Dynamically defined topological entropy of co-compact open covers

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Abstract. In this paper, the concept of dynamically defined topological entropy of co-compact open sets for iteration of a continuous system on non-compact Hausdorff space is introduced. Dynamically defined topological entropy is introduced as an invariant of topological conjugation for perfect mappings defined on any Hausdorff space (compactness and metrizability are not necessarily required). This is achieved through the consideration of a topological dynamic system and co-compact covers of the space. An example in the genetic space is presented.

Keywords: co-compact open set, dynamically defined co-compact entropy, topological dynamical system, genetic space.

1. Introduction

Entropy was first introduced into the theory of dynamical systems by Kolmogorov and Sinai [7] in 1958. Later, on the basis of this work, in 1965, Adler, Konheim and McAndrew [1] defined topological entropy for a continuous map on a compact topological space, using open covers. From then on, entropy soon

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became the center of attention. The importance of entropy arises from it's invariance under conjugacy. Therefore, systems with different entropies cannot be conjugate. This article is an attempt to present an approach to the entropy for iteration of a continuous system $f : X \to X$ on a non-compact Hausdorff space X.

Instead of using all open covers of the space to define entropy, we consider the open covers consisting of the co-compact open sets. The advantages of dynamically defined topological entropy include:

(i) It does not require the space to be metrizable, compared to Bowen's entropy, which is metric dependent.

(ii) It does not require the space to be compact.

(iii) It is an invariant of topological conjugation.

2. Basic notions

Let (X, f) be a topological dynamical system, where X is a Hausdorff and $f : X \to X$ is a continuous mapping.

Definition 2.1. Let X be a Hausdorff space. For an open subset, U of X, if $X \setminus U$ is a compact subset of X, then U is called a co-compact open subset. If every element of an open cover α of X is co-compact, then α is called a co-compact open cover of X (see [13]).

Theorem 2.1. *The intersection of finitely many co-compact open subsets is co-compact, and the union of any collection of co-compact open sets is co-compact open.*

Proof. See [13]. □

Theorem 2.2. *Let X be Hausdorff. Then, any co-compact open cover has a finite subcover.*

Proof. See [13]. □

Definition 2.2. Let X and Y be Hausdorff spaces and let $f : X \to Y$ be a continuous mapping. If f is a closed mapping and all fibers, $f^{-1}(x), x \in Y$, are co-compact, then f is called a perfect mapping.

In particular, if X is compact Hausdorff and Y is Hausdorff, every continuous mapping from X into Y is perfect. If $f : X \to Y$ is perfect, then $f^{-1}(F)$ is compact for each compact subset, $F \subset Y$ (see [6]).

Theorem 2.3. *Let X, Y be two Hausdorff spaces and let $f : X \to Y$ be a perfect mapping. If U is co-compact open in Y, then $f^{-1}(U)$ is co-compact open in X. Moreover if α is a co-compact open cover of Y, then $f^{-1}(\alpha)$ is a co-compact open cover of X.*

Proof. See [13]. □

3. Dynamically defined topological entropy of co-compact open covers

For compact topological systems, Adler, Konheim and McAndrew introduced the concept of topological entropy and studied their properties [1]. But in many physical systems the state space is not compact. For this goal we would like to present another approach to topological entropy for non-compact Hausdorff spaces. Therefore, in the remainder of the paper, a space is assumed to be Hausdorff and a mapping is assumed to be perfect.

Let X be Hausdorff and α be an open co-compact cover of X. Moreover let $N(f^n(X), \alpha)$ be the minimum cardinal number of subsets of α which are a cover for $f^n(X)$, where n is a natural number. By Theorem 2.3, when α is a co-compact open cover of X, α has a finite subcover. Hence, $N(f^n(X), \alpha)$ is a finite positive integer. We define $H(f^n(X), \alpha)$ as the number $\log N(f^n(X), \alpha)$.

Let α, β be two co-compact open covers of X. Define

$$\alpha \vee \beta = \{U \cap V; U \in \alpha \quad \text{and} \quad V \in \beta\}.$$

If for any $U \in \alpha$, there exists $V \in \beta$, such that $U \subset V$, then α is said to be a refinement of β and is denoted by $\beta \prec \alpha$.

The following are some obvious facts:

(i) For any open covers α, β of X, $\alpha \prec \alpha \vee \beta$.

(ii) For any open covers, α and β of X, if β is a subcover of α, then $\alpha \prec \beta$.

(iii) $H(f^n(X), \alpha) \geq 0$.

