Solving oscillation problems using optimized integrator method

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Abstract. In this paper, an explicit optimized integrator method of order four is developed for solving second order ordinary differential equations with oscillatory solutions. The new optimized integrator method (NOIM) depend on the existing hybrid methods with dissipative of order infinity. The constant coefficients of new method is found after using the phase-lag, the amplification error (dissipative error) and the first derivative of the phase-lag. Numerical results are presented to illustrate the robustness and competency of the proposed integrator method compared with the existing methods in the scientific literature for solving oscillatory problems.

Keywords: oscillatory problem, explicit hybrid method, dissipative error, phase-lag, numerical solution.

1. Introduction

This paper devotes to the initial value problems (IVPs) of the form

$$z''(t) = g(t, z), \quad z(t_0) = z_0, \quad z'(t_0) = z'_0. \tag{1}$$

Where the solution of (1) shows an obvious oscillatory behaviour. This problems arise in a several of sciences fields like theoretical physics, quantum mechanics, electronics, celestial mechanics (see [1, 2]). Many numerical methods are derived to approximate the solutions for (1) such as Runge-Kutta method [3] after reducing the problem to an equivalent system of the first order (ODEs) and then solved it. But several authors are proposed direct integration method, for instance Runge-Kutta Nyström method and multistep methods for second order ODEs to obtain the best approximation for the solution of second order ODEs (1), we can find such work in Simos ([4, 5, 6]), Ming et al. [7], and Tsitouras [8]. Franco [10] proposed a class up to six order hybrid methods using the order conditions constructed by Coleman [9]. Searching for the dispersion and dissipation errors of numerical methods for solving oscillatory or periodic problems is extremely significant. Several authors have proposed hybrid methods with the objective to make the dispersion of the method smaller (see [11, 12, 13, 14]) In this paper, we construct a new optimized integrator method based on existing fourth order hybrid method which is dispersive of order six and zero dissipative proposed by Franco [10] and using the technique suggested by Kosti et al. [15]. The paper is organized as follows: In section 2, we present the dispersion and dissipation analysis of the integrator method. In section 3 we describe the derivation of the method. In section 4, we give numerical results that prove the efficiency of our new optimized integrator method by the well known oscillating problems. In Section 5 we observe some conclusions.

2. Phase-Lag analysis of NOIM method

The hybrid method for solving second order ODEs (1) can be written in following form

$$z_{n+1} = 2z_n - z_{n-1} + h^2 \sum_{i=1}^{s} b_i g(t_n + c_i h), \tag{2}$$

$$z_i = (1 + c_i)z_n - c_i z_{n-1} + +h^2 \sum_{j=1}^{s} a_{ij} g(t_n + c_j h). \quad i = 1, \cdots, s. \tag{3}$$

where the parameters b_i, a_{ij}, and c_i of the hybrid methods are expressed using Butcher notation as follows we can defined the methods of the form (2)–(3) as

$$\begin{array}{c|c} \mathrm{c} & \mathrm{A} \\ \hline & b^T \end{array} = \begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

follows

$$Z_1 = z_{n-1}, \quad Z_2 = z_n, \tag{4}$$

$$z_{n+1} = 2z_n - z_{n-1} + h^2 \left[b_1 g_{n-1} + b_2 g_n + \sum_{i=3}^{s} b_i g(t_n + c_i h, Z_i) \right], \tag{5}$$

$$Z_i = (1 + c_i)z_n - c_i\, z_{n-1} + h^2 \sum_{j=3}^{i-1} a_{ij} g(t_n + c_j h, Z_j). \quad i = 3, \cdots, s. \tag{6}$$

where $g_n = g(t_n, z_n)$ and $g_{n-1} = g(t_{n-1}, z_{n-1})$ and the two first nodes are $c_1 = -1$ and $c_2 = 0$.

