Hybrid of shifted Legendre and rational Legendre spectral methods on a semi-infinite interval

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Abstract. In this study, spectral and pseudospectral methods on the half-line are studied using orthogonal systems of shifted Legendre polynomials on a subinterval $[0, S]$ and rational Legendre functions on $[S, \infty)$. Based on this idea a hybrid orthogonal system is introduced. We establish primary results on hybrid approximations of interpolations and some orthogonal projections. The obtained results organize the theory of developing spectral and pseudospectral methods for solving differential equations on a semi-infinite interval. Error analysis for a model problem are established. Numerical results are included to support the theoretical results and shows the effectiveness and performance of this method.

Keywords: hybrid legendre function, semi-infinite interval, spectral and pseudospectral.

1. Introduction

In recent years, considerable growths in the fields of physics, engineering, chemistry and other sciences have been made that can be explained successfully with models that use mathematical tools on infinite intervals. A direct method is to use the spectral method dependent on orthogonal systems on infinite intervals (see, Hermite and Laguerre methods [2, 4, 9, 12, 14, 28, 29]). In these methods, main problems with infinite intervals are transformed to the problems with finite intervals. Jacobi's polynomials are used to approximate the results. For instance, Maleki [19], Boyd [13, 14] and Christov [8] used mutually orthogonal systems of hybrid functions for developing several spectral methods with infinite

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intervals. Some of the recent methods include hybrid functions [17, 20, 26, 27], Chebyshev methods on semi-infinite intervals [6, 13, 21, 22, 23] and Legendre methods [20, 24] and rational methods [10, 18].

In this study, we examine the spectral and pseudospectral methods on a combination of finite and semi-infinite intervals. We use a new hybrid mutual orthogonal system of shifted Legendre polynomials and rational Legendre functions with the weight function $(x-S+1)^{-2}$. We also provide a detailed theoretical analysis of the above hybrid approximation in weighted Sobolev space and assess some error estimates for a model problem.

This method has the following advantages. First, the proposed domain decomposition scheme enables us to approximate problems with oscillatory behaviour with more accuracy compared to the usual rational approximations. Second, this hybrid approach is flexible in that the value of S could be tuned according to the structure of the problem. Third, we do not need variable transformations to solve differential equations on a semi-infinite interval and the problem is solved directly.

The system of hybrid functions is obtained by the Legendre polynomials and its primary properties are introduced in the next Section. In Section 3, we denote several results for various orthogonal projections. In Section 4, we use a Legendre hybrid interpolation approximation. In Section 5, we discuss a model problem and prove error estimates for the Legendre hybrid spectral and pseudospectral methods. In Section 6, the numerical results are compared with rational method [5]. Finally, Section 7 includes the conclusion of the study.

2. Properties of hybrid Legendre functions

Let $S > 0$ and $I = [0, \infty)$ is partitioned into $(n+1)$ sub-intervals as following

$$\Lambda_{S,n} = \coprod_{i=1}^{n+1} \Lambda_i$$

with

$$\Lambda_i = \begin{cases} \left[\frac{S(i-1)}{n}, \frac{Si}{n}\right), & i = 1, 2, \dots, n \\ [S, \infty), & i = n+1. \end{cases}$$

For $1 \leq p < \infty$, let

$$L^p_w(\Lambda_{S,n}) = \left\{ v \mid v \text{ is measurable and } \|v\|_{L^p_{w_i}(\Lambda_i)} < \infty, \quad (i = 1, \dots, n, n+1) \right\}$$

where w_i's are positive weight functions

$$w(x) > 0, \quad w\,|_{\Lambda_i} = w_i \quad, \forall x \in \Lambda_i, \qquad w_i(x) > 0,$$

$$w_i(x) = \begin{cases} 1, & i = 1, 2, \dots, n \\ (x-S+1)^{-2}, & i = n+1 \end{cases}$$

and

$$\|v\|_{L^p_w} = \begin{cases} \left(\sum_{i=1}^{n+1} \int_{\Lambda_i} |v(x)|^p w_i(x) dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ ess \sup_{x \in \Lambda_{S,n}} |v(x)|, & p = \infty. \end{cases} \tag{2.1}$$

Inner product of the space $L^2_w(\Lambda_{S,n})$ is defined as follows:

$$(f, g)_w = \sum_{i=1}^{n+1} \int_{\Lambda_i} f(x) g(x) w_i(x) dx.$$

For any integer $m \geq 0$, we set

$$H^m_w(\Lambda_{S,n}) = \left\{ u \mid \partial^k_x u = \frac{d^k u}{dx^k} \in L^2_w(\Lambda_{S,n}), \quad 0 \leq k \leq m, m > 0 \right\}.$$

In this space, the semi-norm, the norm and the inner product are as follows:

$$(f, g)_{m,w}(\Lambda_{S,n}) = \sum_{k=0}^{m} (\partial^k_x f, \partial^k_x g)_w, \ |f|_{m,w} = \|\partial^m_x f\|_w, \ \|f\|_{m,w} = (f, f)^{\frac{1}{2}}_{m,w}.$$

We consider the space $H^r_w(\Lambda_{S,n})$ with the norm $\|f\|_{r,w}$ by space interpolation as in Adams [25] for any real number $r > 0$. As usual w will be omitted from the relations if $w_i(x) = 1$.

