Some results on uniqueness of certain types of difference polynomials

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Abstract. In this article we investigate the uniqueness of certain type of difference polynomials that share a small function and obtain some results which improve and extend some recent results of Sujoy Majumder ([12]).
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1. Introduction

Let $\mathbb{C}$ be a open complex plane and two functions f and g are non-constant and meromorphic in $\mathbb{C}$. In this article standard notations like $T(r, f)$, $m(r, f)$, $N(r, f)$ of Value Distribution Theory are used see ([1], [2], [3]). In this paper, we denote $\mathbb{C} \setminus \{0\}$ as $\mathbb{C}'$. For $a \in \mathbb{C} \cup \{\infty\}$, if f and g have the same point a with same multiplicities then f and g share a CM. If we do not take the multiplicities into account then f and g share the value a IM. A meromorphic

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function a is said to be a small function of f provided $T(r,a) = S(r,f)$ i.e., $T(r,a) = O\{T(r,f)\}$ as $r \to \infty, r \notin E$.

Definition 1.1 ([14]). *Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. We note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.2 ([8]). *Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r,a;f| \leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f| \leq p)$ we denote the corresponding reduced counting function.*

Similarly we can define $N(r,a;f| \geq p)$ and $\overline{N}(r,a;f| \geq p)$.

Definition 1.3 ([14]). *Let $p \in \mathbb{N} \cup \{\infty\}$. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then*

$$N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f| \geq 2) + \ldots + \overline{N}(r,a;f| \geq p).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

In 2011, K. Liu, X. L Liu and T. B Cao [10] studied the uniqueness of the difference monomials and obtained the following results

Theorem 1 ([10]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C}'$ and $n \in N$. If $n \geq 14$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.*

Theorem 2 ([10]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C}'$ and $n \in \mathbb{N}$. If $n \geq 26$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.*

In 2015, Y. Liu, J. P Wang and F. H Liu [13] improved Theorems 1 and 2 and obtained the following results.

Theorem 3 ([13]). *Let $c \in \mathbb{C}'$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 14)$, $k(\geq 3)$ be two positive integers. If $E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 4 ([13]). *Let $c \in \mathbb{C}'$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 16)$ be a positive integers. If $E_2(1, f^n f(z+c)) = E_2(1, g^n g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 5 ([13]). *Let $c \in \mathbb{C}'$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 22)$ be a positive integers. If $E_1(1, f^n f(z+c)) = E_1(1, g^n g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2017, S. Majumder [12] proved following theorem by taking non-zero polynomial in place of sharing value 1 CM and came out with the following results.

Theorem 6 ([12]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < \frac{n-1}{2}$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0,2)$, then one of the following two cases holds:*

(i) $f(z) \equiv tg(z)$ for some constant t such that $t^{n+1} = 1$;

(ii) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a non-zero constant c and t is a constant such that $t^{n+1} = c^2$.

Theorem 7 ([12]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < \frac{n-1}{2}$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0,1)$. Then conclusion of Theorem 6 holds.*

Theorem 8 ([12]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < \frac{n-1}{2}$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0,0)$. Then conclusion of Theorem 6 holds.*

Regarding Theorems 6-8, one may ask the following question.

Question 1. What happens if $f^n(z)f(z+c)$ is replaced by $f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ in Theorems 6-8?

To answer the above question affirmatively, we prove the following results which is the main results of this article.

Theorem 9. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n > 7 + 6m + 7\sigma$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < \frac{n+m-\sigma}{2}$. Suppose $f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} - p(z)$ and $g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} - p(z)$ share $(0,2)$, then one of the following two cases holds:*

(i) $f(z) \equiv tg(z)$ for some constant t such that $t^{n+m+\sigma} = 1$;

(ii) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a non-zero constant c and t is a constant such that $t^{n+m+\sigma} = c^2$.

