Fractional Bessel differential equation and fractional Bessel functions

Ma'mon Abu Hammad
Department of Mathematics
AlZaytoonah University of Jordan
Amman
Jordan
M.abuhammad@zuj.edu.jo

Sh. AlSharif*
Department of Mathematics
Yarmouk University
Irbed
Jordan
sharifa@yu.edu.jo

R. Khalil
Department of Mathematics
Jordan University
Amman
Jordan
roshdi@ju.edu.jo

A. Shmasneh
Department of Mathematics
Jordan University
Amman
Jordan
shmasnehalaa@gmail.com

Abstract. Using a new simple and well-behaved definition of the fractional derivative which is different from the Caputo and Riemann-Liouville fractional derivative and recently introduced by Khalil and others, we reformulate the second order Bessel differential equation in this new setting. In this article by the use of power series, one of the solution of the fractional differential equation is obtained. Moreover, we find the generating function and use it to prove some nice standard results and recurrence relations. Finally, we present some application and integral representations of Bessel functions of fractional type including sines and cosines.

Keywords: Bessel equation, fractional derivative, Bessel functions.

$*$. Corresponding author

1. Introduction

Special functions [10, 12, 14] are of extreme importance in the study of mathematics and physics. At some level in quantum mechanics spherical harmonic are important in problems with spherical symmetry in electricity and magnetism, where they play the role of cosines and sines in the Fourier expanding of functions. Bessel functions and integrals involving them are in constant demand in applied mathematics and physics. They arise in problems with spherical symmetry. A typical example from Fourier analysis is the Fourier transform of radial functions in the Euclidean space $\mathbb{R}^n$.

Bessel functions which form the standard solution of the Bessel differential equation

$$x^2y'' + xy' + \left(x^2 - p^2\right) = 0,$$

where p is the order of the differential equation are named for the great mathematician Wilhelm Bessel $(1784 - 1846)$. However Daniel Bernoulli (1732), is credited to be the first mathematician to introduce the concept of Bessel functions. It is known [11], that these functions can be obtained by separating the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

in spherical coordinates. That's why Bessel functions plays a significant role in spherical harmonic and Fourier analysis. Bessel equation has two linearly independent solutions $J_n(x)$ and $Y_n(x)$, being a second order differential equation. For more on Bessel functions we refer the reader to the excellent books by Luke [10] and Watson [14].

Fractional calculus owes it's origin to the question raised by L'Hospital in 1695 of whether the derivative of an integer n could be extended and still be valid when n in not an integer. In the last decays fractional calculus caught attention of great mathematicians like Euler, Laplace, Fourier, Riemann, Liouville and others whodirectly or indirectly contributed to it's development, [9, 11, 13]. Recently, [8] a new definition has been simply formulated and called conformable fractional derivative depending on the basic limit definition of derivative. It is defined as: for a function $f : (0, \infty) \to \mathbb{R}$, the conformable fractional derivative of order $\alpha \in (0, 1)$ of f at $t > 0$ is given by

$$D^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}.$$

The fractional derivative of f at $t = 0$ is defined as $D^\alpha f(0) = \lim_{t \to 0} D^\alpha f(t)$. However the definition of fractional derivative either of Caputo or of Riemann-Liouville, both used an expression involving Gamma function and integral form.

The new fractional derivative (conformable fractional derivative) satisfies the product rule, the quotient rule and have results which are natural extension of the corresponding Roll's Theorem and Mean Value Theorem, but unfortunately it does not obey the natural chain rule, see [3, 8].

Note that if a function f is differentiable, then $D_\alpha(f)(t) = (t)^{1-\alpha} f'(t)$. It is clear that the conformable fractional derivative of the constant function is zero. On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives. One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $D_{\frac{1}{2}}(f)(t) = 1$. Hence $D_{\frac{1}{2}}(f)(0) = 1$. But $D_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives. For more on conformable fractional derivative and its applications we refer to [1]$-$[8] and references therein.

Using this new derivative we reformulate the second order Bessel differential equation namely, conformable Bessel fractional differential equation,

$$x^{2\alpha} D^\alpha D^\alpha y + \alpha x^\alpha D^\alpha y + (x^{2\alpha} - \alpha^2 p^2)y = 0. \tag{1.1}$$

With the use of power series, the first solution of the fractional differential equation is obtained. Then we find the generating function and use it to prove some nice standard results and recurrence relations. Finally we present some application and integral representations of Bessel functions of fractional type including sines and cosines.

