CLR property in Menger spaces and related common fixed point theorems

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Abstract. In this paper, we have proved some common fixed point theorems for weakly compatible mappings in Menger space using the notion of CLR and JCLR property and control functions. Some illustrative examples are also given to show the usability of the presented results.

Keywords: Menger space, compatible mappings, weakly compatible mappings, CLR property and JCLR property.

1. Introduction

The notion of probabilistic metric space as a generalization of metric space was introduced by Menger [16]. In Menger theory, the notion of probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In this note he explained how to replace the numerical distance between two points p and q by a function $F(p, q, t)$ whose value $F(p, q, t)$ at the real number t is interpreted as the probability that the distance between p and q is less than

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t. In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [20]. The theory of probabilistic metric space is of paramount importance in Probabilistic Functional Analysis especially due to its extensive applications in random differential as well as random integral equations.

Now, we give preliminaries and basic definitions in Menger space which are useful in this paper.

Definition 1.1 ([20]). *A mapping $F : \mathbb{R}^+ \to \mathbb{R}^+$ is called distribution function if it is non decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}^+\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}^+\} = 1$. We will denote the set of all distribution functions by $\mathcal{L}$.*

Let $\mathcal{L}$ be the set of all distribution functions whereas H be the set of specific distribution functions (also known as Heaviside functions) defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.2 ([16]). *A probabilistic metric space is a pair (K, F), where K is a nonempty set and $F : K \times K \times [0, \infty) \to \mathcal{L}$ is a mapping satisfying the following conditions for all $p, q, r \in K$ and $t, s \geq 0$,*

(p_1) $F(p, q, t) = 1$ *if and only if* $p = q$;

(p_2) $F(p, q, 0) = 0$;

(p_3) $F(p, q, t) = F(q, p, t)$;

(p_4) $F(p, q, t) = 1$ *and* $F(q, r, s) = 1$, *then* $F(p, r, (t + s)) = 1$.

Every metric space (K, d) can always be realized as a Probabilistic metric space by $F(p, q, t) = H(t - d(p, q))$, for all $p, q \in K$, where H be the set of specific distribution functions (also known as Heaviside functions) defined in [20, Definition 1.1].

Probabilistic metric space offers a wider framework than that of the metric space and cover even wider statistical situations.

Definition 1.3 ([20]). *A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if for all $a_1, b_1, c_1, d_1 \in [0, 1]$,*

(1) $\Delta(a_1, 1) = a_1$, $\Delta(0, 0) = 0$;

(2) $\Delta(a_1, b_1) = \Delta(b_1, a_1)$;

(3) $\Delta(c_1, d_1) \geq \Delta(a_1, b_1)$ *for* $c_1 \geq a_1$, $d_1 \geq b_1$;

(4) $\Delta(\Delta(a_1, b_1), c_1) = \Delta(a_1, \Delta(b_1, c_1))$.

Example 1.1. Following are the four basic t-norms:

(i) The minimum t-norm: $\Delta_M(a_1, b_1) = \min\{a_1, b_1\}$.

(ii) The product t-norm: $\Delta_P(a_1, b_1) = a_1 b_1$.

(iii) The Lukasiewicz t-norm: $\Delta_L(a_1, b_1) = \max\{a_1 + b_1 - 1, 0\}$.

(iv) The weakest t-norm, the drastic product:

$$\Delta_D(a_1, b_1) = \begin{cases} \min\{a_1, b_1\} & \text{if} \max\{a_1, b_1\} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have the following ordering in the above stated norms:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M.$$

Definition 1.4 ([16]). *A Menger space is a triplet (K, F, Δ), where (K, F) is a probabilistic metric space and Δ is a t-norm with the following condition:*
For all $p, q, r \in K$ and $t, s \geq 0$,

(p_5) $F(p, r, (t + s)) \geq \Delta(F(p, q, t), F(q, r, s))$.

Example 1.2. Let $K = \mathbb{R}$, $\Delta(a_1, b_1) = \min(a_1, b_1)$, for all a_1, b_1 in $[0, 1]$ and

$$F(p, q, t) = \begin{cases} H(t), & \text{if } p \neq q \\ 1, & \text{if } p = q \end{cases}; \quad \text{where} \quad H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Then (K, F, Δ) is a Menger space.

Definition 1.5. *A sequence $\{p_n\}$ in Menger space (K, F, Δ) is said to be:*

(i) *Convergent at a point $p \in K$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $F(p_n, p, \epsilon) > 1$ for all $n \geq N_{\epsilon,\lambda}$.*

(ii) *Cauchy sequence in K if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $F(p_n, p_m, \epsilon) > 1$ for all $n, m \geq N_{\epsilon,\lambda}$.*

(iii) *Complete if every Cauchy sequence in K is convergent in K.*

In 1996, Jungck [12] introduce the notion of weakly commuting mappings.

Definition 1.6 ([12]). *Two self-mapping f_1 and g_1 of a Menger space (K, F, Δ) are said to be weakly commuting if $F(f_1 g_1 p, g_1 f_1 p, t) \geq F(f_1 p, g_1 p, t)$, for each $p \in K$ and for each $t > 0$.*

In 1982, Sessa [21] weakened the concept of commutativity to weakly commuting mappings. Afterwords, Jungck [13] enlarged the concept of weakly commuting mappings to compatible mappings. In 1991, Mishra [17] introduced the notion of compatible mappings in the setting of probabilistic metric space.

Definition 1.7 ([17]). *Let* (K, F, Δ) *be a Menger space such that the* t*-norm* Δ *is continuous and* f_1, g_1 *be mappings from* K *into itself. Then* f_1 *and* g_1 *are said to be compatible if* $\lim_{n\to\infty} F(f_1g_1p_n, g_1f_1p_n, t) = 1$, *whenever* $\{p_n\}$ *is a sequence in* K *such that* $\lim_{n\to\infty} f_1p_n = \lim_{n\to\infty} g_1p_n = u_1$, *for some* $u_1 \in K$ *and for all* $t > 0$.