The method need only the evaluation of $g(t_n, z_n)$, $g(t_n + c_3 h, Z_3), \dots, g(t_n + c_s h, Z_s)$ for every step after the proceeding start. Thus, hybrid method is considered with $s-1$ stages for each step which can express in the following Butcher notation The construction of NOIM method is based on the using simple test equation

$$z'' = -\omega^2 z, \quad \omega > 0 \tag{7}$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccccc} -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ c_3 & a_{31} & a_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \dots & b_{s-1} & b_s \end{array}$$

Applying the two step hybrid method (2)–(3) to the test equation (7) we obtain

$$z_{n+1} = 2z_n - z_{n-1} - h^2 \sum_{i=1}^{s} b_i \omega^2 z_i, \tag{8}$$

$$z_i = (1 + c_i)z_n - c_i z_{n-1} - h^2 \sum_{j=1}^{s} a_{ij} \omega^2 z_j. \quad i = 1, \cdots, s. \tag{9}$$

The hybrid method (8)–(9) can be expressed in the following vector form

$$z_{n+1} = 2z_n - z_{n-1} - v^2 b^T Z, \quad v = h\omega, \tag{10}$$

$$Z = (c + e)z_n - c\, z_{n-1} + v^2 AZ, \tag{11}$$

where $e = (1, \dots, 1)^T, c = (c_1, \dots, c_s)^T, Z = (Z_1, \dots, Z_s)^T, b = (b_1, \dots, b_s)$ and

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{bmatrix}.$$

Solving equation (11) we get

$$Z = \left(1 + v^2 A\right)^{-1} (c + e)z_n - \left(1 + v^2 A\right)^{-1} c\, z_{n-1}. \tag{12}$$

Substituting equation (12) into equation (10) we get the following difference equation

$$z_{n+1} - P\left(v^2\right) z_n + D\left(v^2\right) z_{n-1} = 0, \tag{13}$$

where

$$P\left(v^2\right) = 2 - v^2 b^T \left(1 + v^2 A\right)^{-1} (c + e), D\left(v^2\right) = 1 - v^2 b^T \left(1 + v^2 A\right)^{-1} c. \tag{14}$$

Definition. The dispersion (or phase-lag error) and the dissipation (or amplification error) are defined respectively as follows (see [10, 15])

$$\alpha(v) = 1 - \sqrt{D(v^2)}, \quad \Psi(v) = v - \cos^{-1}\left(\frac{P(v^2)}{2\sqrt{D(v^2)}}\right), \tag{15}$$

If $\alpha(v) = \mathrm{O}(v^{r+1})$ and $\Psi(v) = \mathrm{O}(v^{p+1})$ therefore the hybrid method is said to have dispersion of order p and dissipative of order r respectively.

3. Derivation of the NOIM method

New optimized integrator method (NOIM) of order four with four stage will be derived based on existing hybrid method proposed by Franco [10] (see Table 1). Alternatively (14) can be written in polynomial form as follows:

$$P\left(v^2\right) = 2 - \beta_1 v^2 + \beta_2 v^4 - \beta_3 v^6 + \beta_4 v^8 + \ldots + \beta_n v^{2i}, \tag{16}$$

$$D\left(v^2\right) = 1 - \gamma_1 v^2 + \gamma_2 v^4 - \gamma_3 v^6 + \gamma_4 v^8 + \ldots + \gamma_n v^{2i}. \tag{17}$$