2. Conformable fractional Bessel's differential equation

Analogous to second order homogeneous linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

we formulate the second order homogeneous linear fractional differential equation as

$$P(x) D^\alpha D^\alpha y + Q(x) D^\alpha y + R(x)y = 0. \tag{2.1}$$

Definition 2.1. *A point $x = x_0$ is called $\alpha-$**regular** singular point of equation* (2.1) *if:*

$$\lim_{x\to x_0} (x - x_0)^\alpha Q(x) \text{ exists and } \lim_{x\to x_0} (x - x_0)^{2\alpha} R(x) \text{ exists.}$$

If P, Q and R are polynomials with no common factors, then the singular points of the differential equation are those for which $P(x) = 0$. The point $x = 0$ is an $\alpha-$regular singular point for equation (1.1).

Fractional Bessel functions form the standard solution of the fractional Bessel differential equation

$$x^{2\alpha} D^\alpha D^\alpha y + \alpha x^\alpha D^\alpha y + (x^{2\alpha} - \alpha^2 p^2)y = 0,$$

where p is the order of the differential equation. To find the first solution for equation (1.1), we begin by taking the fractional power series

$$y = \sum_{m=0}^{\infty} a_m x^{\alpha m + \beta}, \tag{2.2}$$

where β is any real number which will be plugged into the Bessel equation (1.1) and solve for necessary component. For convenience the first and second fractional derivatives for equations (2.2) are

$$\begin{aligned} D^{\alpha}y &= \sum_{m=0}^{\infty}(\alpha m+\beta)a_m x^{\alpha m+\beta-\alpha}, \\ D^{\alpha}D^{\alpha}y &= \sum_{m=0}^{\infty}(\alpha m+\beta)(\alpha m+\beta-\alpha)a_m x^{\alpha m+\beta-2\alpha}. \end{aligned}$$

Substituting into the differential equation (1.1) , define a_{-1} and a_{-2} to be zero and collecting terms, we get

$$\sum_{m=0}^{\infty}\left(\left((\alpha m+\beta)(\alpha m+\beta-\alpha)+\alpha(\alpha m+\beta)-\alpha^2p^2\right)a_m+a_{m-2}\right)x^{\alpha m+\beta}=0.$$

Compare the coefficients of $x^{\alpha m+\beta}$ to yield

$$\begin{aligned} \left((\alpha m+\beta)(\alpha m+\beta-\alpha)+\alpha(\alpha m+\beta)-\alpha^2p^2\right)a_m+a_{m-2} &= 0, \\ \left((\alpha m+\beta)^2-\alpha(\alpha m+\beta)+\alpha(\alpha m+\beta)-\alpha^2p^2\right)a_m+a_{m-2} &= 0, \end{aligned}$$

$$\left((\alpha m+\beta)^2-\alpha^2p^2\right)a_m+a_{m-2}=0. \tag{2.3}$$

Letting $m=0$ in (2.3) and assuming $a_{m-2}=0$, we obtain the indicial equation

$$\left(\beta^2-\alpha^2p^2\right)a_0=0.$$

Since we have no condition on a_0, we can choose a convenient value as needed. Assuming $a_0\neq 0$, we have $\beta^2-\alpha^2p^2 = 0$ or $\beta=\pm\alpha p$ and so we have two cases:

Case 1. $\beta=\alpha\rho$. In this case (2.3) becomes

$$\begin{aligned} \left((\alpha m+\alpha\rho)^2-\alpha^2p^2\right)a_m+a_{m-2} &= 0, \\ \left(\alpha^2m^2+2\alpha^2mp+\alpha^2p^2-\alpha^2p^2\right)a_m+a_{m-2} &= 0, \end{aligned}$$

$$\alpha^2 m\ (m+2p)\,a_m+a_{m-2}=0. \tag{2.4}$$

Putting $m=1,2$ successively in (2.4), we get

$$\alpha^2(1+2p)a_1+a_{-1}=0,\ 2\alpha^2(2+2p)a_2+a_0=0.$$

So, $a_1=0$ and

$$a_2=\frac{-a_0}{2\alpha^2(2+2p)}=\frac{-a_0}{(2\alpha)^2(1+p)}.$$

Since $a_1 = 0$, we can infer that $a_3 = 0$. Continuing in this manner to obtain $a_{2k-1} = 0$, for all positive integers k. For $m = 4$, we have

$$4\alpha^2(4+2p)a_4 + a_2 = 0,$$
$$a_4 = \frac{-a_2}{4\alpha^2(2)(2+p)} = \frac{a_0}{(2\alpha)^4(2)(1+p)(2+p)}.$$

