# On a Diophantine inequality with different powers of primes

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**Abstract.** Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative. Assume that $\frac{\lambda_1}{\lambda_2}$ is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. In this paper, we proved that, for any $\varepsilon > 0$, the number of $\upsilon \in \mathcal{V}$ with $1 \le \upsilon \le X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - \upsilon| < \upsilon^{-\delta}$$

has no solution in primes $p_1, p_2, p_3, p_4, p_5, p_6$ does not exceed $O(X^{1-\frac{1}{8}+2\delta+\varepsilon})$.
**Keywords:** Waring-Goldbach problem, Diophantine inequality, primes.

## 1. Introduction

In 1970, Vaughan [14] proved that every large integer $N$ can be represented as the sum of two squares, two cubes and two biquadrates of natural numbers by showing that

$$\sum_{N=x_1^2+x_2^2+x_3^3+x_4^3+x_5^4+x_6^4} 1 = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{5}{4})}{\Gamma(\frac{13}{6})}\mathfrak{S}'(N)N^{\frac{7}{6}} + O(N^{\frac{7}{6}-\frac{1}{96}+\varepsilon}),$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ are natural numbers and

$$\mathfrak{S}'(N) = \sum_{q=1}^{\infty} \frac{1}{q^6} \sum_{\substack{a=1\\(a,q)=1}}^{q} \prod_{i=1}^{3} \left( \sum_{x_i=1}^{q} e\Big(\frac{a x_i^{i+1}}{q}\Big)\right)^2 e\left(-\frac{aN}{q}\right).$$

In 2015, Lü [12] proved that every large even integer $N$ can be represented in the form

$$N = x^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4,$$

where $x$ is an almost-prime $\mathcal{P}_6$ and $p_2, p_3, p_4, p_5, p_6$ are primes. Very recently, Liu [11] obtained a stronger result with $\mathcal{P}_6$ replaced by $\mathcal{P}_4$. It seems reasonable to conjecture that every sufficiently large even integer $n$ can be expressed in the form

$$n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4, \tag{1.1}$$

where $p_1, p_2, p_3, p_4, p_5, p_6$ are primes. Recently, Zhang and Li [16] considered the set of possible exceptions of the representation (1.1). Let $E(N)$ denote the number of positive even integers $n \le N$, which can not be represented in the form (1.1). Zhang and Li [16] showed that

$$E(N) \ll N^{\frac{13}{16}+\varepsilon}.$$

Later, Zhang [18] improved this result to

$$E(N) \ll N^{\frac{17}{192}+\varepsilon}.$$

Zhang and Li [17] also considered the exceptional problem of the form (1.1) in short intervals.

Typically, when one can handle a representation problem, it is of interest to study the analogous forms with real coefficients (see e.g. [1, 3, 4, 6, 7, 8, 9, 10] ). We call a set of positive real numbers $\mathcal{V}$ a *well-spaced set* if there exists a $c > 0$ such that

$$u, v \in \mathcal{V},\ u \neq v \Rightarrow |u - v| > c.$$

We further assume that, for any $\varepsilon > 0$,

$$|\{v \in \mathcal{V} : 0 \le v \le X\}| \gg X^{1-\varepsilon}.$$

In this paper, we consider a Diophantine inequality with two squares, two cubes and two biquadrates of primes. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative and $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Denote by $E(\mathcal{V}, X, \delta)$ the number of $v \in \mathcal{V}$ with $1 \le v \le X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - v| < v^{-\delta} \tag{1.2}$$

has no solutions in primes $p_1, p_2, p_3, p_4, p_5, p_6$. Our results are as follows:

**Theorem 1.1.** *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative. Suppose that $\frac{\lambda_1}{\lambda_2}$ is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence and $\delta > 0$. Then, we have, for any $\varepsilon > 0$,*

$$E(\mathcal{V}, X, \delta) \ll X^{1-\frac{1}{8}+2\delta+\varepsilon}.$$

**Theorem 1.2.** *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative. Suppose that $\frac{\lambda_1}{\lambda_2}$ is irrational. Let $\mathcal{V}$ be a well-spaced sequence and $\delta > 0$. Then, there is a sequence $X_j \to \infty$ such that*

$$E(\mathcal{V}, X_j, \delta) \ll X_j^{1-\frac{1}{8}+2\delta+\varepsilon} \tag{1.3}$$

*for any $\varepsilon > 0$. Moreover, if the convergent denominators $q_j$ for $\lambda_1/\lambda_2$ satisfy*

$$q_{j+1}^{1-w} \ll q_j \text{ for some } w \in [0, 1), \tag{1.4}$$

*then we have, for all $X \geq 1$ and any $\varepsilon > 0$,*

$$E(\mathcal{V}, X, \delta) \ll X^{2\chi+2\delta+\varepsilon} \tag{1.5}$$

*with*

$$\chi = \max\left(\frac{2-w}{6-4w}, \frac{7}{16}\right). \tag{1.6}$$

Theorem 1.1 follows immediately from Theorem 1.2, since, in the case of $\lambda_1/\lambda_2$ algebraic, we can take $w = \varepsilon$ and then $\chi = \frac{7}{16}$. Thus, we focus on proving Theorem 1.2 in the following.

**Notation.** Throughout this paper, the letter $p$, with or without a subscript, always denotes a prime. The letter $\varepsilon$ denotes a sufficiently small positive number, and the value of $\varepsilon$ may change from statement to statement. Constants, both explicit and implicit, in Vinogradov symbols may depend on $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$. We abbreviate $e^{2\pi i x}$ to $e(x)$.