Theorem 10. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n > 8 + 6m + 8\sigma$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < \frac{n+m-\sigma}{2}$. Suppose $f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} - p(z)$ and $g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} - p(z)$ share $(0,1)$. Then conclusion of Theorem 9 holds.*

Theorem 11. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order,$c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n > 13 + 6m + 13\sigma$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $deg(p) < \frac{n+m-\sigma}{2}$. Suppose $f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} - p(z)$ and $g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} - p(z)$ share $(0,0)$. Then conclusion of Theorem 9 holds.*

Remark 1. If $m = 0$ and $\sigma = 1$ in Theorems 9-11, then Theorems 9-11 reduces to Theorems 6-8.

2. Some lemmas

In this segment, we present few Lemmas which will be used further.

Let F, G be two non-constant meromorphic functions. From now onwords let us use H to denote the following functions.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

Lemma 2.1 ([11]). *Suppose $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$ and the polynomial is defined by $P(z) = a_m f^m + a_{m-1}f^{m-1} + \ldots + a_1 f + a_0$, where $a_m(\not\equiv 0)$, $a_0,\ a_1,\ \ldots,\ a_{m-1}$ are small functions of $f(z)$. Then*

$$T(r, P(f)) = mT(r, f) + S(r, f).$$

Lemma 2.2 ([5]). *Let $f(z)$ be a meromorphic function of finite order ρ, and let $c \in \mathbb{C}'$ be fixed. Then for each $\varepsilon > 0$,we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\varepsilon}).$$

Lemma 2.3 ([5]). *Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C}'$ be fixed. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.4 ([6]). *Let $f(z)$ be non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then:*

$$N(r, 0; f(z+c)) \le N(r, 0; f(z)) + S(r, f),$$
$$N(r, \infty; f(z+c)) \le N(r, \infty; f(z)) + S(r, f),$$
$$\overline{N}(r, 0; f(z+c)) \le \overline{N}(r, 0; f(z)) + S(r, f),$$
$$\overline{N}(r, \infty; f(z+c)) \le \overline{N}(r, \infty; f(z)) + S(r, f).$$

Lemma 2.5 ([16]). *Let $f(z)$ be a transcendental meromorphic function of hyper order, $\rho_2 < 1$ and $F(z) = f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}$, where $m,\ n \in N$. Then we have,*

$$(n+m-\sigma)T(r,f) \le T(r,F) + S(r,f) \le (n+m+\sigma)T(r,f).$$

Lemma 2.6. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ such that $n \ge 2$. Let $p(z)$ be a non-zero polynomial such that $deg(p) < \frac{n+m-\sigma}{2}$. Then:*

(i) if $deg(p) \ge 1$, then $f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \not\equiv p^2(z)$;

(ii) if $p(z) = c \in \mathbb{C}'$, then the relation

$$f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \equiv p^2(z)$$

always implies that $fg = t$ where t is a constant such that $t^{n+m+\sigma} = c^2$.

Proof. Suppose

$$f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \equiv p^2(z). \tag{2}$$

Let $h = fg$ then from equation (2) becomes,

$$\begin{aligned} &h^{n+m}\left(a_m\frac{1}{g^m} + \ldots + \frac{a_0}{h^m}\right)(a_m g^m + a_{m-1}g^{m-1} + \cdots + a_0) \\ &= \frac{p^2(z)}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}. \end{aligned} \tag{3}$$

Let us discuss following two cases.

Case 1. Suppose h is a transcendental meromorphic function. Now, by Lemmas 2.1, 2.2 and 2.4, we get

$$\begin{aligned} (n+m)T(r,h) &= T(r,h^{n+m}) + S(r,h) \\ &= T\left(r, \frac{p^2(z)}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right) + S(r,h) \\ &\le \sigma T(r,h) + S(r,h) \end{aligned}$$

which is a contradiction.