As we know, the Legendre polynomial of degree z, is the eigenfunction of the singular Sturm-Lioville problem

$$\partial_x \left((1 - x^2) \partial_x L_z(x) \right) + z(z + 1) L_z(x) = 0, \quad z = 0, 1, 2, \ldots. \tag{2.2}$$

We consider the Legendre hybrid functions of degree z in sub-intervals $[0, S]$ and in interval $[S, \infty)$, as:

$$F_{z,i}(x) = \begin{cases} L_z \left(\frac{2n}{S} x - 2i + 1 \right), & x \in \Lambda_i, i = 1, 2, \ldots, n \\ L_z \left(\frac{x - S - 1}{x - S + 1} \right), & x \in \Lambda_{n+1}, i = n + 1 \\ 0, & \text{otherwise.} \end{cases}$$

The functions $F_{z,i}(x), i = 1, \ldots, n + 1$ are the eigenfunctions of the following singular Sturm-Lioville problems

$$\partial_x \left(\left(\frac{(2i-1)Sx}{n} - x^2 + (i - i^2) \frac{S^2}{n^2} \right) \partial_x v \right) + z(z+1)v = 0, x \in \Lambda_i, i = 1, \ldots, n,$$
$$(x - S + 1)^2 \partial_x((x - S) \partial_x v) + z(z + 1)v = 0, \ x \in \Lambda_{n+1}. \tag{2.3}$$

Clearly, we have

$$\lim_{x\to\infty} F_{z,n+1}(x) = 1, \qquad \lim_{x\to\infty} (x-S)\partial x F_{z,n+1}(x) = 0. \tag{2.4}$$

From Legendre's polynomials recursive formulas, the following relationships are obtained.

$$\begin{cases} F_{0,i}(x) = 1, \qquad x \in \Lambda_i, (i = 1, \dots, n), \\ F_{1,i}(x) = x, \\ F_{z+1,i}(x) = \dfrac{2z-1}{z}\left(\dfrac{2n}{S}x - 2i + 1\right) F_{z,i}(x) - \dfrac{z-1}{z} F_{z-1,i}(x), \qquad z \geq 1, \\ F_{z,i}(x) = \dfrac{S}{2n}\left[\dfrac{\partial x F_{z+1,i}(x) - \partial x F_{z-1,i}(x)}{2z+1}\right], \end{cases}$$

and

$$\begin{cases} F_{0,n+1}(x) = 1, \qquad x \in \Lambda_{n+1} \\ F_{1,n+1}(x) = \dfrac{x-T-1}{x-T+1}, \\ F_{z+1,n+1}(x) = \dfrac{2z-1}{z}\dfrac{x-S-1}{x-S+1} F_{z,n+1}(x) - \dfrac{z-1}{z} F_{z-1,n+1}(x), \qquad z \geq 1, \\ F_{z,n+1}(x) = \left[\dfrac{\partial x F_{z+1,n+1}(x) - \partial x F_{z-1,n+1}(x)}{2z+1}\right]\left(\dfrac{(x-S+1)^2}{2}\right). \end{cases}$$

By using orthogonal relations of Legendre polynomials, we get

$$\int_{\Lambda_{S,n}} F_{z,i}(x) F_{m,i}(x) w_i(x) dx = (z + \frac{1}{2})^{-1} \delta_{z,m}\delta_{i,j}, \tag{2.5}$$

where $\delta_{z,m}$ and $\delta_{i,j}$ are the Kronecker functions.

According to the above definitions, the functions $\{F_{z,i}(x) \mid i{=}1, \dots, n{+}1; z = 0, 1, 2, \dots\}$ are orthogonal. For $v \in L^2_w(\Lambda_{S,n})$, we have

$$v(x) = \sum_{z=0}^{\infty}\sum_{i=1}^{n+1} \hat{v}_{z,i} F_{z,i}(x)$$

with

$$\hat{v}_{z,i} = \left(z + \frac{1}{2}\right)\int_{\Lambda_i} v(x) F_{z,i}(x) w_i(x) dx.$$

Let

$$l_i(x) = \begin{cases} \dfrac{(2i-1)Sx}{n} - x^2 + \dfrac{(i-i^2)S^2}{n^2}, & i = 1, \dots, n \\ x - S, & i = n+1. \end{cases}$$

By virtue of Eqs. (2.3) and (2.4), we know that $\{\partial x F_{z,i}(x),\ i = 1, \dots, n+1\}$ are mutually orthogonal in $L^2_{l_i}(\Lambda_i) \quad (i = 1, \dots, n+1)$, namely

$$\begin{aligned} &\int_{\Lambda_i} \partial_x F_{z,i}(x) \partial_x F_{m,i}(x) l_i(x) dx \\ &= z(z+1)(z + \frac{1}{2})^{-1}\delta_{z,m}, (i = 1, \dots, n+1), (x \in \Lambda_i). \end{aligned} \tag{2.6}$$

For any integer $N > 0$, we consider

$$\mathcal{H}_{N,n} = \text{span}\left\{F_{z,i} : z = 0, 1, \ldots, N\ ,\ i = 1, 2, \ldots, n+1\right\}.$$

We set $\tilde{\Lambda} = (-1, 1)$ and derive some useful inequalities and embedding inequalities in the following theorems.

Theorem 2.1. *Let $\phi \in \mathcal{H}_{N,n}$ and $1 \leq p \leq q < \infty$, then*

$$\|\phi\|_{L^q_\omega} \leq \left(\frac{S+1}{2}\right)^{\frac{p-q}{pq}} [(p+1)N^2]^{\frac{1}{p}-\frac{1}{q}} \|\phi\|_{L^p_\omega}.$$

Proof. Let $y \in \tilde{\Lambda}, x_{n+1} = \frac{y(S-1)-S-1}{y-1}, x_i = \frac{S}{2n}(y + 2i - 1)(i = 1, \ldots, n)$. For any $\phi \in \mathcal{H}_{N,n}$, we set

$$\psi(y) = \begin{cases} \phi(x_i) = \phi(\dfrac{S}{2n}(y + 2i - 1)), & i = 1, \ldots, n \\ \phi(x_{n+1}) = \phi(\dfrac{y(S-1) - S - 1}{y-1}). \end{cases}$$