## 2. Outline of the method and preliminary lemmas

We follow the adaption of the Hardy-Littlewood method first introduced by Davenport and Heilbronn [2]. Let $X$ be some large positive quantity to be chosen later (we shall take $X = q^{\frac{4}{3}}$ in Lemma 4.2) and $\eta$ be a fixed sufficiently small positive integer. Let $0 < \tau < 1$ (we shall chose $\tau = X^{-\delta}$ in Section 6). Let

$$K_\tau(\alpha) = \left(\frac{\sin(\pi\tau\alpha)}{\tau\alpha}\right)^2$$

for $\tau > 0$ and $\alpha \neq 0$. By continuity, we define $K_\tau(0) = \tau^2$. Then we have, by [15],

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \tag{2.1}$$

and

$$A(x) := \int_{-\infty}^{+\infty} e(x\alpha)K_\tau(\alpha)d\alpha = \max(0, \tau - |x|). \tag{2.2}$$

We define, for $i = 2, 3, 4$,

$$S_i(\alpha) = \sum_{p \in I_i} (\log p)e(p^i\alpha), \quad U_i(\alpha) = \sum_{n \in I_i} e(n^i\alpha), \tag{2.3}$$

where

$$I_i = \left[(\eta X)^{\frac{1}{i}}, X^{\frac{1}{i}}\right].$$

If we write

$$N_\upsilon = \frac{1}{\tau} \sum_{\substack{p_1,p_2\in I_2\\ p_3,p_4\in I_3\\ p_5,p_6\in I_4}} \left(\prod_{j=1}^{6} \log p_j\right) A(\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - \upsilon),$$

then $0 \le N_\upsilon \le \Psi(\upsilon)$, where $\Psi(\upsilon)$ counts the number of solutions to

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - \upsilon| < \tau,$$

weighted by a term $\prod_{j=1}^{6} \log p_j$. We shall restrict our attention to those $\upsilon$ satisfying $X/2 \le \upsilon \le X$. In general, one can consider $2^{-j}X \le \upsilon \le 2^{1-j}X,\ j = 1, 2, \ldots,$ and obtain a satisfactory bound for the exceptional set. Then, from (2.2), we have

$$N_\upsilon = \frac{1}{\tau} \int_{-\infty}^{+\infty} S_2(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_3(\lambda_4\alpha) S_4(\lambda_5\alpha) S_4(\lambda_6\alpha) \cdot K_\tau(\alpha) e(-\upsilon\alpha) d\alpha. \tag{2.4}$$

To estimate the integral in (2.4), we divide the real line into the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$, which are defined by

$$\mathfrak{M} = \{\alpha : |\alpha| \le \phi\},\ \mathfrak{m} = \{\alpha : \phi < |\alpha| < \xi\},\ \mathfrak{t} = \{\alpha : |\alpha| \ge \xi\}, \tag{2.5}$$

with $\phi = X^{-\frac{3}{4}}$ and $\xi = \tau^{-2} X^{\frac{1}{6}+2\varepsilon}$. The main contribution to the integral (2.4) comes from the major arc. In order to handle integrals on these arcs, we need the following lemmas.

**Lemma 2.1** ([5, Theorem 3]). *Let $k \ge 2$ and define*

$$\rho(k) = \begin{cases} \frac{1}{8}, & \text{if } k = 2,\\ \frac{1}{14}, & \text{if } k = 3,\\ \frac{2}{3} \times 2^{-k}, & \text{if } k \ge 4. \end{cases}$$

*Suppose that $\alpha$ is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \le q \le Q, \quad (a, q) = 1, \quad |q\alpha - a| < Q^{-1} \tag{2.6}$$

*with*

$$Q = \begin{cases} X^{\frac{3}{4}}, & \text{if } k = 2,\\ X^{\frac{k-2\rho(k)}{2k-1}}, & \text{if } k \ge 3. \end{cases}$$

*Then, for any $\varepsilon > 0$, we have*

$$\sum_{1 \le p^k \le X} (\log p) e(p^k \alpha) \ll X^{\frac{1}{2}(1-\rho(k))+\varepsilon} + \frac{q^\varepsilon X^{\frac{1}{2}} (\log X)^c}{(q + X|q\alpha - a|)^{\frac{1}{2}}}, \tag{2.7}$$

*where $c > 0$ is an absolute constant and the constant implied by $\ll$ depends at most on $k$ and $\varepsilon$.*

The following lemma shows an improvement of [3, Corollary 2.2].

**Lemma 2.2.** *Suppose that* $X^{\frac{1}{2}} \geq Z \geq X^{\frac{1}{2}-\frac{1}{16}+\varepsilon}$ *and* $|S_2(\lambda\alpha)| > Z$. *Then there two coprime integers* $a, q$ *satisfying*

$$1 \leq q \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^2, \quad |q\lambda\alpha - a| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^2.$$

**Proof.** We prove this lemma by following a similar argument of [1, Corollary 3]. By Dirichlet's theorem in Diophantine approximation, there exist coprime integers $a, q$ satisfying

$$1 \leq q \leq X\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^{-2}, \quad |q\lambda\alpha - a| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^2.$$

Then from (2.7), we have

$$S_2(\lambda\alpha) \ll X^{\frac{1}{2}-\frac{1}{16}+\varepsilon} + \frac{X^{\frac{1}{2}+\varepsilon}}{(q + X|q\lambda\alpha - a|)^{\frac{1}{2}}}.$$