Case 2. Suppose h is rational function. Let

$$h = \frac{h_1}{h_2}. \tag{4}$$

where h_1 and h_2 are two non-zero relatively prime polynomials. By equation (4), we have

$$T(r,h) = max\{deg(h_1),\ deg(h_2)\}log\ r + O(1). \tag{5}$$

Now, by equations (3) - (5), we have

$$\begin{aligned}
&(n+m)\max\{deg(h_1),\ deg(h_2)\}\log r \\
&= T(r,h^{n+m}) + O(1) \\
&\leq T(r,h^{\sigma}) + T(r,p^2(z)) + S(r,f) \\
&\leq \sigma T(r,h) + 2T(r,p) + S(r,f) \\
&= \max\{deg(h_1), deg(h_{2)}\}\log r + 2deg(p) \\
&+ \log r + O(1).
\end{aligned} \tag{6}$$

This implies,

$$(n+m-\sigma)T(r,h) \leq 2T(r,p) + S(r,f).$$

Hence,

$$(n+m-\sigma) \leq 2deg(p).$$

We see that $\max\{deg(h_1),\ deg(h_2)\} \geq 1$. Now by see (6), we deduce that $(n+m-\sigma)/2 \leq deg(p)$, which contradicts our assumption that $deg(p) < \frac{n+m-\sigma}{2}$. Hence h must be a non-zero constant. Let

$$h = t \in \mathbb{C}'. \tag{7}$$

Now, when $deg(p) \geq 1$. By see (3) and (7) we arrive at a contradiction. Therefore $f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \not\equiv p^2(z)$. Suppose $p(z) = c \in \mathbb{C}'$. So by 3 we see that $h^{n+m+\sigma} \equiv c^2$. By (7) we get $t^{n+m+\sigma} \equiv c^2$. This completes the proof. □

Lemma 2.7 ([14]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions sharing (1,2). Then one of the following holds:*

(i) $T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r,f) + S(r,g)$;

(ii) $fg \equiv 1$;

(iii) $f \equiv g$.

Lemma 2.8 ([4]). *Let F and G be two non-constant meromorphic functions sharing (1,1) and $H \not\equiv 0$. Then*

$$\begin{aligned}
T(r,F) \leq{}& N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\
&+ \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F) + S(r,G).
\end{aligned}$$

Lemma 2.9 ([4]). *Let F and G be two non-constant meromorphic functions sharing (1,0) and $H \not\equiv 0$. Then*

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G).$$

Lemma 2.10 ([15]). *Let H be defined as in (1). If $H \equiv 0$ and*

$$\overline{\lim_{r\to\infty}} \frac{\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;F)}{T(r)} < 1, \ \ r \in I,$$

where I is a set of infinite linear measure, then $F \equiv G$ or $F.G \equiv 1$.

3. Proof of main results

Proof of Theorem 9.

Proof. Let $F(z) = \frac{f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}}{p(z)}$ and $G(z) = \frac{g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}}{p(z)}$. Then, F and G share $(1,2)$ except for the zeros of $p(z)$. Now by Lemma 2.7, we see that one of the following three cases holds:

Case 1. Suppose

$$T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+S(r,F)+S(r,G).$$

Now, by applying Lemmas 2.1 and 2.4, we have

$$\overline{N}(r,0;F) \le (1+m+\sigma)T(r,f) + S(r,f).$$

Similarly,

$$\overline{N}(r,0;G) \le (1+m+\sigma)T(r,g) + S(r,f),$$
$$\overline{N}(r,\infty;F) \le (1+m+\sigma)T(r,f) + S(r,f),$$
$$\overline{N}(r,\infty;G) \le (1+m+\sigma)T(r,g) + S(r,f).$$

Therefore,

$$\begin{aligned} T(r,F) &\le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) \\ &+ N_2(r,\infty;G) + S(r,F) + S(r,G) \\ &\le 2\overline{N}(r,0;F) + 2\overline{N}(r,0;G) \\ &+ \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,f) + S(r,g) \\ &\le 2(1+m+\sigma)T(r,f) + 2(1+m+\sigma)T(r,g) + (1+m+\sigma)T(r,f) \\ &+ (1+m+\sigma)T(r,g) + S(r,f) + S(r,g) \\ &\le 3(1+m+\sigma)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned}$$

By Lemma 2.5,

(8) $(n+m-\sigma)T(r,f) \leq 3(1+m+\sigma)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$

Similarly,

(9) $(n+m-\sigma)T(r,g) \leq 3(1+m+\sigma)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$

Combining equation (8),(9)

$$(n+m-\sigma)T(r) \leq 6(1+m+\sigma)T(r)+S(r).$$

where $T(r)=\max\{T(r,f),\ T(r,g)\}$ and $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite logarithmic measure. Therefore $n\leq 6+5m+7\sigma$ which contradicts with $n>7+5m+7\sigma$.