Definition 1.8. *Two self mappings* f_1 *and* g_1 *on a Menger space* (K, F, Δ) *are said to be non-compatible if either* $\lim_{n\to\infty} F(f_1g_1p_n, g_1f_1p_n, t)$ *is non-existent or* $\lim_{n\to\infty} F(f_1g_1p_n, g_1f_1p_n, t) \neq 1$, *whenever* $\{p_n\}$ *is a sequence in* K *such that* $\lim_{n\to\infty} f_1p_n = \lim_{n\to\infty} g_1p_n = u_1$, *for some* $u_1 \in K$ *and for all* $t > 0$.

Further, Singh and Jain [22] proved some fixed point theorems for weakly compatible maps in the setting of Menger space.

Definition 1.9 ([22]). *Two maps* f_1 *and* g_1 *are said to be weakly compatible if they commute at their coincidence points i.e., if* $f_1p = g_1p$, *for some* $p \in K$, *then* $f_1g_1p = g_1f_1p$.

Definition 1.10 ([1]). *A pair of self-mappings* (f_1, g_1) *on a Menger space* (K, F, Δ) *are said to satisfy the property* $(E.A)$ *if there exists a sequence* $\{p_n\}$ *in* K *such that*

$$\lim_{n\to\infty} F(f_1p_n, u_1, t) = \lim_{n\to\infty} F(g_1p_n, u_1, t) = 1, \text{ for some } u_1 \in K \text{ and for all } t > 0.$$

Remark 1.1. It is easy to see that two non-compatible self-mappings of a Menger space satisfy the property (E.A), but the converse is not true in general.

Definition 1.11. *The pairs* (A, S) *and* (B, T) *on a Menger space* (K, F, Δ) *are said to satisfy the common property* $(E.A)$ *if there exists two sequence* $\{p_n\}$ *and* $\{q_n\}$ *in* K *such that*

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Bq_n = \lim_{n\to\infty} Tq_n = u_1, \text{ for some } u_1 \in K.$$

If $B = A$ *and* $T = S$ *in the definition we get the definition of the property* $(E.A)$.

Definition 1.12. *A pair of self-mappings* A *and* S *of a Menger space* (K, F, Δ) *is said to satisfy the common limit range property with respect to the mapping* S *(briefly CLRs property) if there exists a sequence* $\{p_n\}$ *in* K *such that*

$$\lim_{n\to\infty} F(Ap_n, u_1, t) = \lim_{n\to\infty} F(Sp_n, u_1, t) = 1, \text{ for some } u_1 \in S(K) \text{ and for all } t > 0.$$

Now, we give an example of self-mappings A and S satisfying the CLRs property (see [2])

Example 1.3. Let (K, F, Δ) be a Menger space with $K = [0, \infty)$ and for all $p, q \in K$ by $F(p, q, t) = H(t - |p - q|)$, $t > 0$ and $F(p, q, 0) = 0$ where $\Delta(a_1, b_1) = \min\{a_1, b_1\}$ for all $a_1, b_1 \in [0, 1]$. Define self-mappings A and S on K by $Ap = p + 3$, $Sp = 4p$. Let a sequence $\{p_n = 1 + \frac{1}{n}\}$, $n \in N$ in K.

Since $\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = 4$, then

$$\lim_{n\to\infty} F(Ap_n, 4, t) = \lim_{n\to\infty} F(Sp_n, 4, t) = 1,$$

where $4 \in K$. Therefore, the mappings A and S satisfy the CLRs property.

Remark 1.2. From Example 1.3, it is clear that a pair (A, S) satisfying the property (E.A) property with the closeness of the subspace $S(K)$ always verify the CLRs property.

Definition 1.13. *The pairs (A, S) and (B, T) on a Menger space (K, F, Δ) are said to satisfy common limit range property with respect to mappings S and T (briefly CLR_{ST} property) if there exists two sequence $\{p_n\}$ and $\{q_n\}$ in K such that for all $t > 0$*

$$\begin{aligned}\lim_{n\to\infty} F(Ap_n, u_1, t) &= \lim_{n\to\infty} F(Sp_n, u_1, t) = \lim_{n\to\infty} F(Bq_n, u_1, t) \\ &= \lim_{n\to\infty} F(Tq_n, u_1, t) = 1, \text{ where } u_1 \in S(K) \cap T(K).\end{aligned}$$

Remark 1.3. If $B = A$ and $T = S$ in the definition we get the definition of CLRs property.

Remark 1.4. The CLR_{ST}property implies the common property (E.A), but the converse is not true in general (see [5]).

Proposition 1.1 ([15])**.** *If the pairs (A, S) and (B, T) satisfy the common property (E.A) and $S(K)$ and $T(K)$ are closed subsets of K, then the pairs satisfy also the CLR_{ST} property.*

In 2012, Imdad et al. [9] introduce the CLR_{ST} property in Menger space as follows:

Definition 1.14 ([9])**.** *The pairs (A, S) and (B, T) on a Menger space (K, F, Δ) are said to satisfy common limit range property with respect to mappings S and T (briefly CLR_{ST} property) if there exists two sequence $\{p_n\}$ and $\{q_n\}$ in K such that for all $t > 0$*

$$\begin{aligned}\lim_{n\to\infty} F(Ap_n, u_1, t) &= \lim_{n\to\infty} F(Sp_n, u_1, t) = \lim_{n\to\infty} F(Bq_n, u_1, t) \\ &= \lim_{n\to\infty} F(Tq_n, u_1, t) = 1,\end{aligned}$$

where $u_1 = Sr = Tr$, for some $r \in K$.