Since, we have $|S_2(\lambda\alpha)| > Z \geq X^{\frac{1}{2}-\frac{1}{16}+\varepsilon}$, then

$$Z < |S_2(\lambda\alpha)| \ll \frac{X^{\frac{1}{2}+\varepsilon}}{(q + X|q\lambda\alpha - a|)^{\frac{1}{2}}} \ll X^{\frac{1}{2}+\varepsilon}q^{-\frac{1}{2}},$$

which means that

$$1 \leq q \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^2.$$

Therefore, we complete the proof of this lemma. □

**Lemma 2.3.** *Suppose that* $X^{\frac{5}{24}-1-\varepsilon} \leq |\alpha| \leq X^{-\frac{3}{4}}$. *Then we have*

$$|S_4(\lambda\alpha)| \ll X^{\frac{1}{4}-\frac{1}{2\cdot 4^4}+\varepsilon}.$$

**Proof.** This lemma is [3, Corollary 2.3] with $k = 4$. □

**Lemma 2.4.** *Suppose that* $F \in \{S_2^4, S_3^8, S_2^2S_3^4, S_2^2S_4^4\}$. *Then we have*

$$\int_{-1}^{1} |F(\alpha)|d\alpha \ll X^{-1}(F(0))^{1+\varepsilon}, \quad \int_{-\infty}^{\infty} |F(\alpha)|K_\tau(\alpha)d\alpha \ll \tau X^{-1}(F(0))^{1+\varepsilon}.$$

**Proof.** All of these results follow from [14] by using Hua's Lemma. The similar proof can be found in in [3]. □

For $r \geq 1$, we set

$$J_r(X,h) = \int_{\varepsilon X}^{X} \Big(\theta\big((x+h)^{\frac{1}{r}}\big) - \theta\big(x^{\frac{1}{r}}\big) - \big((x+h)^{\frac{1}{r}} - x^{\frac{1}{r}}\big)\Big)^2 dx,$$

where $\theta(x) = \sum_{p \leq x} \log p$ is the Chebyshev function. The following Lemmas 2.5 and 2.6 are Theorems 3.1 and 3.2 in [6], respectively.

**Lemma 2.5.** *Let $r \geq 1$ be a real number. For $0 < Y \leq 1/2$ we have*

$$\int_{-Y}^{Y} |S_r(\alpha) - U_r(\alpha)|^2 d\alpha \ll \frac{X^{\frac{2}{r}-2}\log^2 X}{Y} + Y^2 X + Y^2 J_r\left(X, \frac{1}{2Y}\right).$$

**Lemma 2.6.** *Let $r \geq 1$ be a real number. There exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on $r$, such that*

$$J_r(X,h) \ll h^2 X^{\frac{2}{r}-1} \exp\left(-c_1 \left(\frac{\log X}{\log\log X}\right)^{\frac{1}{3}}\right)$$

*uniformly for $X^{1-\frac{5}{6r}+\varepsilon} \leq h \leq X$.*

Combining Lemmas 2.5 and 2.6, we can deduce the following lemma easily.

**Lemma 2.7.** *Let $r \geq 1$ be a real number. For any fixed real number $A \geq 6$, we have*

$$\int_{|\alpha| \leq X^{\frac{5}{6r}-1-\varepsilon}} |S_r(\alpha) - U_r(\alpha)|^2 d\alpha \ll X^{\frac{2}{r}-1}(\log X)^{-A}.$$

## 3. The major arc

In this section, we handle the contribution from the major arc and show a positive lower bound. To do this, we divide the major arc into two regions: $\mathfrak{M}_1$ and $\mathfrak{M}_2$, which are defined by

$$\mathfrak{M}_1 = \{\alpha : |\alpha| \leq X^{\frac{5}{24}-1-\varepsilon}\}, \quad \mathfrak{M}_2 = \{\alpha : X^{\frac{5}{24}-1-\varepsilon} < |\alpha| \leq X^{-\frac{3}{4}}\}.$$

In the region $\mathfrak{M}_1$, which can be seen as the standard major arc, we use the idea of Languasco and Zaccagnini [6]. While in the region $\mathfrak{M}_2$, we follow the idea of Ge and Wang [3] (see also Harman [4]), which can enlarge the standard major arc.

### 3.1 The region $\mathfrak{M}_1$

In this subsection, we give the lower bound of the integral on the region $\mathfrak{M}_1$. For $i = 2, 3, 4$, we set

$$T_i(\alpha) = \int_{I_i} e(t^i \alpha) dt.$$

Then, by the first derivative estimate for trigonometric integrals (see Titchmarsh [13]), one has

$$T_i(\alpha) \ll X^{\frac{1}{i}-1}\min(X, |\alpha|^{-1}). \tag{3.1}$$

We have the following lemma.

