathcal{D}^c = \phi$, then x is not su. $\widehat{\omega}_p$-limit point to $\mathcal{D}$, so it is contradicts with $x \in \mu\widehat{\omega}_p - d(\mathcal{D})$, then $\mu\widehat{\omega}_p - d(\mathcal{D}) \subseteq \mathcal{D}$. Sufficiency: To show that $\mathcal{D}$ is su. $\widehat{\omega}_p$-closed, if $x \in \mathcal{D}^c$ and we have $\mu\widehat{\omega}_p - d(\mathcal{D}) \subseteq \mathcal{D}$, then x is not su. $\widehat{\omega}_p$-limit point to $\mathcal{D}$, so there is a su. $\widehat{\omega}_p$-open U with $x \in \mathcal{D}$ and $(U - \{x\}) \cap \mathcal{D} = \phi$, but $x \notin \mathcal{D}$, then $U \cap \mathcal{D} = \phi$ and $U \subseteq \mathcal{D}^c$, so $\mathcal{D}^c = \cup\{U_x : x \in \mathcal{D}^c$. Thus $\mathcal{D}^c$is su. $\widehat{\omega}_p$-open (by Lemma 2.5), therefor $\mathcal{D}$ is su. $\widehat{\omega}_p$-closed. □

Proposition 2.25. *Any set G in a su. space X is su. $\widehat{\omega}_p$-open set if and only if all its points are su. $\widehat{\omega}_p$-interior points to G.*

Proof. Suppose G is su. $\widehat{\omega}_p$-open subset of X and $x \in G$, since each set is a subset of itself, so x is su. $\widehat{\omega}_p$-interior point to G, but x is arbitrary point in G, hence all the points in G are su. $\widehat{\omega}_p$-interior points to G. Conversely, since $G = \bigcup_{x \in G}\{x\}$, and every point in G is su. $\widehat{\omega}_p$-interior point to it, thus for all $x \in G$ we can find su. $\widehat{\omega}_p$-open set U_x in X with $x \in U_x \subseteq G$, in order that, G is a union of the su. $\widehat{\omega}_p$-open sets U_x for any $x \in G$, hence G is su. $\widehat{\omega}_p$-open set (by Lemma 2.5). □

Proposition 2.26. *Whenever H is an infinite set in a su. $\widehat{\omega}_p$-compact space X, hence H has a su. $\widehat{\omega}_p$-limit point.*

Proof. Suppose H is an infinite subset of su. $\widehat{\omega}_p$-compact space X. Let H has no su. $\widehat{\omega}_p$-limit point, then there exist a su. $\widehat{\omega}_p$-open set $U_{(x_i)}$ for each $x_i \in X$ such that $(U_{x_i} - \{x\}) \cap H = \phi$. But the hole space is a union of its points, so $X = \cup\{U_{x_i} : x_i \in H\}$ and $\{U_{x_i} : x_i \in X\}$ form an su. $\widehat{\omega}_p$-open cover to X, which is su. $\widehat{\omega}_p$-compact space, then $X = \bigcup_{i=1}^{n} U_{x_i}$. So X is finite, then H is also finite which is contracted with the fact that H is infinite, then H has an su. $\widehat{\omega}_p$-limit point. □

Definition 2.27. *A function h from a su. space X into a su. space Y is called su. ${}^*\widehat{\omega}_p$-open (su. ${}^*\widehat{\omega}_p^*$-open) if $h(\mathfrak{L})$ is su. $\widehat{\omega}_p$-open (su. open) in Y for any su. open (su. $\widehat{\omega}_p$-open) set $\mathfrak{L}$ in X.*

Proposition 2.28. *If a bijective function h from a su. space X into a su. space Y is su. ${}^*\widehat{\omega}_p^*$-open function, then X is su. $\widehat{\omega}_p$-Lindelöf (su. $\widehat{\omega}_p$-compact) whenever Y is su. Lindelöf (su. compact).*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a su. $\widehat{\omega}_p$-open cover to X, thus $X = \bigcup_{\alpha \in \Lambda} V_\alpha$, so $h(X) = h(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} h(V_\alpha)$. But Y is su. Lindelöf, so $Y = \bigcup_{\alpha \in \delta} h(V_\alpha)$ for some countable set $\delta \in \alpha$, then

$$X = h^{-1}(Y) = h^{-1} \bigcup_{\alpha \in \delta} h(V_\alpha) = \bigcup_{\alpha \in \delta} h^{-1}(h(V_\alpha)) = \bigcup_{\alpha \in \delta} V_\alpha,$$

so X is su. $\widehat{\omega}_p$-Lindelöf.