Remark 1.5. If $B = A$ and $T = S$ in the above definition we get the definition of CLRs property.

Definition 1.15 ([8])**.** *Two families of self-mappings $\{A_i\}$ and $\{S_j\}$ are said to be pair wise commuting if*

(1) $A_iA_j = A_jA_i$, $i, j \in \{1, 2, \ldots, m\}$;

(2) $S_kS_l = S_lS_k$, $k, l \in \{1, 2, \ldots, n\}$;

(3) $A_iS_k = S_kA_i$, $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$.

Lemma 1.1 ([22]). *Let $\{p_n\}$ be a sequence in a Menger space (K, F, Δ) with continuous t-norm Δ and $\Delta(t, t) \geq t$. If there exists a constant $k \in (0, 1)$ such that*

$$F(p_n, p_{n+1}, kt) \geq F(p_{n-1}, p_n, t),$$

for all $t > 0$ and $n = 1, 2, 3 \ldots$, then $\{p_n\}$ is a Cauchy sequence in K.

Lemma 1.2 ([22]). *Let (K, F, Δ) be a Menger space. If there exists $k \in (0, 1)$ such that*

$$F(p, q, kt) \geq F(p, q, t), \text{ for all } p, q \in K \text{ and } t > 0, \text{ then } p = q.$$

2. Main results

Now, we prove the main results as follows.

Lemma 2.1. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the following conditions:*

(2.1) *the pair (A, S) satisfies the CLR_S property or the pair (B, T) satisfies the CLR_T property,*

(2.2) *$A(K) \subseteq T(K)$, $B(K) \subseteq S(K)$,*

(2.3) *$T(K)$ or $S(K)$ is a closed subset of K,*

(2.4) *$B(q_n)$ converges for every sequence $\{q_n\}$ in K whenever $T(q_n)$ converges or $A(p_n)$ converges for every sequence $\{p_n\}$ in K whenever $S(p_n)$ converges*

(2.5) *$(1 + \alpha F(Sp, Tq, t))F(Ap, Bq, t) > \alpha \min\{F(Ap, Sp, t)F(Bq, Tq, t), F(Sp, Bq, t)F(Ap, Tq, t)\} + \min\{F(Sp, Tq, t), \sup_{t_1+t_2=\frac{2t}{k}} \min\{F(Ap, Sp, t_1), F(Bq, Tq, t_2), \sup_{t_3+t_4=2t} \min\{F(Sp, Bq, t_3), F(Ap, Tq, t_4)\}$, for all $p, q \in K$, $t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.*

Proof. Suppose that the pair (A, S) satisfies the CLR_S property and $T(K)$ is closed subset of K. Then there exists a sequence $\{p_n\}$ in K such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = r, \text{ where } r \in S(K).$$

Since $A(K) \subseteq T(K)$, there exists a sequence $\{q_n\}$ in K such that $A(p_n) = T(q_n)$. So,

$$\lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Ap_n = r, \text{ where } r \in S(K) \cap T(K).$$

Thus, $Ap_n \to r$, $Sp_n \to r$ and $Tq_n \to r$. Now, we show the $Bq_n \to r$.

Let $\lim_{n\to\infty} F(Bq_n, l, t_0) = 1$. Now, we show that $l = r$. Assume that $l \neq r$. We prove that there exists $t_0 > 0$ such that

$$F\left(r, l, \frac{2}{k}t_0\right) > F(r, l, t_0). \tag{2.6}$$

Suppose the contrary. Therefore for all $t > 0$, we have

$$F\left(r, l, \frac{2}{k}t\right) \leq F(r, l, t). \tag{2.7}$$

Using repeatedly (2.7), we obtain

$$F(r, l, t) \geq F\left(r, l, \frac{2}{k}t\right) \geq \cdots \geq F\left(r, l, \left(\frac{2}{k}\right)^n t\right) \to 1 \text{ as } n \to \infty,$$

this shows that $F(r, l, t) = 1$ for all $t > 0$, which contradicts $l \neq r$.

Without loss of generality, we may assume that t_0 in (2.6) is a continuous point of $F(r, l)$. Since every distance distribution function is left-continuous, (2.6) implies that there exists $\epsilon > 0$ such that (2.6) holds for all $t \in (t_0 - \epsilon, t_0)$. Since $F(r, l)$ is non-decreasing, the set of all discontinuous points of $F(r, l)$ is a countable set at most. Thus, when t_0 is a discontinuous point of $F(r, l)$, we can choose a continuous point t_1 of $F(r, l)$ and $F(r, l)$ in $(t_0 - \epsilon, t_0)$ to replace t_0.

Using the inequality (2.5) with $p = p_n$, $q = q_n$, we get for some $t_0 > 0$

$$\begin{aligned}
&(1 + \alpha F(Sp_n, Tq_n, t_0))F(Ap_n, Bq_n, t_0) \\
&> \alpha \min\{F(Ap_n, Sp_n, t_0)F(Bq_n, Tq_n, t_0), F(Sp_n, Bq_n, t_0)F(Ap_n, Tq_n, t_0)\} \\
&\quad + \min\left\{F(Sp_n, Tq_n, t_0), \min\left\{F(Ap_n, Sp_n, \epsilon), F\left(Bq_n, Tq_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right.\right., \\
&\qquad \left.\left. \min\{F(Sp_n, Bq_n, (2t_0 - \epsilon)), F(Ap_n, Tq_n, \epsilon)\right\}\right\}.
\end{aligned}$$

For all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $n \to \infty$, we have

$$\begin{aligned}
&F(r, l, t_0) + \alpha F(r, l, t_0) \\
&\quad \geq \alpha F(r, l, t_0) + \min\left\{F\left(r, l, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F\left(r, l, (2t_0 - \epsilon)\right)\right\}.
\end{aligned}$$

As $\epsilon \to 0$, we obtain $F(r, l, t_0) \geq F\left(r, l, \frac{2}{k}t_0\right)$, which contradicts (2.6). Thus, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. □

Remark 2.1. The converse of Lemma 2.1 is not true.