**Lemma 3.1.** *We have*

$$\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha \gg \tau^2 X^{\frac{7}{6}}.$$

**Proof.** It is easy to show that

$$\begin{aligned}
&\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&=\int_{\mathfrak{M}_1} T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&+\int_{\mathfrak{M}_1} \Big(S_2(\lambda_1\alpha)-T_2(\lambda_1\alpha)\Big)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&+\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)\Big(S_2(\lambda_2\alpha)-T_2(\lambda_2\alpha)\Big)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&+\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)\Big(S_3(\lambda_3\alpha)-T_3(\lambda_3\alpha)\Big)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&+\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)\Big(S_3(\lambda_4\alpha)-T_3(\lambda_4\alpha)\Big)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&+\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)\Big(S_4(\lambda_5\alpha)-T_4(\lambda_5\alpha)\Big)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&+\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)\Big(S_4(\lambda_6\alpha)-T_4(\lambda_6\alpha)\Big)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&:= J_1+J_2+J_3+J_4+J_5+J_6+J_7.
\end{aligned}$$

In the following, we will show that $J_1 \gg \tau^2 X^{\frac{7}{6}}$ and $J_i = o(\tau^2 X^{\frac{7}{6}})$, for $i = 2,3,4,5,6,7$. Thus, this lemma follows. □

Since the estimates for $J_3, J_5$ and $J_7$ are similar to $J_2, J_4$ and $J_6$, respectively, we restrict our attention to estimate $J_1, J_2, J_4$ and $J_6$,

### 3.1.1 Lower bound for $J_1$

We first establish the lower bound for $J_1$. Note that

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}} T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha\\
&+ O\left(\int_{|\alpha|>\phi} |T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha\right). \qquad (3.2)
\end{aligned}$$

Togethers with (2.1) and (3.1), we obtain that the error term in (3.2) is

$$\ll \tau^2 X^{-\frac{23}{6}} \int_{|\alpha|>X^{\frac{5}{24}-1-\varepsilon}} \frac{d\alpha}{|\alpha|^6} \ll \tau^2 X^{\frac{95}{24}-\frac{23}{6}+5\varepsilon} = o(\tau^2 X^{\frac{7}{6}}). \tag{3.3}$$

For convenience, we write $\mathbf{y} = (y_1, y_2, \ldots, y_6)$. Changing the order of integration and substituting variables, we have

$$\begin{aligned}
&\int_{\mathbb{R}} T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha \\
&= \frac{1}{576(\lambda_1\lambda_2)^{\frac{1}{2}}(\lambda_3\lambda_4)^{\frac{2}{3}}(\lambda_5\lambda_6)^{\frac{3}{4}}} \int (y_1y_2)^{-\frac{1}{2}}(y_3y_4)^{-\frac{2}{3}}(y_5y_6)^{-\frac{3}{4}} \\
&\times \int_{\mathbb{R}} e\Big( (\sum_{j=1}^{6} y_j - \upsilon)\alpha \Big) K_\tau(\alpha)e(-\upsilon\alpha)d\alpha d\mathbf{y} \\
&\gg \int (y_1y_2)^{-\frac{1}{2}}(y_3y_4)^{-\frac{2}{3}}(y_5y_6)^{-\frac{3}{4}} A\Big( \sum_{j=1}^{6} y_j - \upsilon \Big) d\mathbf{y},
\end{aligned}$$

where the domain of integration for $\mathbf{y} = (y_1, y_2, \ldots, y_6)$ are $[\lambda_i\eta X, \lambda_i X]$, respectively, and the integral satisfies $|\sum_{j=1}^{6} y_j - \upsilon| < \tau$. Taking $|\sum_{j=1}^{6} y_j - \upsilon| < \frac{\tau}{2}$, and noting that $\lambda_6\eta X \le y_6 \le \lambda_6 X$, then we can get a lower bound for the last integral:

$$\begin{aligned}
\gg\ & \tau X^{-\frac{3}{4}} \int_{\lambda_1\eta X}^{\lambda_1 X} \cdots \int_{\lambda_5\eta X}^{\lambda_5 X} \int_{\sum_{j=1}^{5} y_j-\upsilon-\frac{\tau}{2}}^{\sum_{j=1}^{5} y_j-\upsilon+\frac{\tau}{2}} (y_1y_2)^{-\frac{1}{2}}(y_3y_4)^{-\frac{2}{3}}(y_5)^{-\frac{3}{4}} \\
& \times A\Big( \sum_{j=1}^{6} y_j - \upsilon \Big) dy_1 \cdots dy_6 \\
\gg\ & \tau^2 X^{\frac{7}{6}},
\end{aligned}$$

which means that

$$\begin{aligned}
&\int_{\mathbb{R}} T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha \\
&\gg \tau^2 X^{\frac{7}{6}}.
\end{aligned} \tag{3.4}$$

Thus, combining (3.2), (3.3) and (3.4), we have $J_1 \gg \tau^2 X^{\frac{7}{6}}$.

### 3.1.2 Upper bounds for $J_2$ and $J_3$

By Euler's summation formula, we have

$$U_j(\lambda\alpha) - T_j(\lambda\alpha) \ll 1 + |\alpha|X, \quad j = 2, 3, 4. \tag{3.5}$$

Using (2.1), we have

$$J_2 \ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1\alpha) - T_2(\lambda_1\alpha)|$$

$$
\begin{aligned}
&\cdot|T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha\\
&\ll \tau^2\int_{\mathfrak{M}_1}|S_2(\lambda_1\alpha)-U_2(\lambda_1\alpha)|\\
&\cdot|T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \qquad (3.6)\\
&+\tau^2\int_{\mathfrak{M}_1}|U_2(\lambda_1\alpha)-T_2(\lambda_1\alpha)|\\
&\cdot|T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha\\
&:=\tau^2(A_2+B_2).
\end{aligned}
$$