By definition of $\mathcal{H}_{N,n}$, we have $\psi(y) \in \mathcal{P}_N$, which is the space of polynomials of degree not exceeding N. By using $\mathcal{P}_N$ [1], we have

$$\left(\int_{\hat{\Lambda}} |\psi(y)|^q dy\right)^{\frac{1}{q}} \leq \left((p+1)N^2\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\hat{\Lambda}} |\psi(y)|^p dy\right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned} \|\phi\|^q_{L^q_w} &= \sum_{i=1}^{n+1} \int_{\Lambda_i} |\phi(x)|^q \omega_i(x) dx = (\frac{S+1}{2}) \int_{\hat{\Lambda}} |\psi(y)|^q dy \\ &\leq (\frac{S+1}{2})[(p+1)N^2]^{\frac{q}{p}-1} \left(\int_{\hat{\Lambda}} |\psi(y)|^p dy\right)^{\frac{q}{p}} \\ &= (\frac{S+1}{2})^{\frac{p-q}{p}} [(p+1)N^2]^{\frac{q}{p}-1} \|\phi\|^q_{L^p_w}. \quad \square \end{aligned}$$

Theorem 2.2. *For any integer $m \geq 0$, $2 \leq p < \infty$, $\phi \in \mathcal{H}_{N,n}$ and $r > 0$, we have*

$$\|\partial^m_x \phi\|_{L^p_w} \leq cN^{2m} \|\phi\|_{L^p_w},$$

and

$$\|\phi\|_{r,w} \leq cN^{2r} \|\phi\|_w.$$

Proof. Let $y \in \tilde{\Lambda}, x_{n+1} = \frac{y(S-1)-S-1}{y-1}, x_i = \frac{S}{2n}(y + 2i - 1)(i = 1, \ldots, n)$. For any $\phi \in \mathcal{H}_{N,n}$, we set

$$\psi(y) = \begin{cases} \phi(x_i) = \phi(\dfrac{S}{2n}(y + 2i - 1)), & i = 1, \ldots, n \\ \phi(x_{n+1}) = \phi(\dfrac{y(S-1) - S - 1}{y-1}). \end{cases}$$

$$\begin{aligned}\|\partial_x\phi\|^p_{L^p_w} &= \sum_{i=1}^{n+1}\int_{\Lambda_i}|\partial_x\phi|^p w_i(x)dx\\ &= n(\frac{2n}{S})^{p-1}\int_{\tilde{\Lambda}}|\partial_y\psi|^p dy+\frac{1}{2^{p+1}}\int_{\tilde{\Lambda}}|\partial_y\psi|^p(y-1)^{2p}dy.\end{aligned}$$

According to Theorem 2.2 in [5], we have

$$\begin{aligned}\|\partial_x\phi\|^p_{L^p_w} &\leq n\left(\frac{2n}{S}\right)^{p-1}\int_{\tilde{\Lambda}}|\partial_y\psi|^p dy+2^p\int_{\tilde{\Lambda}}|\partial_y\psi|^p dy\\ &\leq \left[\left(\frac{2^{(p-1)}n^p}{S^{p-1}}\right)+2^p\right]\|\partial_y\psi\|^p_{L^p(\tilde{\Lambda})}.\end{aligned}$$

Since $\psi(y)\in\mathcal{P}_N$ and using Theorem 2.8 of [1] we have

$$\|\partial_y^m\psi\|_{L^p(\tilde{\Lambda})}\leq cN^{2m}\|\psi\|_{L^p(\tilde{\Lambda})}.$$

By setting $m=1$, we obtain

$$\|\partial_x\phi\|^p_{L^p_w}\leq\left[\left(\frac{2^{p-1}n^p}{S^{p-1}}\right)+2^p\right]c^pN^{2p}\|\psi\|^p_{L^p(\tilde{\Lambda})}.$$

Thus

$$\|\partial_x\phi\|^p_{L^p_w}\leq\left[\left(\frac{2^{p-1}n^p}{S^{p-1}}\right)+2^p\right]c^pN^{2p}(\frac{2}{S+1})\|\phi\|^p_{L^p_w}.$$

By setting

$$c=\left[\left[\frac{2^{p-1}n^p}{S^{p-1}}+2^p\right]\left(\frac{2c^p}{S+1}\right)\right]^{\dfrac{1}{p}},$$

we get

$$\|\partial_x\phi\|^p_{L^p_w}\leq cN^2\|\phi\|^p_{L^p_w}.$$

If above procedure is repeated, we obtain

$$\|\partial_x^m\phi\|_{L^p_w}\leq cN^{2m}\|\phi\|_{L^p_w}(m\geq 0).$$

By setting $p=2$, we have

$$\|\phi\|_{r,w}\leq cN^{2r}\|\phi\|_w.$$

□

Theorem 2.3. *If* $v\in L^2_{w^2}(\Lambda_{S,n}),\partial_x v\in L^2_w(\Lambda_{S,n})$ *and* $v(\frac{S}{n}(i-1))=0,(i=1,\ldots,n+1),c=\frac{16n^2}{9S^2}$, *then*

$$\|v\|_{w^2}\leq\frac{2}{3}|v|_{1,w}.$$

Proof. Let $y \in \tilde{\Lambda}$,

$$\psi(y) = \begin{cases} v(x_i) = v\left(\dfrac{S}{2n}(y+2i-1)\right), & i = 1, \dots, n \\ v(x_{n+1}) = v\left(\dfrac{y(S-1)-S-1}{y-1}\right). \end{cases}$$

$$\|v\|^2_{w^2} = \sum_{i=1}^{n+1} \int_{\Lambda_i} v^2(x) w_i^2(x) dx = \frac{S}{2} \int_{\tilde{\Lambda}} \psi^2(y) dy + \frac{1}{8} \int_{\tilde{\Lambda}} \psi^2(y)(y-1)^2 dy.$$