(iv) For any co-compact open cover α of X and $n \in \mathbb{N}$, $H(f^n(X), \alpha) = 0 \iff N(f^n(X), \alpha) = 1 \iff f^n(X) \in \alpha$.

Theorem 3.1. *Let α, β be two co-compact open covers of X and n be a natural number then,*

(i) If $\alpha \prec \beta$ then $H(f^n(X), \alpha) \leq H(f^n(X), \beta)$.

(ii) $H(f^n(X), \alpha \vee \beta) \leq H(f^n(X), \alpha) + H(f^n(X), \beta)$.

(iii) If $f : X \to X$ is a continuous map then $H(f^{n-1}(X), f^{-1}(\alpha)) \leq H(f^n(X), \alpha)$.

Proof. One can deduce (i) by straightforward calculations based on the definition. For the proof of (ii), let $\{A_1, A_2, ..., A_l\} \subset \alpha$ and $\{B_1, B_2, ..., B_m\} \subset \beta$ be two minimal subcovers for $f^n(X)$. Then, $\{B_i \cap A_j\}$ is an open subcover for $f^n(X)$. So,

$$H(f^n(X), \alpha \vee \beta) \leq \log(lm) = \log l + \log m = H(f^n(X), \alpha) + H(f^n(X), \beta).$$

(iii) Let $\{A_1, ..., A_m\} \subset \alpha$ be a minimal subcover for $f^n(X)$. Then

$$\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_m)\} \subset f^{-1}(\alpha)$$

is an open cover for $f^{n-1}(X)$. So, $H(f^{n-1}(X), f^{-1}(\alpha)) \leq H(f^n(X), \alpha)$. $\square$

Definition 3.1. Let X be Hausdorff space, $f : X \to X$ be a perfect mapping, and α be a co-compact open cover of X. The non-negative number, $h(f^n(X), \alpha) = \limsup_{m\to\infty} \dfrac{1}{m} H(f^n(X), \bigvee_{i=0}^{m-1} f^{-i}(\alpha))$, is said to be the dynamically defined topological entropy of f^n relative to α.

In particular when X is compact Hausdorff, any open set of X is co-compact, and any continuous mapping $f : X \to X$ is perfect. Note that when X is Hausdorff and $f : X \to X$ is perfect, $f^m : X \to X$ is also a perfect mapping [6]. It should be made aware that the new entropy is well defined for perfect mappings on non-compact spaces, e.g., $\mathbb{R}^n$, But Adler, Konheim and McAndrew's topological entropy requires that the space be compact.

4. An example of dynamically defined topological entropy in the genetic space

Each genetic code consists an alphabet of four letters [12]. DNA's alphabet is the set $\{T, C, A, G\}$ (corresponding to the bases thymine, cytosine, adenine, and guanine), while RNA's is $\{U, C, A, G\}$ (corresponding to the bases uracil, cytosine, adenine, and guanine). To model each genetic code we consider n letters as a mapping from the set of integer numbers to $\{0, 1, 2, 3\}$, where $0, 1, 2$ are identification numbers for adenine, guanine, and cytosine respectively, which 3 is the identification number for either thymine or uracil. If we denote $\omega(i)$ by ω_i, then a genetic code can be denoted by a sequence (ω_n) where $(\omega_n) \in \{0, 1, 2, 3\}$.

So, the set of genetic codes is the set $G = \{(\omega_n) : \omega_n \in \{0, 1, 2, 3\}\}$.

We know that there is a one to one correspondence between G and the set of real numbers. So, in this case G is not a compact metric space. The mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ creates a dynamics on the genetic space. f is clearly a perfect mapping. For this linear system, $(\mathbb{R}, f)$, defined by $f(x) = 2x$, the dynamically defined topological entropy is zero.

5. Dynamically defined topological entropy and conjugacy

Theorem 5.1. *Let X be Hausdorff and $f : X \to X$ be perfect. Then,* $h(f^n(X), \alpha) = h(f^n(X), \bigvee_{j=0}^{k} f^{-j}(\alpha))$.