By Cauchy's inequality and Lemma 2.7, we obtain

$$
\begin{aligned}
A_2 &\ll X^{\frac{7}{6}}\left(\int_{\mathfrak{M}_1}|S_2(\lambda_1\alpha)-U_2(\lambda_1\alpha)|^2d\alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{M}_1}|T_2(\lambda_2\alpha)|^2d\alpha\right)^{\frac{1}{2}} \qquad (3.7)\\
&\ll X^{\frac{7}{6}}(\log X)^{-\frac{A}{2}}\left(\int_0^{\frac{1}{X}}Xd\alpha+\int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}}X^{-1}|\alpha|^{-2}d\alpha\right)^{\frac{1}{2}}\\
&\ll X^{\frac{7}{6}}(\log X)^{-\frac{A}{2}},
\end{aligned}
$$

where we have used (3.1). It follows from (3.1) and (3.5) that

$$
\begin{aligned}
B_2 &\ll \int_0^{\frac{1}{X}}|T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \qquad (3.8)\\
&+X\int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}}|\alpha||T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha\\
&\ll X^{\frac{2}{3}}+X\int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}}|\alpha|X^{-\frac{10}{3}}|\alpha|^{-5}d\alpha\\
&\ll X^{\frac{2}{3}}.
\end{aligned}
$$

Then, from (3.6)-(3.8), we have $J_2=o(\tau^2X^{\frac{7}{6}})$.

Arguing similarly, we can also get $J_3=o(\tau^2X^{\frac{7}{6}})$.

### 3.1.3 Upper bounds for $J_4$ and $J_5$

By (2.1), we have

$$
\begin{aligned}
J_4 &\ll \tau^2\int_{\mathfrak{M}_1}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)||S_3(\lambda_3\alpha)-T_3(\lambda_3\alpha)|\\
&\cdot|T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha\\
&\ll \tau^2\int_{\mathfrak{M}_1}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)||S_3(\lambda_3\alpha)-U_3(\lambda_3\alpha)|\\
&\cdot|T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \qquad (3.9)
\end{aligned}
$$

$$+\tau^2\int_{\mathfrak{M}_1}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)||U_3(\lambda_3\alpha)-T_3(\lambda_3\alpha)|$$
$$\cdot|T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha$$
$$:=\tau^2(A_3+B_3).$$

Applying Hölder's inequality, by Lemmas 2.4 and 2.7, we obtain

$$\begin{aligned}(3.10)\qquad A_3 &\ll X^{\frac{5}{6}}\left(\int_{-1}^{1}|S_2(\lambda_1\alpha)|^4d\alpha\right)^{\frac{1}{4}}\left(\int_{-1}^{1}|S_2(\lambda_2\alpha)|^4d\alpha\right)^{\frac{1}{4}}\\ &\times\left(\int_{\mathfrak{M}_1}|S_3(\lambda_3\alpha)-U_3(\lambda_3\alpha)|^2d\alpha\right)^{\frac{1}{2}}\\ &\ll X^{\frac{5}{6}}X^{\frac{1}{2}}(\log X)X^{\frac{1}{3}-\frac{1}{2}}(\log X)^{-\frac{A}{2}}\\ &\ll X^{\frac{7}{6}}(\log X)^{-2}.\end{aligned}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned}(3.11)\qquad B_3 &\ll \int_0^{\frac{1}{X}}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha\\ &+X\int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}}|\alpha||S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha\\ &\ll X^{\frac{5}{6}}+X^{-\frac{7}{6}}\int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}}|\alpha|^{-2}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)|d\alpha\\ &\ll X^{\frac{5}{6}}+X^{-\frac{7}{6}}\left(\int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}}|\alpha|^{-4}d\alpha\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|S_2(\lambda_1\alpha)|^4d\alpha\right)^{\frac{1}{4}}\\ &\times\left(\int_{-1}^{1}|S_2(\lambda_2\alpha)|^4d\alpha\right)^{\frac{1}{4}}\\ &\ll X^{\frac{5}{6}}\log X,\end{aligned}$$

Then, from (3.9)-(3.11), we have $J_4=o(\tau^2X^{\frac{7}{6}})$.

Arguing similarly, we can also get $J_5=o(\tau^2X^{\frac{7}{6}})$.

### 3.1.4 Upper bounds for $J_6$ and $J_7$

The computation for $J_6$ and $J_7$ are similar to that for $J_4$ and $J_5$. Also by (2.1), we have

$$J_6\ll\tau^2\int_{\mathfrak{M}_1}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)|$$
$$\cdot|S_4(\lambda_5\alpha)-T_4(\lambda_5\alpha)||T_4(\lambda_6\alpha)|d\alpha$$

$$\begin{aligned}
&\ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)| \\
&\cdot |S_4(\lambda_5\alpha) - U_4(\lambda_5\alpha)||T_4(\lambda_6\alpha)|d\alpha \\
&+ \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)| \\
&\cdot |U_4(\lambda_5\alpha) - T_4(\lambda_5\alpha)||T_4(\lambda_6\alpha)|d\alpha \\
&:= \tau^2(A_4 + B_4).
\end{aligned} \tag{3.12}$$