Theorem 2.1. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality (2.5) of Lemma 2.1. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, the pairs (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K.*

Proof. Suppose that the pairs (A,S) and (B,T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

Hence, there exists $u_1, v_1 \in K$ such that $Su_1 = Tv_1 = r$.

Now, we show that $Au_1 = Su_1 = r$.

If $Au_1 \neq Su_1$, putting $p = u_1$ and $q = q_n$ in inequality (2.5), we get for some $t_0 > 0$,

$$\begin{aligned}
&(1+\alpha F(Su_1,Tq_n,t_0))F(Au_1,Bq_n,t_0)\\
&> \alpha \min\{F(Au_1,Su_1,t_0)F(Bq_n,Tq_n,t_0), F(Su_1,Bq_n,t_0)F(Au_1,Tq_n,t_0)\}\\
&\quad + \min\left\{F(Su_1,Tq_n,t_0), \min\left\{F\left(Au_1,Su_1,\left(\frac{2}{k}t_0-\epsilon\right)\right), F(Bq_n,Tq_n,\epsilon),\right.\right.\\
&\qquad \left.\left.\min\{F(Su_1,Bq_n,\epsilon), F(Au_1,Tq_n,(2t_0-\epsilon))\right\}\right\},\\
&(1+\alpha F(r,r,t_0))F(Au_1,r,t_0)\\
&> \alpha \min\{F(Au_1,r,t_0)F(r,r,t_0), F(r,r,t_0)F(Au_1,r,t_0)\}\\
&\quad + \min\left\{F(r,r,t_0), \min\left\{F\left(Au_1,r,\left(\frac{2}{k}t_0-\epsilon\right)\right), F(r,r,\epsilon),\right.\right.\\
&\qquad \left.\left.\min\{F(r,r,\epsilon), F(Au_1,r,(2t_0-\epsilon))\right\}\right\},\\
&F(Au_1,r,t_0)+\alpha F(Au_1,r,t_0)\\
&\quad \geq \alpha F(Au_1,r,t_0) + \min\left\{F\left(Au_1,r,\left(\frac{2}{k}t_0-\epsilon\right)\right), F\left(Au_1,r,(2t_0-\epsilon)\right)\right\},
\end{aligned}$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $n \to \infty$, we have as $\epsilon \to 0$, $F(Au_1,l,t_0) \geq F\left(Au_1,l,\frac{2}{k}t_0\right)$, which contradicts (2.6) and so $Au_1 = Su_1 = r$. Therefore, u_1 is a coincidence point of the pair (A,S).

Now, we assert that $Bv_1 = Tv_1 = r$.

If $r \neq Bv_1$, putting $p = u_1$ and $q = v_1$ in the inequality (2.5), we get for some $t_0 > 0$,

$$\begin{aligned}
&(1+\alpha F(Su_1,Tv_1,t_0))F(Au_1,Bv_1,t_0)\\
&> \alpha \min\{F(Au_1,Su_1,t_0)F(Bv_1,Tv_1,t_0), F(Su_1,Bv_1,t_0)F(Au_1,Tv_1,t_0)\}\\
&\quad + \min\left\{F(Su_1,Tv_1,t_0), \min\left\{F(Au_1,Su_1,\epsilon), F\left(Bv_1,Tv_1,\left(\frac{2}{k}t_0-\epsilon\right)\right),\right.\right.\\
&\quad \left.\left.\min\{F(Su_1,Bv_1,(2t_0-\epsilon)), F(Au_1,Tv_1,\epsilon)\right\}\right\}
\end{aligned}$$

$$
\begin{aligned}
&(1+\alpha F(r,r,t_0))F(r,Bv_1,t_0)\\
&\quad > \alpha \min\{F(r,r,t_0)F(Bv_1,r,t_0), F(r,Bv_1,t_0)F(r,r,t_0)\}\\
&\qquad\qquad + \min\left\{F(r,r,t_0), \min\left\{F(r,r,\epsilon), F\left(Bv_1, r, \left(\frac{2}{k}t_0-\epsilon\right)\right)\right.\right.,\\
&\qquad\qquad\qquad \min\{F(Bv_1,r,(2t_0-\epsilon)), F(r,r,\epsilon)\}\\
&F(r,Bv_1,t_0)+\alpha F(r,Bv_1,t_0)\\
&\quad \geq \alpha F(r,Bv_1,t_0) + \min\left\{F\left(Bv_1,r,\left(\frac{2}{k}t_0-\epsilon\right)\right), F\left(Bv_1,r,(2t_0-\epsilon)\right)\right\},
\end{aligned}
$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. As $\epsilon \to 0$, $F(Bv_1,r,t_0) \geq F\left(Bv_1, r, \frac{2}{k}t_0\right)$, which contradicts (2.6) and so $Bv_1 = Tv_1 = r$. Therefore v_1 is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au_1 = Su_1$ we obtain $Ar = Sr$.