To construct the (NOIM) method of order four with four stage, the equations (16)-(17) should be satisfied the order conditions given in [9] up to fourth order. Consequently, we obtain

$$\begin{aligned} P\left(v^2\right) = 2 + \Big(& - b_1 - b_2 - b_3 - b_4 - b_1c_1 - b_2c_2 - b_3c_3 - b_4c_4\Big)v^2 + \Big(b_3a_{31} + \\ & b_3a_{32} + b_4a_{41} + b_4a_{42} + b_4a_{43} + b_3a_{31}c_1 + b_3a_{32}c_2 + \\ & b_4a_{41}c_1 + b_4a_{42}c_2 + b_4a_3c_3\Big)v^4 + \\ & \Big(- b_4a_{43}a_{31} - b_4a_{43}a_{32} - b_4a_{43}a_{31}c_1 - b_4a_{43}a_{32}c_2\Big)v^6, \end{aligned} \tag{18}$$

$$\begin{aligned} D\left(v^2\right) = 1 + \Big(- b_1c_1 - b_2c_2 - b_3c_3 - b_4c_4\Big)v^2 + \Big(b_3a_{31}c_1 + b_3a_{32}c_2 + \\ b_4a_{41}c_1 + b_4a_{42}c_2 + b_4a_{43}c_3\Big)v^4 + \Big(- b_4a_{43}a_{31}c_1 - b_4a_{43}a_{32}c_2\Big)v^6. \end{aligned} \tag{19}$$

Table 1: Franco's hybrid method

-1	0	0	0	0
0	0	0	0	0
$\frac{33}{50}$	0	$\frac{2739}{5000}$	0	0
$-\frac{13}{17}$	$\frac{314860}{20796729}$	$-\frac{1058746}{8268579}$	$\frac{15743000}{686292057}$	0
	$-\frac{89}{1992}$	$\frac{545}{858}$	$\frac{625000}{3316929}$	$\frac{83521}{377832}$

To achieve our goal which is to optimize the Franco's hybrid method given in Table 1, The following conditions should be met

$$1 - \ \eta(v) = v - \cos^{-1}\left(\frac{P(v^2)}{2\sqrt{D(v^2)}}\right) = 0, \tag{20}$$

$$2 - \ \alpha(v) = 1 - \sqrt{D(v^2)} = 0, \tag{21}$$

$$3 - \ \eta(v)' = 0. \tag{22}$$

Substituting the parameters of Franco's hybrid method in Table 1 into equations (20)-(22) and the coefficients a_{41}, a_{42} and a_{43} are chosen as free parameters. Solving the system simultaneously we get the new optimized integrator method (NOIM) with the parameters a_{41}, a_{42} and a_{43} which give by

$$\begin{aligned}
a_{41} =& \frac{800}{1199278039\, v^6}\Big(\sqrt{47229}\, v\sqrt{1413516741 - 1413516741\, (\cos(v))^2 +}\\
& \quad 32682468\, \cos(v) - 32682468 + 8170617\, v^2\Big),\\
a_{42} =& \frac{8}{476821389\, v^6}\Big(1617782166\, \cos(v) v^2 - 2434843866\, v^2 + 539260722\, v^4 -\\
& 27831375\, v^6 - 100\sqrt{47229}\, v\sqrt{1413516741 - 1413516741\, (\cos(v))^2 -}\\
& \quad 3268246800\, \cos(v) + 3268246800 +\\
& \quad 33\sqrt{47229} v^3\sqrt{1413516741 - 1413516741\, (\cos(v))^2}\Big),\\
a_{43} =& \frac{40000}{39576175287\, v^6}\Big(\sqrt{47229}\, v\sqrt{1413516741 - 1413516741\, (\cos(v))^2 +}\\
& \quad 32682468\, \cos(v) - 32682468 + 8170617\, v^2\Big).
\end{aligned}$$

Consequently, the new optimized integrator method (NOIM) of four-order with four stage as follows Taylor series expansion of the free parameters a_{41}, a_{42} and