Proof. We obtain immediately

$$\begin{aligned} h(f^n(X), \bigvee_{j=0}^{k} f^{-j}(\alpha)) &= \limsup_{m\to\infty} \frac{1}{m} H(f^n(X), \bigvee_{i=0}^{m-1} f^{-i}(\bigvee_{j=0}^{k} f^{-j}(\alpha)) \\ &= \limsup_{m\to\infty} \frac{1}{m} H(f^n(X), \bigvee_{t=0}^{m+k-1} f^{-t}(\alpha)) \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{p\to\infty} \frac{p}{p-k}\frac{1}{p} H(f^n(X), \bigvee_{t=0}^{p-1} f^{-t}(\alpha)) \\
&= h(f^n(X), \alpha). \quad \square
\end{aligned}
$$

Theorem 5.2. *Let X be Hausdorff and $f : X \to X$ be perfect. Then for each natural numbers n, k such that $k < n$, we have, $h(f^{n-k}(X), f^{-k}(\alpha)) \leq h(f^n(X), \alpha)$.*

Proof. It suffices to show that $h(f^{n-1}(X), f^{-1}(\alpha)) \leq h(f^n(X), \alpha)$.

By Theorem 3.1 (iii), we have

$$
\begin{aligned}
h(f^{n-1}(X), f^{-1}(\alpha)) &= \limsup_{m\to\infty} \frac{1}{m} H(f^{n-1}(X), \bigvee_{i=0}^{m-1} f^{-i}(f^{-1}(\alpha)) \\
&\leq \limsup_{m\to\infty} \frac{1}{m} H(f^n(X), \bigvee_{i=0}^{m-1} f^{-i}(\alpha)) \\
&= h(f^n(X), \alpha).
\end{aligned}
$$

$\square$

Definition 5.1. Let X be Hausdorff space, $f : X \to X$ be a perfect mapping. The non-negative number, $h(f^n(X)) = \sup_\alpha h(f^n(X), \alpha)$, is said to be the dynamically defined topological entropy of f^n. Where the supremum is taken over all co-compact open covers of X.

Theorem 5.3. *Let X be Hausdorff and $id : X \to X$ be the identity mapping. Then $h(id(X)) = 0$.*

Proof. Obvious. $\square$

Theorem 5.4. *Let X be Hausdorff space, $f : X \to X$ be a perfect mapping. If Ω is a closed subset of X and invariant under f, i.e., $f(\Omega) \subseteq \Omega$, then $h(f^n(\Omega)) \leq h(f^n(X))$.*

Proof. Obvious. $\square$

Theorem 5.5. *Let (X, f) and (Y, g) be two topological dynamical systems, where X and Y are Hausdorff, $f : X \to X$ and $g : Y \to Y$ are perfect mappings. If there exists a topological conjugation, $\varphi : X \to Y$, where φ is also perfect, then $h(f^n(X)) = h(g^n(Y))$.*

Proof. Let α be any co-compact open cover of Y. As φ is perfect and α is co-compact open cover of Y, $\varphi^{-1}(\alpha)$ is co-compact open cover of X by applying

Theorem 2.3. Hence, we have:

$$\begin{aligned} h(g^n(Y),\alpha) &= \limsup_{m\to\infty}\frac{1}{m}H(g^n(Y),\bigvee_{i=0}^{m-1}g^{-i}(\alpha)) \\ &= \limsup_{m\to\infty}\frac{1}{m}H(\varphi^{-1}(g^n(Y)),\bigvee_{i=0}^{m-1}\varphi^{-1}(g^{-i}(\alpha))) \\ &= \limsup_{m\to\infty}\frac{1}{m}H(f^n(\varphi^{-1}(Y)),\bigvee_{i=0}^{m-1}f^{-i}(\varphi^{-1}(\alpha)) \end{aligned}$$

$$\begin{aligned} &= \limsup_{m\to\infty}\frac{1}{m}H(f^n(X),\bigvee_{i=0}^{m-1}f^{-i}(\varphi^{-1}(\alpha))) \\ &= h(f^n(X),\varphi^{-1}(\alpha)). \end{aligned}$$

Therefore, $h(f^n(X)) \geq h(g^n(Y))$. When φ is a topological conjugation, it is, of course, perfect, too. Hence, we have both $h(f^n(X)) \geq h(g^n(Y))$ and $h(g^n(Y)) \geq h(f^n(X))$ from the above proof, implying $h(f^n(X)) = h(g^n(Y))$. $\square$

6. Conclusions

In this paper an extension of the notion of topological entropy for non-compact Hausdorff spaces (compactness and metrizability are not necessarily required) is introduced. The dynamically defined topological entropy is defined based on the co-compact open covers. Interestingly, for non-compact systems this new entropy retains invariant under topological conjugation. The computation of dynamically defined topological entropy of locally compact manifolds for well known examples can be a topic for future research.

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