Now, we prove that r is a common fixed point of A and S. If $r \neq Ar$, applying the inequality (2.5) with $p = r$ and $q = v_1$, we get for some $t_0 > 0$

$$
\begin{aligned}
&t(1+\alpha F(Sr,Tv_1,t_0))F(Ar,Bv_1,t_0)\\
&\quad > \alpha \min\{F(Ar,Sr,t_0)F(Bv_1,Tv_1,t_0), F(Sr,Bv_1,t_0)F(Ar,Tv_1,t_0)\}\\
&\qquad + \min\left\{F(Ar,r,t_0), \min\left\{F\left(Ar,Sr,\epsilon\right), F\left(Bv_1,Tv_1,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right.\right.,\\
&\qquad\qquad \min\{F(Sr,Bv_1,(2t_0-\epsilon)), F(Ar,Tv_1,\epsilon)\}\\
&(1+\alpha F(Ar,r,t_0))F(Ar,r,t_0)\\
&\quad > \alpha \min\{F(Ar,Ar,t_0)F(r,r,t_0), F(Ar,r,t_0)F(Ar,r,t_0)\}\\
&\qquad + \min\left\{F(Ar,r,t_0), \min\left\{F(Ar,r,\epsilon), F\left(r,r,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right.\right.,\\
&\qquad\qquad \min\{F(Ar,r,(2t_0-\epsilon)), F(r,r,\epsilon)\}\\
&F(Ar,r,t_0)+\alpha(F(Ar,r,t_0))^2 \geq \alpha(F(Ar,r,t_0))^2 + F(Ar,r,t_0),
\end{aligned}
$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. $F(Ar,r,t_0) \geq F(Ar,r,t_0)$, which contradicts (2.6) and so $Ar = Sr = r$, which shows that r is a common fixed point of A and S. Since the pair (B,T) is weakly compatible, we get $Br = Tr$. Similarly, we can prove that r is a common fixed point of B and T. Hence r is a common fixed point of A, B, S and T. □

We now give an example to illustrate the above theorem see for details [2].

Example 2.1. Let (K,F,Δ) be a Menger space with $K = [3,11]$ and for all $p,q \in K$ by $F(p,q,t) = H(t-|p-q|)$, $t > 0$ and $F(p,q,0) = 0$, where $\Delta(a_1,b_1) =$

$\min\{a_1, b_1\}$ for all $a_1, b_1 \in [0,1]$. Define self-mappings A and S on K by

$$Ap = \begin{cases} 3, & \text{if } p \in 3 \cup (5,11) \\ 10, & \text{if } p \in (3,5) \end{cases}, \qquad Bp = \begin{cases} 3, & \text{if } p \in 3 \cup (5,11) \\ 9, & \text{if } p \in (3,5) \end{cases},$$

$$Sp = \begin{cases} 3, & \text{if } p = 3 \\ 7, & \text{if } p \in (3,5) \\ \frac{p+1}{3} & \text{if } p \in (5,11) \end{cases}, \qquad Tp = \begin{cases} 3, & \text{if } p = 3 \\ p+4, & \text{if } p \in (3,5) \\ p-2, & \text{if } p \in (5,11). \end{cases}$$

We take $\{p_n = 3\}$, $\left\{q_n = 5 + \frac{1}{n}\right\}$ or $\left\{p_n = 5 + \frac{1}{n}\right\}$, $\{q_n = 3\}$, since

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Bq_n = \lim_{n\to\infty} Tq_n = 3 \in S(K) \cap T(K).$$

Then, the pairs (A,S) and (B,T) satisfy the CLR_{ST} *property.* Thus all the conditions of Theorem 2.1 are satisfied and 3 is the unique common fixed point of the pairs $(A,S)$and (B,T). Remark that all the mappings are even discontinuous at their unique common fixed point 3. In this example $S(K)$ and $T(K)$ are not closed subsets of K.

Lemma 2.2. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions (2.1)-(2.4) of Lemma* 2.1 *and*

$$\begin{aligned}(2.8)\qquad &(1 + \alpha F(Sp, Tq, t))F(Ap, Bq, t) \\ &> \alpha \min\{F(Ap, Sp, t)F(Bq, Tq, t), F(Sp, Bq, t)F(Ap, Tq, t)\} \\ &\quad + \min\Bigg\{F(Sp, Tq, t) \sup_{t_1+t_2=\frac{2t}{k}} \min\{F(Ap, Sp, t_1), F(Sp, Bq, t_2) \\ &\qquad \sup_{t_3+t_4=\frac{2t}{k}} \min\{F(Bq, Tq, t_3), F(Ap, Tq, t_4)\Bigg\},\end{aligned}$$

for all $p, q \in K$, $t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Proof. As in the proof of Lemma 2.1, there exists $t_0 > 0$ such that (2.6) holds. Using the inequality (2.8) with $p = p_n$, $q = q_n$, we have

$$\begin{aligned}&(1 + \alpha F(Sp_n, Tq_n, t_0))F(Ap_n, Bq_n, t_0) \\ &> \alpha \min\{F(Ap_n, Sp_n, t_0)F(Bq_n, Tq_n, t_0), F(Sp_n, Bq_n, t_0)F(Ap_n, Tq_n, t_0)\} \\ &\quad + \min\left\{F(Sp_n, Tq_n, t_0) \min\left\{F(Ap_n, Sp_n, \epsilon), F\left(Sp_n, Bq_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right.\right. \\ &\qquad \left.\left.\min\left\{F\left(Bq_n, Tq_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(Ap_n, Tq_n, \epsilon)\right\}\right.\right\},\end{aligned}$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $n \to \infty$, we have

$$\begin{aligned}
&(1+\alpha F(r,r,t_0))F(r,l,t_0) \\
&\quad > \alpha \min\{F(r,r,t_0)F(l,r,t_0), F(r,l,t_0)F(r,r,t_0)\} \\
&\qquad + \min\left\{F(r,r,t_0)\min\left\{F(r,r,\epsilon), F\left(r,l,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right.\right. \\
&\qquad\quad \left.\left.\min\{F\left(l,r,\left(\frac{2}{k}t_0-\epsilon\right)\right), F\left(r,r,\epsilon\right)\right\} \\
&F(r,l,t_0)+\alpha F(r,l,t_0) \\
&\quad > \alpha F(r,l,t_0) + \min\left\{F(r,r,t_0)\min\left\{F\left(r,r,(\epsilon)\right), F\left(r,l,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right.\right. \\
&\qquad\quad \left.\left.\min\left\{F\left(l,r,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right), F\left(r,r,(\epsilon)\right)\right\}\right. \\
&F(r,l,t_0) > \min\left\{1, F\left(r,l,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right\}.
\end{aligned}$$