According to Poincare inequality, since $\psi(-1) = 0$, we can conclude

$$\|v\|^2_{w^2} \leq \frac{cS}{2} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + \frac{1}{8} \int_{\tilde{\Lambda}} (\psi(y))^2 (y-1)^2 dy.$$

From Theorem 2.3 in [5], we have

$$\begin{aligned} \|v\|^2_{w^2} &\leq \frac{cS}{2} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + \frac{1}{8} \times \frac{4}{9} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 (1-y)^4 dy \\ &\leq \frac{cS}{2n} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + ... + \frac{cS}{2n} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + \frac{1}{8} \times \frac{4}{9} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 (1-y)^4 dy \\ &\leq \frac{cS^2}{4n^2} \sum_{i=1}^{n} \int_{\Lambda_i} (\partial_x v)^2 w_i(x) dx + \frac{4}{9} \int_{\Lambda_{n+1}} (\partial_x v)^2 w_{n+1}(x) dx. \end{aligned}$$

By letting $c = \dfrac{16n^2}{9S^2}$, we obtain

$$\|v\|^2_{w^2} \leq \frac{4}{9} \sum_{i=1}^{n+1} \int_{\Lambda_i} (\partial_x v(x))^2 w_i(x) dx \leq \frac{4}{9} |v|^2_{1,w}. \qquad \square$$

3. Legendre hybrid function approximations

In this section, we present some orthogonal projections for error estimates.

Definition 3.1. *We define the $L^2_w(\Lambda_{S,n})$- orthogonal projection $P_N : L^2_w(\Lambda_{S,n}) \to \mathcal{H}_{N,n}$ by*

$$(P_N u - u, \phi)_w = 0, \qquad \forall \phi \in \mathcal{H}_{N,n}.$$

The space $H^r_{w,A}(\Lambda_{S,n})$ which is considered by space interpolation is introduced to estimate $\|P_N u - u\|_w$, where

$$H^r_{w,A}(\Lambda_{S,n}) = \{u \mid u \text{ is measurable and } \|u\|_{r,w,A} < \infty\} \, (r \geq 0)$$

and

$$\|u\|_{r,w,A} = \left(\sum_{k=0}^{r} \|(x-S+1)^{\frac{r}{2}+k} \partial_x^k u\|^2_{w_{n+1}} + n \|\partial_x^k u\|^2 \right)^{\frac{1}{2}}, \quad (i = 1, \dots, n).$$

By using (2.3), we have

$$Au(x) = \begin{cases} -\partial x\left(\left(\frac{2i-1}{n}Sx - x^2 + \frac{i-i^2}{n^2}S^2\right)\partial_x u(x)\right), & x \in \Lambda_i, \\ -(x-S+1)^2\partial x((x-S)\partial_x u(x)), & x \in \Lambda_{n+1}. \end{cases}$$

where A is Sturm-Liouville operator. Now, by induction, we obtain

$$A^m u(x) = \begin{cases} \sum_{k=1}^{2m} p_k(x)\partial_x^k u(x), & (x \in \Lambda_i), \\ \sum_{k=1}^{2m}(x-S+1)^{m+k} q_k(x)\partial_x^k u(x), & (x \in \Lambda_{n+1}), \end{cases} \tag{3.1}$$

where $p_k(x)$ and $q_k(x)$ are some Legendre hybrid function.

Theorem 3.1. *If $r \geq 0$, for any $u \in H^r_{w,A}(\Lambda_{S,n})$, we have*

$$\|P_N u - u\|_w \leq cN^{-r}\|u\|_{r,w,A}$$

Proof. At first, we set $r = 2m$. By placing

$$F_{z,i}(x) = \begin{cases} -\dfrac{\partial x((\frac{2i-1}{n}Sx - x^2 + \frac{i-i^2}{n^2}S^2)\partial_x F_{z,i}(x))}{z(z+1)}, & i = 1, \dots, n, \\ -\dfrac{(x-S+1)^2\partial x((x-S)\partial_x F_{z,i}(x))}{z(z+1)}, & i = n+1 \end{cases}$$

and using Eqs. (2.3) and (2.5) and integration by parts,

$$\begin{aligned} \hat{u}_{z,i} &= (z+\frac{1}{2})\int_{\Lambda_i} u(x)F_{z,i}(x)w_i(x)dx \\ &= -\frac{2z+1}{2z(z+1)}\int_{\Lambda_i} u(x)\partial_x((\frac{2i-1}{n}Sx - x^2 + \frac{i-i^2}{n^2}S^2)\partial_x F_{z,i}(x))dx \\ &= \frac{2z+1}{2z(z+1)}\int_{\Lambda_i}(\frac{2i-1}{n}Sx - x^2 + \frac{i-i^2}{n^2}S^2)(\partial_x u(x))(\partial_x F_{z,i}(x))dx \\ &= \frac{2z+1}{2z(z+1)}\int_{\Lambda_i} Au(x)F_{z,i}(x)dx = \dots \\ &= \frac{2z+1}{2z^m(z+1)^m}\int_{\Lambda_i} A^m u(x)F_{z,i}(x)dx, (i = 1, \dots, n). \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
\hat{u}_{z,n+1} &= (z+\frac{1}{2})\int_{\Lambda_{n+1}} u(x)F_{z,n+1}(x)w_{n+1}(x)dx\\
&= -\frac{2z+1}{2z(z+1)}\int_{\Lambda_{n+1}} u(x)(x-S+1)^2\partial x((x-S)\partial_x F_{z,n+1}(x))w_{n+1}(x)dx\\
&= -\frac{2z+1}{2z(z+1)}\int_{\Lambda_{n+1}} u(x)\partial x((x-S)\partial_x F_{z,n+1}(x))dx\\
&= \frac{2z+1}{2z(z+1)}\int_{\Lambda_{n+1}} (x-S)\partial_x u(x)\partial_x F_{z,n+1}(x)dx\\
&= -\frac{2z+1}{2z(z+1)}\int_{\Lambda_{n+1}} \partial_x((x-S)\partial_x u(x))F_{z,n+1}(x)dx\\
&= \frac{2z+1}{2z(z+1)}\int_{\Lambda_{n+1}} Au(x)F_{z,n+1}(x)w_{n+1}(x)dx = \dots\\
(3.3)\quad &= \frac{2z+1}{2z^m(z+1)^m}\int_{\Lambda_{n+1}} A^m u(x)F_{z,n+1}(x)w_{n+1}(x)dx.
\end{aligned}
$$