Continue the process for $m = 2r(even)$, a general formula for the coefficients a_m is obtained,

$$a_m = a_{2r} = \frac{(-1)^r a_0}{(2\alpha)^{2r} r!(1+p)(2+p)(3+p)...(r\ +p)}.$$

Substituting these coefficients into (2.2), gives one of the solution to the conformable fractional Bessel equation:

$$y = a_0 \sum_{r\ =0}^{\infty} \frac{(-1)^r}{(2\alpha)^{2r} r!(1+p)(2+p)(3+p)...(r\ +p)}\ x^{2\alpha r+\alpha p}, \tag{2.5}$$

where $a_0 \neq 0$ is arbitrary. With the aid of the gamma function giving a_0 the value

$$a_0 = \frac{1}{(2\alpha)p\Gamma(p+1)},$$

this solution may be written in a nice way and is denoted by $J_{\alpha p}$ called the conformable fractional Bessel function of order αp,

$$J_{\alpha p}(x) = \sum_{r\ =0}^{\infty} (-1)^r \frac{x^{2\alpha r+\alpha p}}{(2\alpha)^{2r\ +p} r!\Gamma(r\ +p\ +1)}. \tag{2.6}$$

Case 2. $\beta = -\alpha p$. Replacing p by $-p$ in case 1, we find

$$J_{-\alpha p}(x) = \sum_{r\ =0}^{\infty} (-1)^r \frac{x^{2\alpha r-\alpha p}}{(2\alpha)^{2r-p} r!\Gamma(r\ -p\ +1)}. \tag{2.7}$$

For $p = 0$ in (2.6) , the fractional Bessel function $J_0(x)$ has the form

$$J_0(x) = 1 - \frac{x^{2\alpha}}{\alpha^2(2)^2} + \frac{x^{4\alpha}}{\alpha^4(2)^2(4)^2} - \frac{x^{6\alpha}}{\alpha^6(2)^2(4)^2(6)^2} + \frac{x^{8\alpha}}{\alpha^8(2)^2(4)^2(6)^2(8)^2} + ...$$

The functions $J_{\alpha p}(x)$and $J_{-\alpha p}(x)$ in (2.6) and (2.7) are called conformable fractional Bessel functions. If $\alpha p = n$, for some integer n, then $J_n(x)$ is just the classical Bessel functions.

3. Fractional Bessel functions and generating function

Most of the properties of the fractional Bessel functions can be proved using their generating functions.

A generating function of a function J_n is another function $G(J_n, t)$ whose power series expansion has the function J_n as the coefficient of t^n. That means

$$G(J_n, t) = \Sigma\, t^n J_n(x).$$

In this section we find the generating function of the fractional Bessel functions of the first kind $J_{\alpha p}$.

Theorem 3.1. *Let p be an integer. Then the generating function of the fractional Bessel functions of the first kind is*

$$G(x,t) = \sum_{p=-\infty}^{\infty} J_{\alpha p}(x) t^p = e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})}.$$

Proof. Recall that the power series representation $e^t = \sum_{r=0}^{\infty} \frac{t^r}{r!}$. So, we have

$$\begin{aligned} e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} &= e^{\frac{x^\alpha t}{2\alpha}} \,.\, e^{-\frac{x^\alpha}{2\alpha}t^{-1}} \\ &= \sum_{r=0}^{\infty} \frac{x^{\alpha r}\, t^r}{(2\alpha)^r\, r!} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{\alpha k}\, t^{-k}}{(2\alpha)^k\, k!}. \end{aligned}$$

Since the multiplication of two infinite power series can be written as a Cauchy Product, we have

$$e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{\alpha r + \alpha k}\, t^{r-k}}{(2\alpha)^{r+k}\, k!\, r!}.$$

Letting $p = r - k$ so that p varies from $-\infty$ to ∞, the power series expansion can be written as

$$\begin{aligned} e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} &= \sum_{p=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{\alpha(p+k)+\alpha k} t^p}{(2\alpha)^{p+2k}\, k!\, (p+k)!} \\ &= \sum_{p=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2\alpha k + \alpha p}}{(2\alpha)^{p+2k}\, k!\, \Gamma(p+k+1)} \right) t^p. \end{aligned}$$