Applying Hölder's inequality, by Lemmas 2.4 and 2.7, we obtain

$$\begin{aligned}
A_4 &\ll X^{\frac{11}{12}} \left(\int_{-1}^{1} |S_2(\lambda_1\alpha)|^4 d\alpha\right)^{\frac14} \left(\int_{-1}^{1} |S_2(\lambda_2\alpha)|^4 d\alpha\right)^{\frac14} \\
&\times \left(\int_{\mathfrak{M}_1} |S_4(\lambda_5\alpha) - U_4(\lambda_5\alpha)|^2 d\alpha\right)^{\frac12} \\
&\ll X^{\frac{11}{12}} X^{\frac12} (\log X) X^{\frac14 - \frac12} (\log X)^{-\frac{A}{2}} \\
&\ll X^{\frac76} (\log X)^{-2}.
\end{aligned} \tag{3.13}$$

From (3.1) and (3.5), we have

$$\begin{aligned}
B_4 &\ll \int_0^{\frac1X} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)T_4(\lambda_6\alpha)|d\alpha \\
&+ X \int_{\frac1X}^{X^{\frac{5}{24}-1-\varepsilon}} |\alpha||S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)T_4(\lambda_6\alpha)|d\alpha \\
&\ll X^{\frac{11}{12}} + X^{\frac14} \int_{\frac1X}^{X^{\frac{5}{24}-1-\varepsilon}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)|d\alpha \\
&:= X^{\frac{11}{12}} + B_{41}.
\end{aligned} \tag{3.14}$$

Using Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned}
B_{41} &\ll X^{\frac14} \left(\int_{-1}^{1} |S_2(\lambda_1\alpha)|^4 d\alpha\right)^{\frac14} \left(\int_{-1}^{1} |S_2(\lambda_2\alpha)|^4 d\alpha\right)^{\frac14} \\
&\times \left(\int_{-1}^{1} |S_3(\lambda_3\alpha)|^8 d\alpha\right)^{\frac18} \left(\int_{-1}^{1} |S_3(\lambda_4\alpha)|^8 d\alpha\right)^{\frac18} \left(\int_{\frac1X}^{X^{\frac{5}{24}-1-\varepsilon}} 1 d\alpha\right)^{\frac14} \\
&\ll X^{\frac{93}{96}+\varepsilon}.
\end{aligned} \tag{3.15}$$

Then, from (3.12)-(3.15), we have $J_6 = o(\tau^2 X^{\frac76})$.

Arguing similarly, we can also get $J_7 = o(\tau^2 X^{\frac76})$.

### 3.2 The region $\mathfrak{M}_2$

In this subsection, we give the upper bound for the integral on the region $\mathfrak{M}_2$.

**Lemma 3.2** ([3, Lemma 3.2]). *We have*

$$\int_{|\alpha|\le X^{-\frac{2}{3}}} |S_2(\lambda\alpha)|^2 d\alpha \ll 1.$$

Then, we have the following lemma:

**Lemma 3.3.** *We have*

$$\int_{\mathfrak{M}_2} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha = o(\tau^2 X^{\frac{7}{6}}).$$

**Proof.** By Lemmas 2.3, 2.4 and 3.2, we have

$$\begin{aligned}
&\int_{\mathfrak{M}_2} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha \\
&\ll \tau^2 X^{2(\frac{1}{4}-\frac{1}{2\cdot 4^4})+\varepsilon}\left(\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^2 d\alpha\right)^{\frac{1}{2}}\left(\int_{-1}^{1} S_2^2(\lambda_2\alpha)S_3^4(\lambda_3\alpha)d\alpha\right)^{\frac{1}{4}} \\
&\times\left(\int_{-1}^{1} S_2^2(\lambda_2\alpha)S_3^4(\lambda_4\alpha)d\alpha\right)^{\frac{1}{4}} \\
&\ll \tau^2 X^{\frac{7}{6}-\frac{1}{256}+\varepsilon} = o(\tau^2 X^{\frac{7}{6}}).
\end{aligned}$$

□

Therefore, combining Lemmas 3.1 and 3.3, we complete the estimate of the integral from the major arc. We have the following lemma:

**Lemma 3.4.** *We have*

$$\int_{\mathfrak{M}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)K_\tau(\alpha)e(-\upsilon\alpha)d\alpha \gg \tau^2 X^{\frac{7}{6}}.$$

## 4. The minor arc

In this section, we give the estimate of the integral on the minor arc. Let $\widetilde{\mathfrak{m}} = \mathfrak{m}_1 \cup \mathfrak{m}_2$, where

$$\mathfrak{m}_j = \{\alpha \in \mathfrak{m} : |S_2(\lambda_j\alpha)| \le X^{\frac{7}{16}+2\varepsilon}\}, \quad \text{for } j = 1, 2.$$

**Lemma 4.1.** *We have*

$$\int_{\widetilde{\mathfrak{m}}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.$$

**Proof.** Using Hölder's inequality and Lemma 2.4, we can obtain

$$\begin{aligned}&\int_{\mathfrak{m}_1} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha\\
&\ll \left(\sup_{\alpha\in\mathfrak{m}_1} |S_2(\lambda_1\alpha)|\right)^2 X^{\frac{4}{3}} \left(\int_{\mathbb{R}} |S_2^2(\lambda_2\alpha)S_4^4(\lambda_5\alpha)|K_\tau(\alpha)d\alpha\right)^{\frac{1}{2}}\\
&\times \left(\int_{\mathbb{R}} |S_2^2(\lambda_2\alpha)S_4^4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha\right)^{\frac{1}{2}}\\
&\ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.\end{aligned}$$

By symmetry we can get the same bound for the integral on $\mathfrak{m}_2$. Thus, we have

$$\int_{\widetilde{\mathfrak{m}}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.$$