Therefore, we derive from (3.1) and (3.2) and (3.3) and the definition of $H^r_{w,A}(\Lambda_{S,n})$ that

$$
\begin{aligned}
\|P_N u-u\|_w^2 = &\sum_{z=N+1}^{\infty}\sum_{i=1}^{n+1}\hat{u}_{z,i}^2\|F_{z,i}\|_{w_i}^2\\
&= \sum_{z=N+1}^{\infty}\frac{1}{z^{2m}(z+1)^{2m}}\left[\frac{\int_{\Lambda_1} A^m u(x)F_{z,1}(x)dx}{\|F_{z,1}\|_{w_1}^2}\right]^2\|F_{z,1}\|_{w_1}^2+\dots\\
&+\frac{1}{z^{2m}(z+1)^{2m}}\left[\frac{\int_{\Lambda_n} A^m u(x)F_{z,n}(x)dx}{\|F_{z,n}\|_{w_n}^2}\right]^2\|F_{z,n}\|_{w_n}^2\\
&+\frac{1}{z^{2m}(z+1)^{2m}}\left[\frac{\int_{\Lambda_{n+1}} A^m u(x)F_{z,n+1}(x)w_{n+1}(x)dx}{\|F_{z,n+1}\|_{w_{n+1}}^2}\right]^2\|F_{z,n+1}\|_{w_{n+1}}^2\\
&\le cN^{-4m}(n\|A^m u\|^2+\|A^m u\|_{w_{n+1}}^2).
\end{aligned}
$$

Next, we set $r = 2m+1$. By using (2.3) and (2.5) and integration by parts,

$$
\begin{aligned}
\hat{u}_{z,i} &= \frac{2z+1}{2z^m(z+1)^m}\int_{\Lambda_i} A^m u(x)F_{z,i}(x)dx\\
(3.4)\quad &= -\frac{2z+1}{2z^{m+1}(z+1)^{m+1}}\int_{\Lambda_i} A^m u(x)\partial_x((\frac{2i-1}{n}Sx-x^2+\frac{i-i^2}{n^2}S^2)\partial_x F_{z,i}(x))dx\\
&= \frac{2z+1}{2z^{m+1}(z+1)^{m+1}}\int_{\Lambda_i} \partial_x(A^m u(x))\partial_x F_{z,i}(x)l_i(x)dx, (i=1,\dots,n).
\end{aligned}
$$

and

$$\begin{aligned}
\hat{u}_{z,n+1} &= \frac{2z+1}{2z^m(z+1)^m}\int_{\Lambda_{n+1}} A^m u(x)F_{z,n+1}(x)w_{n+1}(x)dx \\
&= -\frac{2z+1}{2z^{m+1}(z+1)^{m+1}}\int_{\Lambda_{n+1}} A^m u(x)\partial_x((x-S)\partial_x F_{z,n+1}(x))dx \\
&= \frac{2z+1}{2z^{m+1}(z+1)^{m+1}}\int_{\Lambda_{n+1}} \partial_x(A^m u(x))\partial_x F_{z,n+1}(x)l_{n+1}(x)dx. \qquad (3.5)
\end{aligned}$$

By using (2.6) and (3.4) and (3.5), we have

$$\begin{aligned}
\|P_N u - u\|_w^2 &= \sum_{z=N+1}^{\infty}\sum_{i=1}^{n+1}\hat{u}_{z,i}^2\|F_{z,i}\|_{w_i}^2 \\
&\leq cN^{-2(2m+1)}\sum_{z=N+1}^{\infty}\left(\frac{\int_{\Lambda_1}\partial_x(A^m u)\partial_x F_{z,1}(x)l_1(x)dx}{\|\partial_x F_{z,1}(x)\|_{l_1}^2}\right)^2\|\partial_x F_{z,1}\|_{l_1}^2 + \ldots \\
&+ \left(\frac{\int_{\Lambda_n}\partial_x(A^m u)\partial_x F_{z,n}(x)l_n(x)dx}{\|\partial_x F_{z,n}(x)\|_{l_n}^2}\right)^2\|\partial_x F_{z,n}\|_{l_n}^2 \\
&+ \left(\frac{\int_{\Lambda_{n+1}}\partial_x(A^m u)\partial_x F_{z,n+1}(x)l_{n+1}(x)dx}{\|\partial_x F_{z,n+1}(x)\|_{l_{n+1}}^2}\right)^2\|\partial_x F_{z,n+1}\|_{l_{n+1}}^2 \\
&\leq cN^{-2(2m+1)}(\|\partial_x A^m u\|_{l_1}^2 + \ldots + \|\partial_x A^m u\|_{l_n}^2 + \|\partial_x A^m u\|_{l_{n+1}}^2) \\
&\leq cN^{-2(2m+1)}(n\|A^m u\|^2 + \|A^m u\|_{w_{n+1}}^2) \\
&\leq cN^{-2m(2m+1)}\|u\|_{r,w,A}^2.
\end{aligned}$$