As $\epsilon \to 0$, we obtain, $F(r,l,t_0) \geq F\left(r,l,\frac{2}{k}t_0\right)$, which contradicts (2.6) and so we have $r = l$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property. □

Theorem 2.2. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality (2.8) of Lemma 2.2. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, the pairs (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in K.*

Proof. Suppose that the pairs (A,S) and (B,T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

Hence, there exists $u, v \in K$ such that $Su = Tv = r$. From Theorems 2.1, we can easily prove that r is a common fixed point of A, B, S and T.

If $B = A$ and $T = S$ in Theorems 2.1 and 2.2, we obtain a common fixed point for a set of self-mappings. □

In the proof of the following lemma, we do not need to prove the inequality (2.6).

Lemma 2.3. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions (2.1)-(2.4) of Lemma 2.1 and*

$$(2.9)\quad \begin{aligned}&(1+\alpha F(Sp,Tq,t))F(Ap,Bq,t)\\ &> \alpha \min\{F(Ap,Sp,t)F(Bq,Tq,t), F(Sp,Bq,t)F(Ap,Tq,t)\}\\ &+ \min\left\{F(Sp,Tq,t)\sup_{t_1+t_2=\frac{2t}{k}}\max\{F(Ap,Sp,t_1),F(Bq,Tq,t_3)\}\right.\\ &\left.\sup_{t_3+t_4=2t}\max\{F(Sp,Bq,t_2),F(Ap,Tq,t_4)\}\right\},\end{aligned}$$

for all $p, q \in K$, $t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. Suppose that the pair (A, S) satisfies the CLR_S property and $T(K)$ is closed subset of K. Then there exists a sequence $\{p_n\}$ in k such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = r,$$

where $r \in S(K)$.

Since $A(K) \subseteq T(K)$, there exists a sequence $\{q_n\}$ in K such that $A(p_n) = T(q_n)$. So,

$$\lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Ap_n = r,$$

where $r \in S(K) \cap T(K)$. Thus, $Ap_n \to r$, $Sp_n \to r$ and $Tq_n \to r$. Now, we show the $Bq_n \to r$. Let $\lim_{n\to\infty} F(Bq_n, l, t_0) = 1$. Now, we show that $l = r$. Assume that $l \neq r$.

Using the inequality (2.9) with $p = p_n$, $q = q_n$, we get for some $t_0 > 0$,

$$\begin{aligned}&(1+\alpha F(Sp_n,Tq_n,t_0))F(Ap_n,Bq_n,t_0)\\ &> \alpha \min\{F(Ap_n,Sp_n,t_0)F(Bq_n,Tq_n,t_0), F(Sp_n,Bq_n,t_0)F(Ap_n,Tq_n,t_0)\}\\ &+ \min\left\{F(Sp_n,Tq_n,t_0)\max\left\{F(Ap_n,Sp_n,\epsilon), F\left(Bq_n,Tq_n,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right.\right.\\ &\min\{F(Sp_n,Bq_n,(2t_0-\epsilon)),F(Ap_n,Tq_n,\epsilon)\},\end{aligned}$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $n \to \infty$, we have

$$\begin{aligned}&(1+\alpha F(r,r,t_0))F(r,l,t_0)\\ &> \alpha \min\{F(r,r,t_0)F(l,r,t_0), F(r,l,t_0)F(r,r,t_0)\}\\ &+ \min\left\{F(r,r,t_0)\max\left\{F(r,r,\epsilon), F\left(r,l,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right.\right.\\ &\max\{F(r,l,(2t_0-\epsilon)),F(r,r,\epsilon)\}\end{aligned}$$

$$
\begin{aligned}
&F(r,l,t_0) + \alpha F(r,r,t_0) \\
&\quad > \alpha F(r,r,t_0) + \min\left\{F(r,r,t_0) \max\left\{F\left(r,r,(\epsilon)\right), F\left(r,l,\left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right)\right.\right. \\
&\qquad\quad \max\{F(l,r,((2t_0-\epsilon))), F(r,r,(\epsilon))\} \\
&F(r,l,t_0) > 1,
\end{aligned}
$$

for some $t_0 > 0$ and so we have $r = l$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property. □

Theorem 2.3. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality (2.9) of Lemma 2.3. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, the pairs (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in K.*

Proof. Suppose that the pairs (A,S) and (B,T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

Hence, there exists $u, v \in K$ such that $Su = Tv = r$. Now, we show that $Au = Su = r$.

If $Au \neq Su$, putting $p = u$ and $q = q_n$ in inequality (2.9), we get for some $t_0 > 0$,

$$
\begin{aligned}
&(1+\alpha F(Su,Tq_n,t_0))F(Au,Bq_n,t_0) \\
&\quad > \alpha \min\{F(Au,Su,t_0)F(Bq_n,Tq_n,t_0), F(Su,Bq_n,t_0)F(Au,Tq_n,t_0)\} \\
&\qquad + \min\left\{F(Su,Tq_n,t_0), \max\left\{F\left(Au,Su,\left(\frac{2}{k}t_0-\epsilon\right)\right), F\left(Bq_n,Tq_n,\epsilon\right)\right\},\right. \\
&\qquad\quad \max\{F(Su,Bq_n,\epsilon), F(Au,Tq_n,(2t_0-\epsilon))\} \\
&(1+\alpha F(r,r,t_0))F(Au,r,t_0) \\
&\quad > \alpha \min\{F(Au,r,t_0)F(r,r,t_0), F(r,r,t_0)F(Au,r,t_0)\} \\
&\qquad + \min\left\{F(r,r,t_0), \max\left\{F\left(Au,r,\left(\frac{2}{k}t_0-\epsilon\right)\right), F\left(r,r,\epsilon\right),\right.\right. \\
&\qquad\quad \left.\max\{F(r,r,\epsilon), F(Au,r,(2t_0-\epsilon))\}\right\} \\
&F(Au,r,t_0) > 1,
\end{aligned}
$$

and so $Au = Su = r$. Therefore, u is a coincidence point of the pair (A,S).