-1	0	0	0	0
0	0	0	0	0
$\frac{33}{50}$	0	$\frac{2739}{5000}$	0	0
$-\frac{13}{17}$	a_{41}	a_{42}	a_{43}	0
	$-\frac{89}{1992}$	$\frac{545}{858}$	$\frac{625000}{3316929}$	$\frac{83521}{377832}$

a_{43} is practical and is given as follows

$$\begin{aligned}
a_{41} =& \frac{314860}{20796729} - \frac{11245}{20796729}v^2 + \frac{2249}{249560748}v^4 - \frac{2249}{24706514052}v^6\\
&+ \frac{865}{1383564786912}v^8 - \frac{173}{55342591476480}v^{10} + O\left(v^{12}\right),\\
a_{42} =& -\frac{1058746}{8268579} + \frac{11245}{8268579}v^2 - \frac{245141}{992229480}v^4 + \frac{150683}{28891387800}v^6\\
&- \frac{1601807}{27504601185600}v^8 + \frac{46537}{110018404742400}v^{10} + O\left(v^{12}\right),
\end{aligned}$$

$$a_{43}=\frac{15743000}{686292057}-\frac{562250}{686292057}v^2+\frac{56225}{4117752342}v^4-\frac{56225}{407657481858}v^6 +\frac{21625}{22828818984048}v^8-\frac{865}{182630551872384}v^{10}+O\left(v^{12}\right).$$

4. Problems tested

Some oscillatory problems are solved to show the accuracy of NOIM method in this section. We are compared the numerical results with the effective methods in the interval from 0 to 1000. In the comparison we have chosen the following acronyms:

- NOIM4: New optimized integrator method of four order developed in this paper.
- PFRKN4: Existing Phase fitted Runge-Kutta Nyström of four order method presented in [16].
- RKN5: Existing five order Runge-Kutta Nyström method in [2].

The measurement of the eficiency is tested by using the following absolute error

$$\text{Absolute error} = \max\left\{|z(t_n)-z_n|\right\}$$

where z_n is the numerical solution and $z(t_n)$ is the analytic solution. It can be observed from the figures (1)-(3) that the curve of the NOIM method locates below every other curve. Consequently, we conclude that optimized integrator method is more efficiency and effectiveness than the existing methods.

Problem 1 ([17]). $z''(t)=-100z(t), z(0)=1, z'(0)=-2$. The fitted frequency, $v=10$, and the analytic solution is: $z(t)=-(1/5)\sin(10t)+\cos(10t)$.

Problem 2 ([16]). $z''(t)=-v^2z(t)+(v^2-1)\sin(t), z(0)=1, z'(0)=v+1, v=10$. the analytic solution is: $z(t)=\cos(vt)+\sin(vt)+\sin(t)$.

Problem 3 ([18]).

$$\begin{aligned} z_1''(t)+z_1(t)&=0.001\cos(t), \quad z_1(0)=1, \quad z_1'(0)=0,\\ z_2''(t)+z_2(t)&=0.001\sin(t), \quad z_2(0)=0, \quad z_2'(0)=009995, \end{aligned}$$

The fitted frequency, $v=1$, and the analytic solution is:

$$\begin{aligned} z_1&=\cos(t)+0.0005t\sin(t),\\ z_2&=\sin(t)-0.0005t\cos(t), \end{aligned}$$