□

Now, we turn to handle the range $\mathfrak{m}^* = \mathfrak{m}\backslash\widetilde{\mathfrak{m}}$. We have the following lemma:

**Lemma 4.2.** *We have*

$$\int_{\mathfrak{m}^*} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{3}{4}+\varepsilon}.$$

**Proof.** Note that, for any $\alpha \in \mathfrak{m}^*$, we have

$$|S_1(\lambda_1\alpha)| > X^{\frac{7}{16}+2\varepsilon} \text{ and } |S_2(\lambda_2\alpha)| > X^{\frac{7}{16}+2\varepsilon}.$$

We divide $\mathfrak{m}^*$ into disjoint sets $S(Z_1, Z_2, y)$, such that for $\alpha \in S(Z_1, Z_2, y)$, we have

$$Z_1 < |S_1(\lambda_1\alpha)| \le 2Z_1,\ Z_2 < |S_2(\lambda_2\alpha)| \le 2Z_2,\ y < |\alpha| \le 2y,$$

where $Z_1 = 2^{k_1}X^{\frac{7}{16}+2\varepsilon}, Z_2 = 2^{k_2}X^{\frac{7}{16}+2\varepsilon}$ and $y = 2^{k_3}\phi$ for some non-negative integers $k_1, k_2, k_3$. Then, from Lemma 2.2, there exist two pairs of coprime integers $(a_1, q_1)$ and $(a_2, q_2)$ satisfying

$$a_1a_2 \ne 0, 1 \le q_i \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_i}\right)^2, |q_i\lambda_i\alpha - a_i| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_i}\right)^2, i = 1, 2. \tag{4.1}$$

Furthermore, we subdivide $S(Z_1, Z_2, y)$ into sets $S(Z_1, Z_2, y, Q_1, Q_2)$, where $Q_j < q_j \le 2Q_j$ on each set. Then, we have

$$\begin{aligned}\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| &= \left|(q_1\lambda_1\alpha - a_1)\frac{a_2}{\lambda_2\alpha} - (q_2\lambda_2\alpha - a_2)\frac{a_1}{\lambda_2\alpha}\right|\\
&\ll Q_2X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2 + Q_1X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2\\
&\ll X^{1+4\varepsilon}Z_1^{-2}Z_2^{-2}\\
&\ll X^{-\frac{3}{4}-4\varepsilon}.\end{aligned}$$

We take $X = q^{\frac{4}{3}}$. Thus,

$$\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| = o(q^{-1}).$$

We also have $|a_2 q_1| \ll y Q_1 Q_2$. Thus, if $|a_2 q_1|$ took $R$ distinct values, we could apply the pigeon-hole principle to deduce the existence of $n$ satisfying

$$\left\|n\frac{\lambda_1}{\lambda_2}\right\| \ll X^{-\frac{3}{4}-4\varepsilon}, \quad n \ll \frac{yQ_1Q_2}{R}.$$

This would contradict $a/q$ being a convergent to $\lambda_1/\lambda_2$ if $q$ is sufficiently large, unless

$$R \ll \frac{yQ_1Q_2}{q}.$$

From the well-known bound for the divisor function, each value of $|a_2 q_1|$ corresponds to $O(X^{\varepsilon})$ values of $a_2, q_1$. Then, we conclude that each set of $S(Z_1, Z_2, y, Q_1, Q_2)$ is made up of $O(RX^{\varepsilon})$ intervals of length

$$\begin{aligned}
&\ll \min\left(Q_1^{-1}X^{-1}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\Big)^2, Q_2^{-1}X^{-1}\Big(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\Big)^2\right)\\
&\ll \frac{X^{2\varepsilon}}{Z_1 Z_2 Q_1^{1/2} Q_2^{1/2}}.
\end{aligned}$$

Let $\mathcal{A}$ denote such a set $S(Z_1, Z_2, y, Q_1, Q_2)$. Then, integrating over $\mathcal{A}$ gives

$$\begin{aligned}
(4.2)\qquad &\int_{\mathcal{A}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha\\
&\ll \min(\tau^2, y^{-2}) Z_1^2 Z_2^2 X^{\frac{4}{3}+1} \frac{X^{2\varepsilon}}{Z_1 Z_2 Q_1^{1/2} Q_2^{1/2}} \frac{X^{\varepsilon} y Q_1 Q_2}{q}\\
&\ll \tau X^{\frac{4}{3}+2+5\varepsilon} q^{-1}\\
&\ll \tau X^{\frac{31}{12}+5\varepsilon}.
\end{aligned}$$

Then, summing over all possible values of $Z_1, Z_2, y, Q_1, Q_2$, we conclude that

$$\int_{\mathfrak{m}^*} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{3}{4}+\varepsilon}.$$

□

Combining Lemmas 4.1 and 4.2 we can deduce the following lemma immediately.