Using (2.6) and (2.7) we get

$$G(x,t) = e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} = \sum_{p=-\infty}^{\infty} J_{\alpha p}(x) t^p.$$

□

As an application of the generating function we have the following nice recurrence relations:

Theorem 3.2. *The following results are valid for all values of* αp :
(*i*) $J_{-\alpha p}(x) = (-1)^p J_{\alpha p}(x)$;
(*ii*) $\frac{d^\alpha}{dx^\alpha}\left[x^{\alpha \rho} J_{\alpha p}(x)\right] = x^{\ \alpha p} J_{\alpha(p-1);}$
(*iii*) $\frac{d^{\ \alpha}}{dx^\alpha}\left[x^{-\alpha p} J_{\alpha p}(x)\right] = -x^{\ -\alpha p} J_{\alpha p\ +\alpha}.$

Proof. (*i*)

$$\begin{aligned} J_{-\alpha p}(x) &= \sum_{r\ =0}^{\infty}(-1)^r \frac{x^{2\alpha r\ -\alpha p}}{(2\alpha)^{2r-p}\ \ r!\ \Gamma(r-p+1)} \\ &= \sum_{r\ =0}^{p-1}(-1)^r \frac{x^{2\alpha r-\alpha p}}{(2\alpha)^{2r-p}\ r!\ \Gamma(r-p+1)} \\ &\quad + \sum_{r\ =p}^{\infty}(-1)^r \frac{x^{2\alpha r\ -\alpha p}}{(2\alpha)^{2r\ -p}\ \ r!\ \Gamma(r-p+1)}. \end{aligned}$$

Regarding that $\Gamma(-p+r+1)$ is infinite for $r = 0, 1, 2, 3, ..., p$, the finite sum is zero. Letting $r = p + k$, the infinite series can be written as

$$\begin{aligned} J_{-\alpha p}(x) &= \sum_{k\ =0}^{\infty}(-1)^{k+p} \frac{x^{2\alpha(k+p)-\alpha p}}{(2\alpha)^{2(k+p)-p}\ (k+p)!\ \Gamma(k+1)} \\ &= (-1)^p \sum_{k\ =0}^{\infty}(-1)^k \frac{x^{2\alpha k+\alpha p}}{(2\alpha)^{2k\ +p}\ k!\ \Gamma(k+p+1)} = (-1)^p J_{\alpha p}(x). \end{aligned}$$

To prove (*ii*), wecalculate the $\alpha-$fractional derivative of $x^{\alpha\rho} J_{\alpha p}(x)$ with respect to x to get

$$\begin{aligned} \frac{d^\alpha}{dx^\alpha}\left[x^{\alpha\rho} J_{\alpha p}(x)\right] &= \frac{d^\alpha}{dx^\alpha}\left(\sum_{r=0}^{\infty}(-1)^r \frac{x^{2\alpha r+2\alpha p}}{(2\alpha)^{2r+p}\ r!\ \Gamma(r+p+1)}\right) \\ &= \sum_{r=0}^{\infty}(-1)^r \frac{2\alpha(r+p)\ x^{2\alpha r+2\alpha p-\alpha}}{(2\alpha)^{2r\ +p}\ r!\ (r+p)\Gamma(r+p\ )} \\ &= \sum_{r=0}^{\infty}(-1)^r \frac{x^{\ \alpha p} x^{2\alpha r+\alpha(p-1)}}{(2\alpha)^{2r+p-1}\ \ r!\ \Gamma(r+p-1+1)} = x^{\ \alpha p} J_{\alpha(p-1)}. \end{aligned}$$

Similarly (*iii*) can be proved by taking the $\alpha-$fractional derivative of $x^{-\alpha\rho} J_{\alpha p}(x)$,

$$\begin{aligned} \frac{d^{\ \alpha}}{dx^\alpha}\left[x^{-\alpha p} J_{\alpha p}(x)\right] &= \sum_{r\ =1}^{\infty}(-1)^r \frac{(2\alpha r)\ x^{2\alpha r-\alpha}}{(2\alpha)^{2r\ +\rho}\ r!\ \Gamma(r+\rho+1)} \\ &= \sum_{r\ =1}^{\infty}(-1)^r \frac{x^{\ 2\alpha r-\alpha}}{(2\alpha)^{2r+p-1}\ (r-1)!\ \Gamma(r+p+1)} \end{aligned}$$

$$\begin{aligned} &= -\sum_{r=0}^{\infty}(-1)^r \frac{x^{2\alpha r+\alpha(p+1)}}{(2\alpha)^{2r+\alpha(p+1)}\, r!\, \Gamma((p+1)+r+1)}\, x^{-\alpha p} \\ &= -x^{-\alpha\rho} J_{\alpha(p+1)}(x). \end{aligned}$$

□

4. Some applications

In this section some classical nice formulas of fractional type has been deduced and presented.