Now, we assert that $Bv = Tv = r$. If $r \neq Bv$, putting $p = u$ and $q = v$ in the inequality (2.9), we get for some $t_0 > 0$

$$
\begin{aligned}
&(1+\alpha F(Su,Tv,t_0))F(Au,Bv,t_0)\\
&\quad > \alpha \min\{F(Au,Su,t_0)F(Bv,Tv,t_0), F(Su,Bv,t_0)F(Au,Tv,t_0)\}\\
&\qquad + \min\{F(Su,Tv,t_0), \max\left\{F(Au,Su,\epsilon), F\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\\
&\qquad\qquad \max\{F(Su,Bv,(2t_0-\epsilon)), F(Au,Tv,\epsilon)\}\}\\
&(1+\alpha F(r,r,t_0))F(r,Bv,t_0)\\
&\quad > \alpha \min\{F(r,r,t_0)F(Bv,r,t_0), F(r,Bv,t_0)F(r,r,t_0)\}\\
&\qquad + \min\left\{F(r,r,t_0), \max\left\{F(r,r,\epsilon), F\left(Bv,r,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\right.\\
&\qquad\qquad \left.\max\{F(Bv,r,(2t_0-\epsilon)), F(r,r,\epsilon)\}\right\}\\
&F(Bv,r,t_0) > 1
\end{aligned}
$$

and so $Bv = Tv = r$. Therefore, v is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au = Su$ we obtain $Ar = Sr$. Now, we prove that r is a common fixed point of A and S. If $r \neq Ar$, applying the inequality (2.9) with $p = r$ and $q = v$, we get for some $t_0 > 0$,

$$
\begin{aligned}
&(1+\alpha F(Sr,Tv,t_0))F(Ar,Bv,t_0)\\
&\quad > \alpha \min\{F(Ar,Sr,t_0)F(Bv,Tv,t_0), F(Sr,Bv,t_0)F(Ar,Tv,t_0)\}\\
&\qquad + \min\left\{F(Ar,r,t_0), \max\left\{F(Ar,Sr,\epsilon), F\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\},\right.\\
&\qquad\qquad \left.\max\{F(Sr,Bv,\epsilon), F(Ar,Tv,(2t_0-\epsilon))\}\right\}\\
&(1+\alpha F(Ar,r,t_0))F(Ar,r,t_0)\\
&\quad > \alpha \min\{F(Ar,Ar,t_0)F(r,r,t_0), F(Ar,r,t_0)F(Ar,r,t_0)\}\\
&\qquad + \min\left\{F(Ar,r,t_0), \max\left\{F(Ar,Ar,\epsilon), F\left(r,r,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\right.\\
&\qquad\qquad \left.\max\{F(Ar,r,\epsilon), F(Ar,r,(2t_0-\epsilon))\}\right\}\\
&F(Ar,r,t_0)+\alpha(F(Ar,r,t_0))^2 \geq \alpha(F(Ar,r,t_0))^2 + F(Ar,r,t_0)\\
&F(Ar,r,t_0) > F(Ar,r,t_0)
\end{aligned}
$$

which is impossible and so $Ar = Sr = r$, which shows that r is a common fixed point of A and S. Since the pair (B,T) is weakly compatible, we get $Br = Tr$. Similarly, we can prove that r is a common fixed point of B and T. Hence r is a common fixed point of A, B, S and T. □

Let φ be the set of all non-decreasing and continuous functions $\varphi : (0,1] \to (0,1]$ such that $\varphi(t) > t$ for all $t \in (0,1]$.

Lemma 2.4. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions (2.1)-(2.4) of Lemma 2.1 and*

$$(2.10)\qquad \begin{aligned}&(1+\alpha F(Sp,Tq,t))F(Ap,Bq,t)\\ &\quad > \alpha \min\{F(Ap,Sp,t)F(Bq,Tq,t), F(Sp,Bq,t)F(Ap,Tq,t)\}\\ &\quad + \varphi\Big\{F(Sp,Tq,tt) \sup_{t_1+t_3=\frac{2t}{k}} \min\{F(Ap,Sp,t_1), F(Bq,Tq,t_3)\}\\ &\qquad \sup_{t_2+t_4=} \frac{2t}{k} \min\{F(Sp,Bq,t_2), F(Ap,Tq,t_4)\}\Big\},\end{aligned}$$

for all $p, q \in K$, $t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$, $\varphi \in \varphi$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Proof. From Lemma 2.3 we have $Ap_n \to r$, $Sp_n \to r$ and $Tp_n \to r$. Now we show the $Bq_n \to r$.