**Lemma 4.3.** *We have*

$$\int_{\mathfrak{m}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.$$

## 5. The trivial arc

Applying Hölder's inequality, we have

$$
\begin{aligned}
&\int_{\mathfrak{t}}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha \\
&\ll\left(\int_{\xi}^{\infty}|S_2^2(\lambda_1\alpha)S_3^4(\lambda_3\alpha)|K_\tau(\alpha)d\alpha\right)^{\frac14}\left(\int_{\xi}^{\infty}|S_2^2(\lambda_1\alpha)S_3^4(\lambda_4\alpha)|K_\tau(\alpha)d\alpha\right)^{\frac14} \\
&\times\left(\int_{\xi}^{\infty}|S_2^2(\lambda_2\alpha)S_4^4(\lambda_5\alpha)|K_\tau(\alpha)d\alpha\right)^{\frac14}\left(\int_{\xi}^{\infty}|S_2^2(\lambda_2\alpha)S_4^4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha\right)^{\frac14}.
\end{aligned}\tag{5.1}
$$

These four integrals at the right side of (5.1) can be handled in a similar argument. Using (2.1) and Lemma 2.4, we have, for $j=3,4$,

$$
\begin{aligned}
&\int_{\xi}^{\infty}|S_2^2(\lambda_1\alpha)S_3^4(\lambda_j\alpha)|K_\tau(\alpha)d\alpha \\
&\ll\sum_{n=[\xi]}^{\infty}\int_{n}^{n+1}S_2^2(\lambda_1\alpha)S_3^4(\lambda_j\alpha)\frac{1}{\alpha^2}d\alpha \\
&\ll\sum_{n=[\xi]}^{\infty}\frac{1}{n^2}\int_0^1 S_2^2(\lambda_1\alpha)S_3^4(\lambda_j\alpha)d\alpha \\
&\ll\xi^{-1}X^{\frac43+\varepsilon}.
\end{aligned}
$$

Similarly, we can get, for $j=5,6$,

$$
\int_{\xi}^{\infty}|S_2^2(\lambda_1\alpha)S_4^4(\lambda_j\alpha)|K_\tau(\alpha)d\alpha\ll\xi^{-1}X^{1+\varepsilon}.
$$

Thus, inserting these bounds into (5.1) we have

$$
\begin{aligned}
&\int_{\mathfrak{t}}|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha \\
&\ll\xi^{-1}X^{\frac43+\varepsilon}\ll\tau^2X^{\frac76-\varepsilon},
\end{aligned}\tag{5.2}
$$

where $\xi=\tau^{-2}X^{\frac16+2\varepsilon}$ is used.

## 6. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2. We take $\tau=X^{-\delta}$. Let $\mathcal{E}=\mathcal{E}(\mathcal{V},X,\delta)$ denote the set of $\upsilon\in[X/2,X]\bigcap\mathcal{V}$ such that the inequality (1.2) has no solution in primes $p_1,p_2,p_3,p_4,p_5,p_6$, and $E=E(\mathcal{V},X,\delta)=|\mathcal{E}(\mathcal{V},X,\delta)|$. Then from Lemma 3.4 and (5.2), we have

$$
\begin{aligned}
&\left|\sum_{\upsilon\in\mathcal{E}}\int_{\mathfrak{m}}S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)e(-\upsilon\alpha)K_\tau(\alpha)d\alpha\right| \\
&\gg\tau^2X^{\frac76}E.
\end{aligned}\tag{6.1}
$$

Applying Cauchy's inequality and Lemma 4.3, we have

$$\begin{aligned}
&\left|\sum_{\upsilon\in\mathcal{E}}\int_{\mathfrak{m}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)e(-\upsilon\alpha)K_\tau(\alpha)d\alpha\right| \\
&\ll \left(\int_{-\infty}^{+\infty}\left|\sum_{\upsilon\in\mathcal{E}} e(-\upsilon\alpha)\right|^2 K_\tau(\alpha)d\alpha\right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathfrak{m}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha\right)^{\frac{1}{2}} \qquad (6.2)\\
&\ll \left(\tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}\right)^{\frac{1}{2}}\left(\sum_{\upsilon_1,\upsilon_2\in\mathcal{E}}\max(0,\tau-|\upsilon_1-\upsilon_2|)\right)^{\frac{1}{2}} \\
&\ll \tau E^{\frac{1}{2}}\left(X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}\right)^{\frac{1}{2}}.
\end{aligned}$$

Then combining (6.1) and (6.2), we have

$$E(\mathcal{V}, X_j, \delta) \ll X_j^{1-\frac{1}{8}+2\delta+\varepsilon}. \qquad (6.3)$$

Since $\lambda_1/\lambda_2$ is irrational, there are infinitely many $q$ we could have taken, and this gives the sequence $X_j \to +\infty$.

Now, if the convergent denominators for $\lambda_1/\lambda_2$ satisfy (1.4), then we can modify our works in Lemmas 4.1 and 4.2. We now assume that

$$\min(Z_1, Z_2) > X^{\chi+2\varepsilon}$$

with $\chi$ given by (1.6). We then obtain

$$\left|a_2 q_1\frac{\lambda_1}{\lambda_2} - a_1 q_2\right| \ll X^{1-4\chi-4\varepsilon}.$$

However, we know from (1.4) that there is a convergent $a/q$ to $\lambda_1/\lambda_2$ with

$$X^{(1-w)(4\chi-1)} \ll q \ll X^{4\chi-1}.$$

The expression corresponding to (4.2) is now

$$\begin{aligned}
&\int_{\mathcal{A}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \\
&\ll \tau X^{\frac{4}{3}+2+\varepsilon}q^{-1} \ll \tau X^{\frac{10}{3}-(1-w)(4\chi-1)+\varepsilon} \\
&\ll \tau X^{\frac{7}{3}+2\chi+\varepsilon}
\end{aligned}$$

by our choice of $\chi$. Thus,

$$\int_{\mathfrak{m}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{\frac{7}{3}+2\chi+\varepsilon}.$$

Working as (6.1)-(6.3), we can complete the proof of Theorem 1.2 easily.

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