Case 2. $F\equiv G$. Then we have

$$f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} \equiv g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \tag{10}$$

Let $h=\frac{f}{g}$. Then $f=gh$, with these equation (10) becomes,

$$(gh)^n(z)P(gh)\prod_{j=1}^{d}(gh)(z+c_j)^{v_j} \equiv g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}.$$

$$h^n(z)[a_m g^m h^m+\ldots+a_0]\prod_{j=1}^{d} h(z+c_j)^{v_j} \equiv P(g).$$

$$[a_m g^m h^{n+m}+\ldots+a_0 h^n] \equiv \frac{P(g)}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}.$$

$$\begin{aligned}(n+m)T(r,h) &\leq T(r,h^{n+m})+S(r,h)\\ &\leq N\left(r,\frac{1}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right)+m\left(r,\frac{1}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right)\\ &\leq T\left(r,\frac{1}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right)+S(r,h)\\ &\leq T(r,h^{\sigma})+S(r,h)\\ &\leq \sigma T(r,h)+S(r,h).\end{aligned}$$

Since $n\geq 2$, we see that h is a constant. Then, we have $h^{n+m+\sigma}=1$. Therefore, $f(z)=tg(z)$, where $t^{n+m+\sigma}=1$.

Case 3. $F.G\equiv 1$. Then we have $f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \equiv p^2(z)$. Hence Theorem 9 follows by Lemma 2.6. This completes the proof. □

Proof of Theorem 10. Let

$$F = \frac{f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}}{p(z)}$$

and

$$G = \frac{g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}}{p(z)}.$$

Then, F and G share $(1,1)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 1. $H \not\equiv 0$. By Lemma 2.3, we have

$$\overline{N}(r,0;F) \le (1+m+\sigma)T(r,f) + S(r,f)$$

Similarly,

$$\overline{N}(r,0;G) \le (1+m+\sigma)T(r,g) + S(r,f)$$
$$\overline{N}(r,\infty;F) \le (1+m+\sigma)T(r,f) + S(r,f)$$
$$\overline{N}(r,\infty;G) \le (1+m+\sigma)T(r,g) + S(r,f).$$

Now by applying Lemmas 2.1, 2.4 and 2.8 we have,

$$\begin{aligned}
T(r,F) \le\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\
&+ \frac{1}{2}\overline{N}(r,\infty;F) + S(r,f) + S(r,g) \\
\le\ & 2\overline{N}(r,0;F) + 2\overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\
&+ \frac{1}{2}\overline{N}(r,\infty;F) + S(r,f) + S(r,g) \\
\le\ & \{2(1+m+\sigma) + (1+m+\sigma) + \frac{1}{2}(1+m+\sigma) + \frac{1}{2}(1+m+\sigma)\}T(r,f) \\
&+ \{2(1+m+\sigma) + (1+m+\sigma)\}T(r,g) + S(r,f) + S(r,g) \\
\le\ & 4(1+m+\sigma)T(r,f) + 3(1+m+\sigma)T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}
(n+m-\sigma)T(r,f) \le\ & 4(1+m+\sigma)T(r,f) + 3(1+m+\sigma)T(r,g) \\
& + S(r,f) + S(r,g).
\end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned}
(n+m-\sigma)T(r,g) \le\ & 4(1+m+\sigma)T(r,g) + 3(1+m+\sigma)T(r,f) \\
& + S(r,f) + S(r,g).
\end{aligned} \tag{12}$$

Adding inequalities (11) and (12) we get,

$$(n+m-\sigma)T(r) \leq 7(1+m+\sigma)T(r)+S(r).$$

where $T(r)=max\{T(r,F),T(r,G)\}$, $S(r)=max\{S(r,f),S(r,g)\}$. Which implies that $n \leq 7+6m+8\sigma$ which contradicts with $n>8+6m+8\sigma$.