Lemma 4.1.

$$\int x^{\alpha p} J_{\alpha p-\alpha}(x) d^{\alpha} x = x^{\alpha p} J_{\alpha p}(x).$$

Proof. This follows immediately from Theorem 3.2(ii). In fact since

$$\frac{d^{\,\alpha}}{dx^{\alpha}}\left[x^{\alpha p} J_{\alpha p}(x)\right] = x^{\alpha p} J_{\alpha p-\alpha},$$

it follows that

$$\int x^{\alpha p} J_{\alpha p-\alpha}(x) d^{\alpha} x = x^{\alpha p} J_{\alpha p}(x).$$

□

Lemma 4.2. (1) $\int x^3 J_{\frac{1}{2}}(x) dx^{\frac{1}{2}} = x^3 J_1(x) - 2x^{\frac{5}{2}} J_{\frac{3}{2}}(x) + x^2 J_2(x)$.

(2) $\int x^2 J_{\frac{5}{2}}(x) d^{\frac{1}{2}} x = -x^2 J_2(x) - 3x^{\frac{3}{2}} J_{\frac{3}{2}}(x) - 12x J_1(x) - 24 x^{\frac{1}{2}} J_{\frac{1}{2}}(x) + 24 J_0(x)$.

Proof. (1) Using Theorem 3.2(ii) and Integration by parts and

$$u = x^2 \Longrightarrow d^{\frac{1}{2}} u = 2x^{\frac{3}{2}} d^{\frac{1}{2}} x,\ d^{\frac{1}{2}} v = x J_{\frac{1}{2}} \Longrightarrow v = x J_1$$

we get

$$\begin{aligned} \int x^3 J_{\frac{1}{2}}(x) d^{\frac{1}{2}} x &= \int x^2 x J_{\frac{1}{2}}(x) d^{\frac{1}{2}} x \\ &= x^3 J_1(x) - 2\int x^{\frac{5}{2}} J_1(x) d^{\frac{1}{2}} x. \end{aligned}$$

Repeating the process for the second integral

$$u = x \Longrightarrow d^{\frac{1}{2}} u = x^{\frac{1}{2}} d^{\frac{1}{2}} x, d^{\,\frac{1}{2}} v = x^{\frac{3}{2}} J_1(x) d^{\frac{1}{2}} x \Longrightarrow v = x^{\frac{3}{2}} J_{\frac{3}{2}}(x),$$

we get

$$\begin{aligned} \int x^3 J_{\frac{1}{2}}(x) d^{\frac{1}{2}} x &= x^3 J_1(x) - 2\left(x^{\frac{5}{2}} J_{\frac{3}{2}}(x) - \int x^2 J_{\frac{3}{2}}(x) d^{\frac{1}{2}} x\right) \\ &= x^3 J_1(x) - 2\, x^{\frac{5}{2}} J_{\frac{3}{2}}(x) + x^2 J_2(x). \end{aligned}$$

For part (2), the proof is similar to that of (1), using integration by parts

$$u = x^4 \Longrightarrow d^{\frac{1}{2}}u \; = 4x^{\frac{7}{2}}d^{\frac{1}{2}}x, d^{\frac{1}{2}}v = x^{-2}J_{\frac{5}{2}}(x)d^{\;\frac{1}{2}}x \Longrightarrow v = -x^{-2}J_2(x),$$

we have

$$\begin{aligned}\int x^2 J_{\frac{5}{2}}(x)d^{\frac{1}{2}}x &= \int x^4 x^{-2}J_{\frac{5}{2}}(x)d^{\frac{1}{2}}x \\ &= -x^2J_2(x) + 4\int x^{\frac{3}{2}}J_2(x)d^{\frac{1}{2}}x \\ &= -x^2J_2(x) + 4\int x^3x^{\frac{-3}{2}}J_2(x)d^{\frac{1}{2}}x.\end{aligned}$$