Which completes the proof. □

Definition 3.2. *We define $H_w^1(\Lambda_{S,n})$-orthogonal projection $P_N^1 : H_w^1(\Lambda_{S,n}) \to \mathcal{H}_{N,n}$ by*

$$(P_N^1 u - u, \phi)_{1,w} = 0, \qquad \forall\phi \in \mathcal{H}_{N,n}, \quad \forall u \in H_w^1(\Lambda_{S,n}).$$

For estimating $\|P_N^1 u - u\|_{1,w}$, we introduce

$$H_{w,B}^r(\Lambda_{S,n}) = \{u \mid u \textit{ is measurable on } \Lambda_{S,n} \textit{ and } \|u\|_{r,w,B} < +\infty\}\ (r \geq 0)$$

where

$$\|u\|_{r,w,B} = \left(\sum_{k=1}^{r}\|(x-S+1)^{\frac{r}{2}+k-\frac{1}{2}}\partial_x^k u\|_{w_{n+1}}^2 + n^2\|\partial_x^k u\|^2\right)^{\frac{1}{2}}. \qquad (3.6)$$

Theorem 3.2. *For any $u \in H_{w,B}^r(\Lambda_{S,n})$, with $r \geq 1$*

$$\|P_N^1 u - u\|_{1,w} \leq cN^{1-r}\|u\|_{r,w,B}.$$

Proof. We have $\|P_N^1 u - u\|_{1,w} \leq \|\phi - u\|_{1,w}$ for any $\phi \in \mathcal{H}_{N,n}$. Let $y \in \tilde{\Lambda} = (-1,1)$,

$$\psi(y) = \begin{cases} u(x_i) = u(\frac{S}{2n}(y+2i-1)) & i = 1,\dots,n \\ u(x_{n+1}) = u(\frac{y(S-1)-S-1}{y-1}). \end{cases}$$

By taking

$$\phi(x) = \begin{cases} \phi(x_i) = \tilde{P}^1_{N,0,0,0,0}\psi(y) & i = 1,\dots,n \\ \phi(x_{n+1}) = \tilde{P}^1_{N,4,0,0,0}\psi(y). \end{cases}$$

By using Theorem 2.5 in [3], we obtain

$$\begin{aligned}
&\|\phi - u\|^2_{1,w} = \sum_{k=0}^{1}\left\{\sum_{i=1}^{n+1}\int_{\Lambda_i}\partial_x^k(\phi-u)\partial_x^k(\phi-u)w_i(x)dx\right\} \\
&= \frac{S}{2n}\int_{\tilde{\Lambda}}(\tilde{P}^1_{N,0,0,0,0}\psi(y)-\psi(y))^2dy + \frac{2n}{S}\int_{\tilde{\Lambda}}(\partial_y\tilde{P}^1_{N,0,0,0,0}\psi(y)-\partial_y\psi(y))^2dy + \dots \\
&+ \frac{S}{2n}\int_{\tilde{\Lambda}}(\tilde{P}^1_{N,0,0,0,0}\psi(y)-\psi(y))^2dy + \frac{2n}{S}\int_{\tilde{\Lambda}}(\partial_y\tilde{P}^1_{N,0,0,0,0}\psi(y)-\partial_y\psi(y))^2dy \\
&+ \frac{1}{2}\int_{\tilde{\Lambda}}(\tilde{P}^1_{N,4,0,0,0}\psi(y)-\psi(y))^2dy + \frac{1}{8}\int_{\tilde{\Lambda}}(\partial_y\tilde{P}^1_{N,4,0,0,0}\psi(y)-\partial_y\psi(y))^2(y-1)^4dy \\
&\leq \frac{2n}{S}\|\tilde{P}^1_{N,0,0,0,0}\psi-\psi\|^2_{1,0,0,0,0} + \dots + \frac{2n}{S}\|\tilde{P}^1_{N,0,0,0,0}\psi-\psi\|^2_{1,0,0,0,0} \\
&+ \|\tilde{P}^1_{N,4,0,0,0}\psi-\psi\|^2_{1,4,0,0,0} \\
&\leq \frac{2c_1n^2}{S}N^{2-2r}\|\psi\|_{r,0,0,*} + c_2N^{2-2r}\|\psi\|^2_{r,4,0,*} \\
&\leq cN^{2-2r}\|\psi\|_{r,0,0,*} + cN^{2-2r}\|\psi\|^2_{r,4,0,*}, \quad (c = \max(c_2, \frac{2c_1n^2}{S})).
\end{aligned}$$

Note that

$$1 - y^2 = \begin{cases} \frac{-4x^2}{S^2}x^2 - 4i^2 + \frac{8ni}{S} + 4i - \frac{4n}{S}x, & x \in \Lambda_i, i = 1,\dots,n, \\ \frac{4x}{(x-S+1)^2}, & x \in \Lambda_{n+1}, \end{cases}$$

$$1 - y = \begin{cases} 2i - \frac{2n}{S}, & i = 1,\dots,n, \\ \frac{2}{x-S+1}, & x \in \Lambda_{n+1} \end{cases}$$

and one can show easily by induction that

$$\partial_y^k\psi(y) = \begin{cases} (\frac{S}{2n})^k\partial_x^k u(x), & x \in \Lambda_i, i = 1,\dots,n \\ \sum_{j=1}^{k} q_j(x)(x-S+1)^{k+j}\partial_x^j u(x), & x \in \Lambda_{n+1}. \end{cases}$$

where $q_j(x)$ are some hybrid polynomials with degree j which are uniformly bounded on $\Lambda_{S,n}$. According to [3], we obtain the following relations for $r \geq 0$