In case of compactness the proof is similar. $\square$

Proposition 2.29. *Every countable space is su. $\widehat{\omega}_p$-Lindelöf.*

Proof. Since every su. $\widehat{\omega}_p$-open cover to it must be countable. $\square$

Lemma 2.30. *If (X, μ) is a su. space and $\mu' \subseteq \mu$, and if G is su. pre-open set in (X, μ'), thus it is su. pre-open in (X, μ).*

Proof. Take G as su. pre-open in (X, μ'), then $G \subseteq \mu' - int^{\mu}(\mu' - cl^{\mu}(G))$. But $\mu' \subseteq \mu$, then $G \subseteq \mu - int^{\mu}(\mu - cl^{\mu}(G))$. So G is su. pre-open set in (X, μ). $\square$

Lemma 2.31. *Suppose (X, μ) is a su. space and $\mu' \subseteq \mu$, whenever $\mathfrak{L}$ is a su. $\widehat{\omega}_p$-open set in (X, μ'), hence it is su. $\widehat{\omega}_p$-open in (X, μ).*

Proof. If $\mathfrak{L}$ is su. $\widehat{\omega}_p$-open set in (X, μ') and let $x \in \mathfrak{L}$, then there is a su. pre-open set $\mathcal{K}$ in (X, μ') where $x \in \mathcal{K}$ and $\mathcal{K} - \mathfrak{L}$ =countable. But $\mathcal{K}$ is also su. pre-open in (X, μ), so $\mathfrak{L}$ is su. $\widehat{\omega}_p$-open in (X, μ). $\square$

Proposition 2.32. *Suppose (X, μ) is a su. $\widehat{\omega}_p$-Lindelöf space, whenever μ' is coarser than μ, hence (X, μ') is su. $\widehat{\omega}_p$-Lindelöf.*

Proof. Give $\mathcal{C} = \{U_\alpha : \alpha \in \Lambda\}$ as a su. $\widehat{\omega}_p$-open cover to (X, μ'), so by Lemma 2.31 $\mathcal{C}$ is su. ω_p-open cover to (X, μ), which is su. $\widehat{\omega}_p$-Lindelöf, then $\mathcal{C}$ owns a countable sub cover, that is (X, μ') is su. $\widehat{\omega}_p$-Lindelöf. $\square$

Corollary 2.33. *Suppose (X, μ) is a su. $\widehat{\omega}_p$-Lindelöf space and if μ' is coarser than μ, hence (X, μ') is Lindelöf.*

Proof. Take $\mathcal{C} = \{U_{\alpha \in \Lambda} : \alpha \in \Lambda\}$ as an μ'-open cover to (X, μ'), but $\mu' \subseteq \mu$ then $\mathcal{C}$ is μ-open cover to (X, μ), so $\mathcal{C}$ is a su. $\widehat{\omega}_p$-open cover to (X, μ) (Each open set is su. $\widehat{\omega}_p$-open). But (X, μ) is su. $\widehat{\omega}_p$-Lindelöf, then $\mathcal{C}$ owns a countable sub cover, that is (X, μ') is su. Lindelöf space. $\square$

Remark 2.34. Not necessary that any su. sub space of su. $\widehat{\omega}_p$-Lindelöf is su. $\widehat{\omega}_p$-Lindelöf space too.