Continue repeating the process for the second integral

$$\begin{aligned} u &= x^3 \Longrightarrow d^{\frac{1}{2}}u \; = 3x^{\frac{5}{2}}d^{\frac{1}{2}}x, \\ d^{\frac{1}{2}}v &= x^{\frac{-3}{2}}J_2(x)d^{\frac{1}{2}}x \Longrightarrow v = -x^{\frac{-3}{2}}J_{\frac{3}{2}}(x)\end{aligned}$$

we have

$$\begin{aligned}\int x^2J_{\frac{5}{2}}(x)d^{\frac{1}{2}}x &= -x^2J_2(x) - 4x^{\frac{1}{2}}J_{\frac{3}{2}}(x) + 12\int x\; J_{\frac{3}{2}}(x)d^{\frac{1}{2}}x \\ &= -x^2J_2(x) - 3x^{\frac{1}{2}}J_{\frac{3}{2}}(x) + 12\int x^2x^{-1}J_{\frac{3}{2}}(x)d^{\frac{1}{2}}x \\ &= -x^2J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) \; - 12xJ_1(x) + 24\int x^{\frac{1}{2}}J_1(x)d^{\frac{1}{2}}x \\ &= -x^2J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) \; - 12xJ_1(x) + 24\int xx^{\frac{-1}{2}}J_1(x)d^{\frac{1}{2}}x \\ &= -x^2J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) \; - 12xJ_1(x) - 24x^{\frac{1}{2}}J_{\frac{1}{2}}(x) \\ &\quad +24\int J_{\frac{1}{2}}(x)d^{\frac{1}{2}}x \\ &= -x^2J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) \; - 12xJ_1(x) - 24x^{\frac{1}{2}}J_{\frac{1}{2}}(x) + 24J_0(x).\end{aligned}$$

□

Lemma 4.3. (1) $\cos(\frac{x^\alpha}{\alpha}\sin\theta) = J_0(x) + 2J_{2\alpha}(x)\cos 2\theta + 2J_{4\alpha}(x)\cos 4\theta + ...$
(2) $\sin(\frac{x^\alpha}{\alpha}\sin\theta) = 2J_\alpha(x)\sin\theta + 2J_{3\alpha}(x)\sin 3\theta + ...$

Proof. Using the generating function, $e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})}$, where $t = e^{i\,\theta}$, we get:

$$\begin{aligned} e^{\frac{x^\alpha}{2\alpha}(e^{i\,\theta}-e^{-i\,\theta})} &= e^{\frac{ix^\alpha}{\alpha}\sin\theta} \\ &= \sum_{-\infty}^{\infty} J_{\alpha p}(x)t^p\end{aligned}$$

$$
\begin{aligned}
&= \sum_{-\infty}^{\infty} J_{\alpha p}(x)\,(\cos p\theta + i \sin p\theta) \\
&= \left\{ \begin{array}{c} J_0(x) + (J_{-\alpha}(x) + J_\alpha(x))\cos\theta \\ + [J_{-2\alpha}(x) + J_{2\alpha}(x)]\cos 2\theta + ... \end{array} \right\} \\
&\quad + i \left\{ \begin{array}{c} (J_\alpha(x) - J_{-\alpha}(x))\sin\theta \\ + (J_{2\alpha}(x) - J_{-2\alpha}(x))\sin 2\theta + ... \end{array} \right\} \\
&= \left\{ \begin{array}{c} J_0(x) + (J_{-\alpha}(x) + J_\alpha(x))\cos\theta \\ + (J_{-2\alpha}(x) + J_{2\alpha}(x))\cos 2\theta + ... \end{array} \right\} \\
&\quad + i \left\{ \begin{array}{c} (J_\alpha(x) - J_{-\alpha}(x))\sin\theta \\ + (J_{2\alpha}(x) - J_{-2\alpha}(x))\sin 2\theta + ... \end{array} \right\} \\
&= \{J_0(x) + 2J_{2\alpha}(x)\cos 2\theta + ...\} \\
&\quad + i\{2J_\alpha(x)\sin\theta + 3J_{3\alpha}(x)\sin 2\theta + ...\}.
\end{aligned}
$$

The proof of (2) is similar. □

Finally, let us solve:

$$x\, D^{\frac{1}{2}} D^{\frac{1}{2}} y + \frac{1}{2} x^{\frac{1}{2}} D^{\frac{1}{2}} y + (x^{2} - 1) y = 0.$$

Here $\alpha = \frac{1}{2}$ and $p = 2$. Using equation (2.6) we get the solution to be.

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{r+1}}{r!\Gamma(r+3)}.$$

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