$$A^{(1)}_{r,0,0}(\psi) \leq \sum_{k=r-[\frac{r}{2}]+1}^{r} (\frac{S}{2n})^{2k-1} \|\partial_x^k u\|^2,$$

$$A^{(2)}_{r,0,0}(\psi) \leq \sum_{k=1}^{[\frac{r+1}{2}]} (\frac{S}{2n})^{2k-1} \|\partial_x^k u\|^2,$$

$$A^{(1)}_{r,4,0}(\psi) \leq c_3 \sum_{k=r-[\frac{r}{2}]+1}^{r} \sum_{j=1}^{k} \|(x-S+1)^{\frac{r}{2}+j-\frac{1}{2}} \partial_x^j u\|^2_{w_{n+1}},$$

$$A^{(2)}_{r,4,0}(\psi) \leq c_4 \sum_{k=r}^{[r+\frac{1}{2}]} \sum_{j}^{k} \|(x-S+1)^{k+j-1} \partial_x^j u\|^2_{w_{n+1}}.$$

Now, by setting $(c = \max(c_2, \dfrac{2c_1 n^2}{S}, c_3, c_4))$, we obtain

$$\begin{aligned}
\|\phi - u\|^2_{1,w} &\leq cN^{2-2r}(A^{(1)}_{r,0,0}(\psi) + A^{(2)}_{r,0,0}(\psi)) + cN^{2-2r}(A^{(1)}_{r,4,0}(\psi) + A^{(2)}_{r,4,0}(\psi)) \\
&\leq cN^{2-2r}(A^{(1)}_{r,4,0}(\psi) + n^2 A^{(1)}_{r,0,0}(\psi) + A^{(2)}_{r,4,0}(\psi) + n^2 A^{(2)}_{r,0,0})(\psi)) \\
&\leq cN^{2-2r}(\sum_{k=1}^{r} \|(x-T+1)^{\frac{r}{2}+k-\frac{1}{2}} \partial_x^k u\|^2_{w_{n+1}} + n^2 \|\partial_x^k u\|^2) \\
&\leq cN^{2-2r} \|u\|^2_{r,w,B}.
\end{aligned}$$

Which completes the proof. □

We define another orthogonal projection for Dirichlet boundary conditions at $x = 0$, in partial differential equations.

Definition 3.3. *Let us denote*

$$\begin{aligned}
H^1_{0,w}(\Lambda_{S,n}) = \big\{ v \mid v \in H^1_w(\Lambda_{S,n}),\ \ v(S(i-1)/n) = 0, i = 1, \dots, n+1, \\
v(x)(x-S+1)^{-\frac{3}{2}} \to 0 \ as\ x \to \infty \big\} \\
\mathcal{H}^0_{N,n} = \{\phi \in \mathcal{H}_{N,n} \mid \phi(S(i-1)/n) = 0, i = 1, \dots, n+1\},
\end{aligned}$$

$$a^v_w(f, v) = (\partial_x f, \partial_x(vw)) + v(f, v)_w. \tag{3.7}$$

We consider the $H^1_{0,w}(\Lambda_{S,n})$-orthogonal projection $P^{1,0}_N : H^1_{0,w}(\Lambda_{S,n}) \to \mathcal{H}^0_{N,n}$ by

$$a^v_w(P^{1,0}_N v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{H}^0_{N,n}, \qquad \forall v \in H^1_{0,w}(\Lambda_{S,n})$$

Lemma 3.1. *For any* $f, v \in H^1_{0,w}(\Lambda_{S,n})$ *and* $v > \frac{3}{4}$,

$$a^v_w(v,v) \geq c\|v\|^2_{1,w},$$
$$|a^v_w(f,v)| \leq c\|f\|_{1,w}\|v\|_{1,w}$$

Proof. Let $v = \frac{3}{4} + \varepsilon$ and $\varepsilon > 0$. By using Theorem 2.3, integrating by parts and Lemma 3.1 in [5], we find that

$$\begin{aligned}
a^v_w(v,v) &= (\partial_x v, \partial_x(vw)) + v(v,v)w \\
&= \sum_{i=1}^{n+1}\int_{\Lambda_i} \partial_x v \partial_x(vw) w_i(x)dx + v\sum_{i=1}^{n+1}\int_{\Lambda_i} v(x)v(x)w_i(x)dx, \\
&= |v|^2_{1,w} + v\|v\|^2_w + \frac{1}{2}\int_{\Lambda_{n+1}} \partial_x(v^2(x))\partial_x w_{n+1}(x)dx, \\
&= |v|^2_{1,w} + v\|v\|^2_w - 3\int_{\Lambda_{n+1}} v^2(x)(x-S+1)^{-4}dx, \\
&\geq |v|^2_{1,w} + v\|v\|^2_w - 3\|v\|^2_{w^2}, \\
&\geq |v|^2_{1,w} - (\frac{9}{4} - \frac{\varepsilon}{2})\|v\|^2_{w^2} + (\frac{3}{4}+\varepsilon)\|v\|^2_w - (\frac{3}{4}+\frac{\varepsilon}{2})\|v\|^2_{w^2}, \\
&\geq \frac{2}{9}\varepsilon|v|^2_{1,w} + \frac{\varepsilon}{2}\|v\|^2_w. \qquad (3.8)
\end{aligned}$$

The first result is obtained. Next, by using Theorem 2.3 and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
a^v_w(f,v) &= |(\partial_x f, \partial_x(vw) + v(f,v)_w| \\
&\leq \|\partial_x f\|_w\|\partial_x v\|_w + 2\|\partial_x f\|_w\|v\|_{w^2} + v\|f\|_w\|v\|_w, \\
&\leq \|f\|_{1,w}\|v\|_{1,w} + \frac{4}{3}\|f\|_{1,w}\|v\|_{1,w} + v\|f\|_{1,w}\|v\|_{1,w}, \\
&\leq c\|f\|_{1,w}\|v\|_{1,w}.
\end{aligned}$$