Case 2. $H \equiv 0$. In view of Lemmas 2.4 and 2.5 we get

$$\begin{aligned}
&\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&\quad\leq 2(1+m+\sigma)T(r,f)+2(1+m+\sigma)T(r,g)+S(r,f)+S(r,g)\\
&\quad\leq \frac{2(1+m+\sigma)}{n+m-\sigma}T(r,F)+\frac{2(1+m+\sigma)}{n+m-\sigma}T(r,G)+S(r,f)+S(r,g)\\
&\quad\leq \frac{4(1+m+\sigma)}{n+m-\sigma}T(r)+S(r),
\end{aligned}$$

where $T(r)=max\{T(r,F),T(r,G)\}$, $S(r)=max\{S(r,f),S(r,g)\}$. Since $n>8+6m+8\sigma$, we have

$$\varlimsup_{r\to\infty}\frac{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)}{T(r)}<1$$

and so by Lemma 2.10, we have either $F \equiv G$ or $F.G \equiv 1$. Hence, Theorem 10 follows by the proof of Theorem 9. This completes the proof.

Proof of Theorem 11. Let

$$F(z)=\frac{f^n(z)P(f)\prod_{j=1}^{d}f(z+c_j)^{v_j}}{p(z)}$$

and

$$G(z)=\frac{g^n(z)P(g)\prod_{j=1}^{d}g(z+c_j)^{v_j}}{p(z)}.$$

Then, F and G share $(1,0)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 1. $H \not\equiv 0$. By Lemma 2.3, we have

$$\overline{N}(r,0;F) \leq (1+m+\sigma)T(r,f)+S(r,f).$$

Similarly,

$$\overline{N}(r,0;G) \leq (1+m+\sigma)T(r,g)+S(r,f),$$

$$\overline{N}(r,\infty;F) \leq (1+m+\sigma)T(r,f)+S(r,f),$$

$$\overline{N}(r,\infty;G) \leq (1+m+\sigma)T(r,g)+S(r,f).$$

Now, by Lemmas 2.1, 2.4 and 2.9 we have

$$\begin{aligned}T(r,F) &\le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\&+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\&\le 2\overline{N}(r,0;F)+2\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+2\overline{N}(r,0;F)\\&+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\&\le \{2(1+m+\sigma)+(1+m+\sigma)+2(1+m+\sigma)\}T(r,f)\\&+\{2(1+m+\sigma)+(1+m+\sigma)+(1+m+\sigma)+(1+m+\sigma)\}T(r,g)\\&\le 7(1+m+\sigma)T(r,f)+5(1+m+\sigma)T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}(n+m-\sigma)T(r,f) &\le 7(1+m+\sigma)T(r,f)+5(1+m+\sigma)T(r,g)\\&+S(r,f)+S(r,g).\end{aligned} \tag{13}$$

Similarly

$$\begin{aligned}(n+m-\sigma)T(r,g) &\le 7(1+m+\sigma)T(r,g)+5(1+m+\sigma)T(r,f)\\&+S(r,f)+S(r,g).\end{aligned} \tag{14}$$

Adding inequalities (13) and (14) we get

$$\begin{aligned}(n+m-\sigma)\{T(r,g)+T(r,f)\} &\le 12(1+m+\sigma)\{T(r,g)+T(r,f)\}\\&+S(r,f)+S(r,g).\end{aligned}$$

i.e., $n \le 12+11m+13\sigma$, which contradicts with $n > 13+11m+13\sigma$.

Case 3. $H \equiv 0$. Hence Theorem 11 follows from the proof of Theorems 9 and 10. This completes the proof.

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