A study on the fuzzy dynamical systems by using of dynamical relations

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Abstract. In this paper, some dynamical relations on fuzzy dynamical systems are introduced and their properties are studied. This relations are special subsets of $X \times X$, where X denotes the base space of the fuzzy dynamical systems $(X, f, M, *)$. By using this relations (i.e. $O_F f, R_F f, N_F f$) a new method for studying fuzzy dynamical systems is introduced.

Keywords: fuzzy dynamical system, orbit relation, recurrent relation, prolongation relation.

1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [8]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories [4, 5, 8]. Among other fields, a progressive developments is made in the field of fuzzy dynamical systems. The theory of fuzzy dynamical systems has been investigated by many authors from different points of view

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[2, 7]. In analogy to the classical case, this paper is an attempt to present a new method for studying fuzzy dynamical systems. To do this, we introduce the concept of three new dynamical relations Of, Rf and Nf. By using this relations the properties of fuzzy dynamical systems are investigated.

2. Preliminary facts

This section is devoted to provide the prerequisites that are necessary for the next section. Let us to recall the definition of fuzzy metric space (see, [2]).

Definition 2.1. *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous t-norm if it satisfies the following condition:*

1. $*$ *is an associative and commutative operation,*
2. $a * 1 = a$, *for all* $a \in (0,1]$,
3. $a * b \leq c * d$, *whenever* $a \leq c, b \leq d$ *where* $a, b, c, d \in (0,1]$,
4. *if* $a * b = a * c$ *then* $b = c$,
5. $*$ *is continuous.*

Definition 2.2. *A fuzzy metric space is a triple* $(X, M, *)$ *where* X *is a non-empty set, * is a continuous t-norm and* $M : X^2 \times [0, \infty) \longrightarrow [0,1]$ *is a mapping which has the following properties:*
For every $x, y, z \in X$ *and* $t, s > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ *if and only if* $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
5. $M(x, y, .) : (0, \infty) \longrightarrow [0,1]$ *is a continuous map.*

Let $(X, M, *)$ be a fuzzy metric space. A set $A \subset X$ is called a fuzzy open set if for any $x \in A$ there exist $0 < r < 1$ and $T_0 \in (0, \infty)$ so that if $M(x, y, t) > 1-r$ then $y \in A$ for all $t > T_0$. Let $(X, M, *)$ be a fuzzy metric space. An open ball $B(x, r, t)$ with center $x \in X$ and radius r ,$0 < r < 1$, $t > 0$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

Let $(X, M, *)$ be a fuzzy metric space. Let $\tau_M = \{A \subset X : x \in A \Longleftrightarrow$ there exist $t > 0$ and $r \in (0,1)$ $s.t$ $B(x, r, t) \subset A\}$. Then, τ_M is a topology on X. A fuzzy metric space is called a compact fuzzy metric space if (X, τ_M) is a compact space [2]. A fuzzy map $f : X \to X$ is said to be fuzzy continuous at x_0, if for each $\epsilon \in (0,1)$ and each $t > 0$ there is $\delta \in (0,1)$ so that for each x with $M(x, x_0, t) > 1 - \delta$, we deduce $M(f(x), f(x_0), t) > 1 - \epsilon$. Also f is called uniformly fuzzy continuous if for any $\epsilon \in (0,1)$ there is $\delta \in (0,1)$ so that for each x and y with $M(x,y,t) > 1-\delta$, we deduce $M(f(x), f(y), t) > 1-\delta$. A fuzzy map $f: X \to X$ is said to be fuzzy homeomorphism, if $f: X \to X$ be a fuzzy continuous bijection map with fuzzy continuous inverse. Let $(X, M, *)$ be a fuzzy metric space and $f: X \to X$ be a fuzzy homeomorphism then $(X, f, M, *)$ is called a fuzzy dynamical system [2].

In the remaining of the paper, we suppose that X is a compact metric space and $f: X \to X$ is a fuzzy homeomorphism.