Example 2.35. Take X as an uncountable set and μ consist of X and the subsets of X which not contain x_0 belong to X. If $\mathcal{M}$ is a su. $\widehat{\omega}_p$-open cover to X, also the only su. ω_p-open set which contain x_0 is just X. So $X \in \mathcal{M}$, thus $\{X\}$ is countable sub cover of X. It follows that X is su. $\widehat{\omega}_p$-Lindelöf. But $Y = X - \{x_0\}$ is not su. $\widehat{\omega}_p$-Lindelöf subspace, since $\{y : y \in Y\} = \mathcal{M}^*$ is a su. $\widehat{\omega}_p$-open cover to Y which has no countable sub cover (since $\{y\}$ is su. $\widehat{\omega}_p$-open set in X and $\{y\} = Y \cap \{y\}$, so $\{y\}$ is su. $\widehat{\omega}_p$-open in Y, then μ_Y is the discrete topology. But $\mathcal{M}^*$ is uncountable su. $\widehat{\omega}_p$-open cover to Y and $\mathcal{M}^*$ has no proper sub cover, where if we select one member of $\mathcal{M}^*$ the reminder is not cover to Y. That is $\mathcal{M}^*$ has no a countable sub cover to Y, therefore Y is not su. $\widehat{\omega}_p$-Lindelöf.

Proposition 2.36. *Any su. $\widehat{\omega}_p$-closed subspace of su. $\widehat{\omega}_p$-Lindelöf space is su. $\widehat{\omega}_p$-Lindelöf.*

Proof. Let Y be su. $\widehat{\omega}_p$-closed subset of a su. $\widehat{\omega}_p$-Lindelöf space X, and $= \{V_\alpha : \alpha \in \Lambda\}$ be an $\mu_Y - \widehat{\omega}_p$-open cover to Y. Then there exist $\mu_X - \widehat{\omega}_p$-open sets U_α such that $V_\alpha = U_\alpha \cap Y$ that is, $\{U_\alpha : \alpha \in\}$ be a $\mu_X - \widehat{\omega}_p$-open cover to Y. Then $X = \{U_\alpha : \alpha \in \Lambda\} \cup Y^c$ is $\mu_X - \widehat{\omega}_p$-open cover to X, but it is su. ω_p-Lindelöf, thus we can find a countable set $j \subseteq \Lambda$ in which $X = \{U_\alpha : \alpha \in j\} \cup Y^c$, then $Y = \{U_\alpha : \alpha \in j\}$. Therefore, Y is su. $\widehat{\omega}_p$-Lindelöf. □

Definition 2.37. *A su. topological space X is said to be su. locally $\widehat{\omega}_p$-compact (for short $\mu\ell\widehat{\omega}_p$-compact) if every $x \in X$ owns a su. neighbourhood its su. closure is su. $\widehat{\omega}_p$-compact.*

Proposition 2.38. *Whenever X is su. $\widehat{\omega}_p$-compact space, then it is $\mu\ell\widehat{\omega}_p$-compact.*

Proof. Take X as a su. $\widehat{\omega}_p$-compact space. If we take any point x in hole su. space X, we have X su. clopen, so X a su. neighbourhood and it is equal to its su. closure, which is su. $\widehat{\omega}_p$-compact, then it is $\mu\ell\widehat{\omega}_p$-compact. □

This proposition is irreversible.

Example 2.39. The su. discrete space $(\mathcal{R}, \mu_D)$ is $\mu\ell\widehat{\omega}_p$-compact space but not su. $\widehat{\omega}_p$-compact.

Proposition 2.40. *Every su. closed subspace of $\mu\ell\widehat{\omega}_p$-compact space is $\mu\ell\widehat{\omega}_p$-compact.*

Proof. Take Y as su. $\widehat{\omega}_p$-closed subspace of $\mu\ell\widehat{\omega}_p$-compact space X. Also $P \in Y \subseteq X$, then $P \in X$. So, there is a su. open neighbourhood G to P in X with su. $cl(G)$ is su. $\widehat{\omega}_p$-compact. Since $G \cap Y$ is su. open neighbourhood of P in Y and $G \cap Y \subset G$ and su. $cl(G \cap Y) \subset cl(G)$. But su. $cl(G \cap Y)$ is su. closed, so it is su. $\widehat{\omega}_p$-closed subset of su. $\widehat{\omega}_p$-compact su. $cl(G)$, that is su. $cl(G \cap Y)$ is su. $\widehat{\omega}_p$-compact since su. $cl(G \cap Y)$ is su. closed in X, and a subset of Y, then $cl_{inX}(G \cap Y) = cl_{inY}(G \cap Y)$ so it is su. $\widehat{\omega}_p$-compact in Y. So we have every point in Y has a su. neighbourhood its su. closure is su. $\widehat{\omega}_p$-compact. □

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