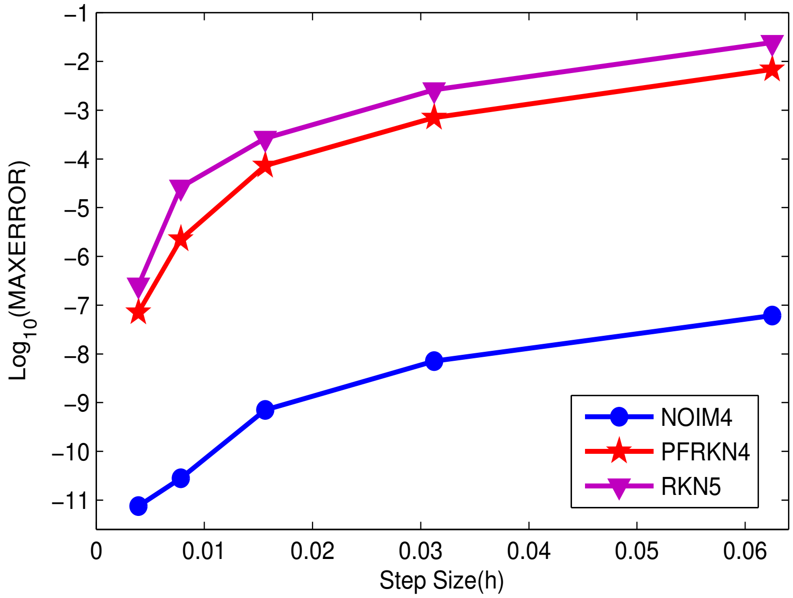

Figure 1: Efficiency Results for Problem 1

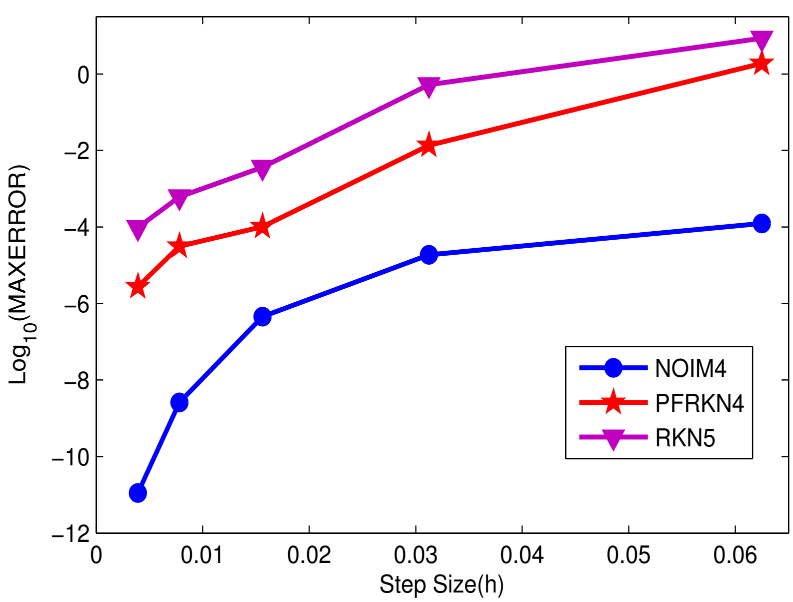

Figure 2: Efficiency Results for Problem 2

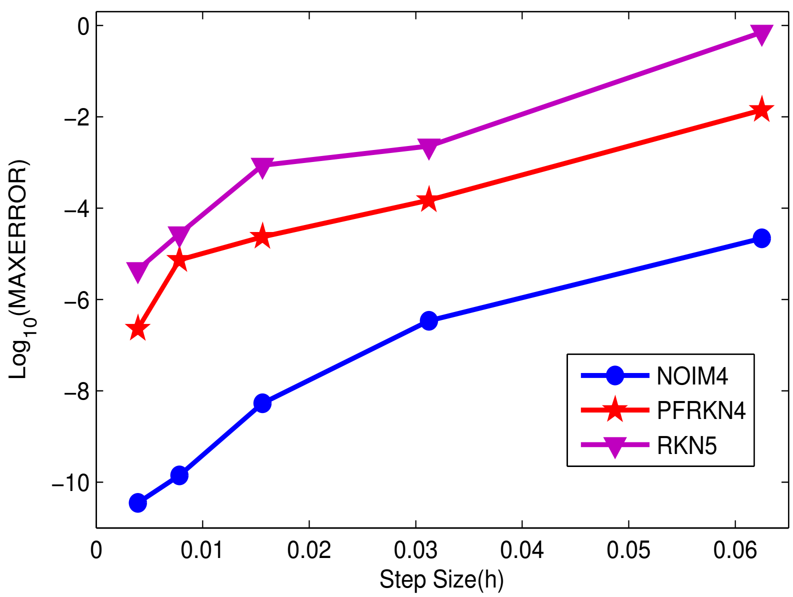

Figure 3: Efficiency Results for Problem 3

5. Conclusion

In this study, optimized integrator method of order four is derived based on the existing hybrid methods. The new method has dissipative of order infinity and dispersive of order six. Numerical results detect that the NOIM method is more competence than the existing methods.