Definition 2.3. *A relation from X to Y, written $F: X \longrightarrow Y$, is an arbitrary subset of the product $X \times Y$. Also, we define $F(x) = \{y : (x,y) \in F\}$. If $X = Y$, we call F a relation on X. Also for any relation F on X we define the cyclic set*

$$| F | := \{x \in X : (x,x) \in F\}.$$

In this paper, we will use the following notations:

$$F^0 = 1_X = \{(x,x) : x \in X\}, \quad F^n = F \circ F \circ \cdots \circ F,$$
$$F^n \circ F^m = F^{n+m}.$$

Definition 2.4. *Suppose that $F: X \longrightarrow Y$ and $G: Y \longrightarrow Z$ are two relations. If $A \subseteq X$, $B \subseteq Y$, then*

1. *We define the inverse relation of F by $F^{-1} := \{(y,x) : (x,y) \in F\}$,*
2. *$F(A) := \bigcup_{x \in A} F(x)$,*
3. *$F^{-1}(B) := \{x : F(x) \cap B \neq \emptyset\}$,*
4. *The composition $G \circ F : X \longrightarrow Z$ is the relation which is defined by $G \circ F := \{(x,z) :$ there exists $y \in Y, (x,y) \in F$ and $(y,z) \in G\}$.*

Definition 2.5. *Let F be a relation on X. We call F a closed relation when it is a closed subset of $X \times X$ (see, [3]).*

Notice that, the composition of closed relations is closed. Also, if F is a closed relation then $| F |$, is a closed subset of X.

Definition 2.6. *A relation F on X is reflexive if $1_X \subseteq F$, symmetric if $F^{-1} = F$ and transitive if $F \circ F \subseteq F$.*

3. Fuzzy dynamical relations

Definition 3.1. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then the orbit relation Of is defined by*

$$Of = \bigcup_{n=1}^{\infty} f^n$$

If $x \in X$,
$$Of(x) = \{f^n(x), n \in \mathbb{Z}^+\}.$$
We recall that for relation F,
$$|F| = \{x; (x, x) \in F\}.$$
So,
$$|Of| = \{x; (x, x) \in Of\} = \{x; \exists n; x = f^n(x)\}.$$
Thus, $|Of|$ *is a set, consisting of the periodic points.*

Definition 3.2. *The point* $y \in X$ *is called* $\omega-$*limit for* $x \in X$ *if there exist sequences* $\{n_k\}$ *of natural numbers, such that if* $k \to \infty$ *then*
$$M(f^{n_k}(x), y, t) \to 1.$$
The set of all limit points of an orbit $Of(x)$ *is denoted by*
$$\omega f(x) = \limsup_n \{f^n(x)\} = \{y, \exists n_i \to \infty, M(f^{n_i}(x), y, t) \to 1\}.$$
$\omega-$*limit sets give fundamental information about asymptotic behaviour of a fuzzy dynamical system.*

In fact, in this case $\omega f(x)$ is the set of $\omega-$limit points of an orbit $x \in X$ that
$$\omega f(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k \geqslant n} f^k(x)}.$$
And the set of $\alpha-$limit points is defined as follows:
$$\alpha f(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k \geqslant n} f^{-k}(x)}.$$

Definition 3.3. *Let* $(X, f, M, *)$ *be a fuzzy dynamical system, the relation* Rf *is defined by* $Rf = Of \cup \omega f$. *If* $x \in X$, *we have*
$$Rf(x) = Of(x) \cup \omega f(x).$$
The relation Rf *is called the recurrent relation.*

Notice that the cyclic set $|Rf| = \{x; (x, x) \in Rf\}$, *consisting of the recurrent points.*

Theorem 3.1. *Let* $(X, f, M, *)$ *be a fuzzy dynamical system then for each* $x \in X$,

1) $\omega f(x) \neq \emptyset$ *and* $Rf(x)$, $\omega f(x)$ *are fuzzy closed relations.*

2) Of *is a transitive relation.*

3) *Rf is a transitive relation.*

Proof. 1) We know that, $\omega f(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k\geqslant n} f^k(x)}$, and the proof is clear.