Let $\lim_{n\to\infty} F(Bq_n, l, t_0) = 1$. Now we show that $l = r$. Assume that $l \neq r$. Using the inequality (2.10) with $p = p_n$, $q = q_n$, we get for some $t_0 > 0$

$$\begin{aligned}&(1+\alpha F(Sp_n,Tq_n,t_0))F(Ap_n,Bq_n,t_0)\\ &\quad > \alpha \min\{F(Ap_n,Sp_n,t_0)F(Bq_n,Tq_n,t_0), F(Sp_n,Bq_n,t_0)F(Ap_n,Tq_n,t_0)\}\\ &\quad + \varphi\Big\{F(Sp_n,Tq_n,t_0) \min\Big\{F(Ap_n,Sp_n,\epsilon), F\Big(Bq_n,Tq_n,\Big(\frac{2}{k}t_0-\epsilon\Big)\Big)\\ &\qquad \min\Big\{F\Big(Sp_n,Bq_n,\Big(\frac{2}{k}t_0-\epsilon\Big)\Big), F(Ap_n,Tq_n,\epsilon)\Big\},\end{aligned}$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $n \to \infty$, we have

$$\begin{aligned}&(1+\alpha F(r,r,t_0))F(r,l,t_0)\\ &\quad > \alpha \min\{F(r,r,t_0)F(l,r,t_0), F(r,l,t_0)F(r,r,t_0)\}\\ &\quad + \varphi\Big\{F(r,r,t_0) \min\Big\{F(r,r,\epsilon), F\Big(l,r,\Big(\frac{2}{k}t_0-\epsilon\Big)\Big)\\ &\qquad \min\Big\{F\Big(r,l,\Big(\frac{2}{k}t_0-\epsilon\Big)\Big), F(r,r,\epsilon)\Big\}\\ &F(r,l,t_0) + \alpha F(r,l,t_0)\\ &\quad > \alpha F(r,l,t_0) + \varphi\Big\{F(r,r,t_0) \min\Big\{F(r,r,(\epsilon)), F\Big(r,l,\Big(\Big(\frac{2}{k}t_0-\epsilon\Big)\Big)\Big)\\ &\qquad \min\Big\{F\Big(l,r,\Big(\Big(\frac{2}{k}t_0-\epsilon\Big)\Big)\Big), F(r,r,(\epsilon))\Big\}\\ &F(r,l,t_0) > \varphi\Big(F\Big(l,r,\Big(\Big(\frac{2}{k}t_0-\epsilon\Big)\Big)\Big)\Big),\end{aligned}$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. As $\epsilon \to 0$, $F(r, l, t_0) > F\left(l, r, \left(\left(\frac{2}{k}t_0\right)\right)\right)$, for some $t_0 > 0$.

By (2.6) so we have $r = l$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. □

Remark 2.2. Lemmas 2.1, 2.2, 2.3 and 2.4 remain true if we assume that the pair (B, T) satisfies the CLR_T property, $B(K) \subseteq S(K)$ and $S(K)$ is a closed subset of K.

Theorem 2.4. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality* (2.10) *of Lemma* 2.4*. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, the pairs (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K.*

Proof. Suppose that the pairs (A, S) and (B, T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

From Theorem 2.3 and using function φ, we can easily prove the theorems. □

Remark 2.3. In Theorems 2.1, 2.2, 2.3 and 2.4 by a similar manner, we can prove that A, B, S and T have a unique common fixed point in K if we assume that the pairs (A, S) and (B, T) verify $JCLR_{ST}$ property or CLR_{AB} property instead of CLR_{ST} property.

Theorem 2.5. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions of Lemma* 2.1*, or Lemma* 2.2*, Lemma* 2.3*, Lemma* 2.4*. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K.*

Proof. In view of Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4, the pairs (A, S) and (B, T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

The rest of the proof follows the proof of Theorems 2.1, 2.2, 2.3 and 2.4. □

Example 2.2. We retain A and B and replace S and T in Example 2.1 by the following mappings

$$Sp = \begin{cases} 3, & \text{if } p = 3 \\ 6, & \text{if } p \in (3, 5) \\ \frac{p+1}{3} & \text{if } p \in [5, 11), \end{cases} \qquad Tp = \begin{cases} 3, & \text{if } p = 3 \\ 9, & \text{if } p \in (3, 5) \\ p - 2, & \text{if } p \in [5, 11). \end{cases}$$

Therefore, $A(K) = \{3, 4\} \subset [3, 9] = T(K)$ and $B(K) = \{3, 5\} \subset [3, 6] = S(K)$.

Thus, all the conditions of Theorem 2.3 are satisfied and 3 is the unique common fixed point of the pairs (A, S) and (B, T). Also, it is noted that Theorem 2.1 cannot be used in the context of this example as $S(K)$ and $T(K)$ are closed subsets of K.

Applying Theorems 2.1, 2.2, 2.3 and 2.4, we deduce a common fixed point for four finite families of self-mappings given by the following corollary.

Corollary 2.1. *Let* $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ *and* $\{T_h\}_{h=1}^q$ *be four finite families of self-mappings of Menger space* (K, F, Δ), *where* Δ *is a continuous* t*-norm with* $A = A_1A_2 \ldots A_m$, $B = B_1B_2 \ldots B_n$, $S = S_1S_2 \ldots S_p$, $T = T_1T_2 \ldots T_q$ *satisfies the inequality* (2.5) *of Lemma* 2.1 *or the inequality* (2.8) *of Lemma* 2.2. *Suppose that the pairs* (A, S) *and* (B, T) *verify the* CLR_{ST} *property. Then* $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ *and* $\{T_h\}_{h=1}^q$ *a unique common fixed point in* K, *provided that the pairs of families* $(\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p)$, $(\{B_r\}_{r=1}^n, \{T_h\}_{h=1}^q)$ *commute pair wise.*

By setting $A = A_1 = A_2 = \cdots = A_m$, $B = B_1 = B_2 = \cdots = B_n$, $S = S_1 = S_2 = \cdots = S_p$, $T = T_1 = T_2 = \cdots = T_q$ *in Corollary* 2.1, *we get that* A, B, S *and* T *have a unique common fixed point in* K, *provided that the pairs* (A^m, S^p) *and* (B^n, T^q) *commute pair wise.*

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Accepted: July 20, 2020