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References

[1] R.L. Liboff, *Introductory quantum mechanics*, Addison-Wesley, Reading, MA, 1980.

[2] E. Hairer, S.P. Nørsett, G. and Wanner, *Solving ordinary differential equations I: Nonstiff problems*, Berlin, Springer, second edition, 2010.

[3] K. Hussain, F. Ismail, N.Senu, F. Rabiei, *Optimized fourth-order Runge-Kutta method for solving oscillatory problems*, AIP Conference Proceedings, 1739 (2016), 020032.

[4] T.E. Simos, *Exponentially-fitted Runge-Kutta-Nyström method for the numerical solution of initial value problems with oscillating solutions*, Applied Mathematics Letters, 15 (2002), 217-225.

[5] T.E. Simos, *Explicit eighth order methods for the numerical integration of initial-value problems with periodic or oscillating solutions*, Computer Physics Communications, 119 (1999), 32-44.

[6] T.E. Simos, *Optimizing a hybrid two-step method for the numerical solution of the Schrödinger equation and related problems with respect to phase-lag*, Journal of Applied Mathematics, 2012, (2012), Article ID 420387.

[7] Q. Ming, Y. Yang, Y. Fang, *An optimized Runge-Kutta method for the numerical solution of the radial Schrödinger equation*, Mathematical Problems in Engineering, 2012, (2012), Article ID 867948.

[8] Ch. Tsitouras, *Explicit two-step methods for second-order linear IVPs*, Computers and Mathematics with Applications, 43 (2002), 943-949.

[9] J.P. Coleman, *Order conditions for a class of two-step Methods for $y''(x) = f(x,y)$*, IMA Journal of Numerical Analysis, 23 (2003), 197-220.

[10] J.M. Franco, *A class of explicit two-step hybrid methods for second-order IVPs*, Journal of Computational and Applied Mathematics, 187 (2006), 41-57.

[11] R. M. Thomas, *Phase Properties of higher order, almost Pstable formulae*, BIT Numerical Mathematics, 24 (1984), 225-238.

[12] H. van de Vyver, *A symplectic Runge-Kutta Nyström method with minimal phase-lag*, Physics Letters A, 367 (2007), 16-24.

[13] R. Ambrosio, M. Ferro, B. Paternoster, *Trigonometrically fitted two-step hybrid methods for special second order ordinary differential equations*, Mathematics and Computers in Simulation, 81 (2011), 1068-1084.

[14] S.Z. Ahmad, F. Ismail, N. Senu, M.B. Suleiman, *Zero-dissipative phase-fitted hybrid methods for solving oscillatory second order ordinary differential equations*, Applied Mathematics and Computation, 219 (2013), 10096-10104.

[15] A.A. Kosti, Z.A. Anastassi, T.E. Simos, *An optimized explicit Runge-Kutta Nyström method for the numerical solution of orbital and related periodical initial value problems*, Computer Physics Communications, 183 (2012), 470-479.

[16] D. F. Papadopoulos, Z. A. Anastassi, T. E. Simos, *A phase-fitted Runge-Kutta Nyström method for the numerical solution of initial value problems*

with oscillating solutions, Computer Physics Communications, 180 (2009), 1839-1846.

[17] M. M. Chawla, P. S. Rao, *High-accuracy P-stable methods for* $y''(x) = f(x, y)$, IMA Journal of Numerical Analysis, 5 (1985), 215-220.

[18] E. Stiefel, D. G. Bettis, *Stabilization of Cowells method*, Numerische Mathematik, 13 (1969), 154-175.

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