2) Suppose $(x, y) \in (Of) \circ (Of)$, then there is $z \in X$ such that ; $(x, z) \in Of$ and $(z, y) \in Of$. By definition Of

$$\exists n \in \mathbb{Z}^+; z = f^n(x) \wedge \exists m \in \mathbb{Z}^+; y = f^m(z).$$

So,

$$y = f^m(f^n(x)) = f^{m+n}(x),$$

therefore $(x, y) \in Of$.

3) Suppose that $(x, z) \in (Rf) \circ (Rf)$, then

$$\exists y; (x, y) \in Rf \wedge (y, z) \in Rf,$$

from definition Rf we have

$$\forall \epsilon_1, t_1 > 0, \exists n; M(f^n(x), y, t_1) > 1 - \epsilon_1$$

and

$$\forall \epsilon_2, t_2 > 0, \exists m; M(f^m(y), z, t_2) > 1 - \epsilon_2.$$

Since, f is a fuzzy homeomorphism on the compact fuzzy metric space $(X, M, *)$, so f is uniformly fuzzy continuous and therefore,

$$\forall \epsilon_3, t_3 > 0, ; M(f^{n+m}(x), f^m(y), t_3) > 1 - \epsilon_3.$$

Now, by triangle inequaliry we have,

$$\begin{aligned} M(f^{n+m}(x), z, t_1 + t_2) &\geqslant M(f^{n+m}(x), f^m(y), t_1) * M(f^m(y), z, t_2) \\ &> (1 - \epsilon_3) * (1 - \epsilon_2) > 1 - e.(e \in (0, 1)). \end{aligned}$$

Thus, $(x, z) \in Rf$. Therefore, Rf is a transitive relation. □

Theorem 3.2. *Let $(X, f, M, *)$ be a fuzzy dynamical system, then*

1) $Of = f \cup ((Of) \circ f) = f \cup (f \circ (Of))$.

2) $Of^{-1} = (Of)^{-1}$.

3) $Rf = f \cup ((Rf) \circ f) = f \cup (f \circ (Rf))$.

Proof. 1) Since Of is a transitive relation and $f \subseteq Of$, we have

$$f \circ (Of) \subseteq (Of) \circ (Of) \subseteq Of.$$

So,

$$f \cup (f \circ (Of) \subseteq Of. \qquad (1)$$

Conversely, for every $(x, y) \in Of$, we have

$$\exists n \in \mathbb{Z}^+; y = f^n(x) = f(f^{n-1}(x)).$$

Let $z = f^{n-1}(x)$, then $(x, z) \in Of$ and $(z, y) \in f$ therefore $(x, y) \in f \circ (Of)$, and we have

$$Of \subseteq f \cup (fo(Of)). \tag{2}$$

Hence, one conclude $Of = f \cup (f \circ (Of))$.

2) Since $Of = \{(x, f^n(x)); x \in X, n \in \mathbb{Z}^+\}$. So,

$$(Of)^{-1} = \{(f^n(x), x); x \in X, n \in \mathbb{Z}^+\}$$

let, $y = f^n(x)$, then

$$\begin{aligned}(Of)^{-1} &= \{(y, f^{-n}(y);); y \in X\} \\ &= \{(y, f^{-n}(y);); y \in X, n \in \mathbb{Z}^+\} \\ &= O(f^{-1}).\end{aligned}$$

3) Since, $f \subseteq R_F f$ and R_F is a transitive relation we have $f \circ (R_F f) \subseteq R_F \circ R_F \subseteq R_F f$. So, $f \cup (f \circ (R_F f)) \subseteq R_F f$. Conversely, let $(x, y) \in R_F f$ and $y \neq f(x)$. Then, the subsequence $\{n_k\}$ of natural numbers is exist such that if $k \to \infty$, $M(f^{n_k}(x), y, t) \to 1$.

So, $M(f^{n_k - 1}(f(x)), y, t) \to 1$. Let $z = f(x)$ then, we have $(x, z) \in f$and$(z, y) \in R_F f$. Therefore, $(x, y) \in (R_F f) \circ f$. which completes the proof.

The proof of the other properties is similar. □

Definition 3.4. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system and $A \subseteq X$ then,*

1) *The set A is called forward invariant if $f(A) \subseteq A$.*

2) *The set A is called invariant respect to f if $f(A) = A$.*

Theorem 3.3. *Let $(X, f, M, *)$ be a fuzzy dynamical system and suppose $x \in X$, then*

1) *If $A = O, R$, then $Af(x)$ respect to f is forward invariant.*

2) *If $A = \alpha, \omega$, then $f(Af(x)) = Af(x)$.*

3) *If $A = O, R$, then $f(\mid Af \mid) = \mid Af \mid$.*

Proof. 1) According to Theorem 3.2, $Af = f \cup (f \circ (Af))$ hence $foAf \subseteq Af$. That is mean for each $x \in X$, $f(Af(x)) \subseteq Af(x)$.

2) According to Theorem 3.2, $Af = f \circ Af = f^{-1} \circ Af = Af \circ f^{-1}$, then for each $x \in X$, $f(Af(x) = Af(x)$.

3) In the case $A = O$, the proof is obvious. For case $A = R$, according to Definition 3.3 and cyclic set be result $f(|Rf|) = \{f(x); x \in |Rf|\} = \{f(x); (x, x) \in Rf\}$. If $x \in |Rf|$ then $f(x) \in |Rf|$. Thus, $f(|Rf|) \subseteq \{f(x); (f(x), f(x)) \in Rf$. Set $z = f(x)$, thus $f(|Rf|) \subseteq \{z; (z, z) \in Rf\} = |Rf|$. On the other hand, similar to process the above we have $f^{-1}(|Rf|) \subseteq |Rf|$. Then, $f(|Rf\ |) = Rf$. □

Theorem 3.4. *Let the set A respect to fuzzy dynamical system f be forward invariant, if A be closed then $Rf(A) \subseteq A$.*

Proof. Suppose A be closed, according to Rf we have $Rf(A) = \overline{Of(A)} \subseteq \overline{A} \subseteq A$, then $Rf(A) \subseteq A$. □

Definition 3.5. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then Ωf is the relation of all $(x, y) \in \Omega f$ if and only if sequence $\{x_n\}$ and subsequence $\{n_k\}$ of sequence $\{n\}$ of natural numbers be exist that as $n \longrightarrow \infty$*

$$M(x_n, x, t) \longrightarrow 1, n_k \longrightarrow \infty, M(f^{n_k}(x_k), y, t) \longrightarrow 1.$$

Theorem 3.5. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, $x, y, z \in X$ then:*

(1) *if $z, y \in \omega f(x)$ then $z \in \omega f(y)$;*

(2) *if $y \in \omega f(x)$ then $y \in\mid \Omega f \mid$.*

Proof. (1) According definition $\omega-$limit relation, sequence $\{n_k\}$ of natural numbers is exist such that for $k \longrightarrow \infty$

$$n_k \longrightarrow \infty, M(f^{n_k}(x), y, t) \longrightarrow 1.$$

Also sequence $\{n'_k\}$ of natural numbers is exist such that if

$$n'_k \longrightarrow \infty, M(f^{n'_k}(x), z, t) \longrightarrow 1.$$

Set $y_k = f^{n_k}(x)$ and $\{m_k\}$ is increasing sequence of natural numbers such that for each $k \longrightarrow \infty$

$$M(y_k, y, t) \longrightarrow 1, m_k \longrightarrow \infty, M(f^{m_k}(y_k), z, t) \longrightarrow 1.$$

Therefore, $z \in \omega f(y)$.

(2) Set in (1) $z = y$. □

Theorem 3.6. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then*

$$\Omega f = f \circ (\Omega f) = f^{-1} \circ (\Omega f) = (\Omega f) \circ f^{-1} = (\Omega f) \circ f.$$

Proof. Let $(x, y) \in \Omega f$. According definition of relation Ωf, sequence $\{x_n\}$ and subsequence $\{n_k\}$ of sequence $\{n\}$ of natural numbers be exist that as $n \longrightarrow \infty$

$$M(x_n, x, t) \longrightarrow 1, n_k \longrightarrow \infty, M(f^{n_k}(x), y, t) \longrightarrow 1.$$

Since X is compact metric fuzzy space, sequence $\{f^{n_k-1}(x)\}$ has convergent subsequence, that is mean,

$$\exists\{n_{i_j}\} \subseteq \{n_k\}, M(f^{n_{i_j}-1}(x), z, t) \longrightarrow 1 \Longrightarrow M(f^{n_{i_j}}(x), f(z), t) \longrightarrow 1.$$

Due to uniqueness of limit we have $z = f(y)$, so $(x, z) \in (\Omega f) \wedge (z, y) \in f$, hence $(x, y) \in f \circ (\omega f)$, so

$$\Omega f \subseteq f \circ \Omega f, \tag{3}$$

On the other hand

$$f \circ (\Omega f) \subseteq \Omega f, \tag{4}$$

From (3) and (4) be result $f \circ \Omega f = \Omega f$. Similar to process the above $\Omega f = f^{-1} \circ \Omega f = (\Omega f) \circ f^{-1}$. □

Theorem 3.7. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system and $x \in X$. Then $\Omega f(x)$ is invariant with respect to f.*

Proof. According Theorem 3.6, $fo\Omega f = \Omega f$. Hence the proof is complete. □

In the following example, we examine the difference between two relations of ωf and ωf.

Example 3.1. Let $X = [0, 1]$, $a * b = ab$ and d be Euclidean meter on $[0, 1]$, then (X, d) is compact metric space and $(X, M, *)$ that $M(x, y, t) = \frac{t}{t+d(x,y)}$ is compact fuzzy metric space. Consider homeomorphism $f(x) = x^2$ on X, for fuzzy dynamical system $(X, f, M, *)$, $\omega f(0) = \{0\}, \omega f(1) = \{1\}$. If $x \in (0, 1)$, $\omega f(x) = (0, 1]$. hence $\omega f = ((0, 1] \times \{1\}) \cup \{(0, 0)\}$, while $\omega f = ((0, 1] \times \{1\}) \cup (\{0\} \times \{[0, 1]\})$.

Definition 3.6. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then relation Nf is defined as the follows:*

$$Nf = Of \cup \Omega f.$$

The relation Nf is called prolongation of f.

Remark 3.1. From definition easily, one can result that Nf is a closed relation. Also

$$\mid Nf \mid = \{x, (x, x) \in Nf\}.$$

The points of set $\mid Nf \mid$ are called non-wandering points.

Theorem 3.8. *Suppose that* $(X, f, M, *)$ *be a dynamical system, then*

$$\{x \in X, \omega f(x) = \omega f(x)\} = \{x \in X, Rf(x) = Nf(x)\}.$$

Proof. For all $x \in X$, $Rf(x) = Of(x) \cup \omega f(x) = Nf(x) = Of(x) \cup \Omega f(x) \Longleftrightarrow \omega f(x) = \Omega f(x)$, that proof is complete. □

Example 3.2. Let $X = S^1$, $a * b = ab$ and $d(x, y)$ be a distance between of x and y that

$$d(x = (\cos 2\pi t, \sin 2\pi t), y = (\cos 2\pi s, \sin 2\pi s)) = \mid t - s \mid, \quad t, s \in [0, 1)$$

then (X, d) is compact metric space and define

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

that $(X, M, *)$ is compact fuzzy metric space. Consider homeomorphism $f(x) = x$ on X, then

$$Of = \bigcup_{n=1}^{\infty} f^n = f = \{(x, x), x \in X\},$$

for all $x \in X$ $\omega f(x) = \omega f(x) = Rf(x) = Nf(x) = \{x\}$ hence $\mid Nf \mid = \mid Rf \mid = \mid Of \mid = \mid f \mid$. In fact fuzzy fixed points, recurrence points and non-wandering points are similar.

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