Which indicates the second result. □

Theorem 3.3. *If* $r \geq 1$, *for any* $v \in H^r_{w,B}(\Lambda_{S,n}) \cap H^1_{0,w}(\Lambda_{S,n})$, $v > \frac{3}{4}$, *we can conclude*

$$\|P^{1,0}_N v - v\|_{1,w} \leq cN^{1-r}\|v\|_{r,w,B}.$$

Proof. By using Lemma 3.1, for any $\phi \in \mathcal{H}^0_{N,n}$, we set

$$\begin{aligned}
\|P^{1,0}_N v - v\|^2_{1,w} &\leq ca^v_w(P^{1,0}_N v - v, P^{1,0}_N v - v) \leq ca^v_w(P^{1,0}_N v - v, \phi - v) \\
&\leq c\|P^{1,0}_N v - v\|_{1,w}\|\phi - v\|_{1,w}.
\end{aligned}$$

Therefore,

$$\|P^{1,0}_N v - v\|_{1,w} \leq c \inf_{\phi\in\mathcal{H}^0_{N,n}} \|\phi - v\|_{1,w}. \qquad (3.9)$$

Next, let $y \in \tilde{\Lambda} = (-1, 1)$,

$$\psi(y) = \begin{cases} v(x_i) = v(\frac{S}{2n}(y + 2i - 1)), & i = 1, \ldots, n \\ v(x_{n+1}) = v(\frac{y(S-1) - S - 1}{y - 1}), \end{cases}$$

and take

$$\phi(x) = \begin{cases} \phi(x_i) = \tilde{P}^{1,0}_{N,0,0,0,0}\psi(y), & i = 1, \ldots, n \\ \phi(x_{n+1}) = \tilde{P}^{1,0}_{N,4,0,0,0}\psi(y), \end{cases}$$

in (3.9).

By using Theorem 2.6 in [3] and similar arguments in the proof of Theorem 3.2, the desired result is obtained. □

Definition 3.4. *Let* $\hat{a}_w(f, v) = \frac{1}{2}(\partial_x f, w^{\frac{-3}{2}}(x)\partial_x v)_w + (f, v)_w$. *We consider* $H^1_{w,A}(\Lambda_{S,n})$*-orthogonal projection* $\hat{P}^1_N : H^1_{w,A}(\Lambda_{S,n}) \to \mathcal{H}_{N,n}$ *by*

$$\hat{a}_w(\hat{P}^1_N v - v, \phi) = 0 \quad , \forall \phi \in \mathcal{H}_{N,n}, \forall v \in H^1_{w,A}(\Lambda_{S,n}). \tag{3.10}$$

Theorem 3.4. *For any* $v \in H^r_{w,A}(\Lambda_{S,n})$ *and* $r \geq 1$,

$$\|\hat{P}^1_N v - v\|_w \leq cN^{-r}\|v\|_{r,w,A}$$

and

$$\|w^{\frac{-3}{4}}(x)\partial_x(\hat{P}^1_N v - v)\|_w \leq cN^{1-r}\|v\|_{r,w,A}.$$

Proof. For $y \in \tilde{\Lambda} = (-1, 1)$, let us denote

$$\psi(y) = \begin{cases} v(x_i) = v(\frac{S}{2n}(y + 2i - 1)), & i = 1, \ldots, n \\ v(x_{n+1}) = v(\frac{y(S-1) - S - 1}{y - 1}), \end{cases}$$

$$\psi^*_N(y) = \hat{P}^1_N v(x) = \begin{cases} \hat{P}^1_N v(x_i) = \tilde{P}^1_{N,0,0,0,0}\psi(y), & i = 1, \ldots, n \\ \hat{P}^1_N v(x_{n+1}) = \tilde{P}^1_{N,1,0,0,0}\psi(y), \end{cases}$$

$$\begin{aligned} \|\hat{P}^1_N v - v\|^2_w &= \sum_{i=1}^{n+1} \int_{\Lambda_i} (\hat{P}^1_N v - v)^2 w_i(x) dx \\ &= \frac{S}{2n} n \int_{\tilde{\Lambda}} (\psi^*_N - \psi) dy + \frac{1}{2} \int_{\tilde{\Lambda}} (\psi^*_N - \psi) dy \\ &= \frac{S}{2n} n \|\tilde{P}^1_{N,0,0,0,0}\psi - \psi\|^2_{0,0,0,0,0} + \frac{1}{2}\|\tilde{P}^1_{N,1,0,0,0}\psi - \psi\|^2_{0,1,0,0,0} \\ &\leq cN^{-2r}(n\|\psi\|^2_{r,0,0,*} + \|\psi\|^2_{r,1,0,*}). \end{aligned} \tag{3.11}$$

For the second formula, we set

$$
\begin{aligned}
&\|w^{\frac{-3}{4}}(x)\partial x(\hat{P}_N^1 v - v)\|_w^2 \\
&= \sum_{i=1}^{n+1} \int_{\Lambda_i} (w_i^{\frac{-3}{4}}(x)\partial_x \hat{P}_N^1 v - \partial_x v)^2 w_i(x)dx \\
&= \frac{2n}{S}\int_{\tilde{\Lambda}} [\partial_y(\psi_N^* - \psi)]^2 dy + \ldots + \frac{2n}{S}\int_{\tilde{\Lambda}} [\partial_y(\psi_N^* - \psi)]^2 dy \\
&+ \frac{1}{4}\int_{\tilde{\Lambda}} [\partial_y(\psi_N^* - \psi)]^2 (1-y)dy \\
&\leq \frac{2n}{S} n\|\tilde{P}_{N,0,0,0,0,0}^1 \psi - \psi\|_{1,0,0,0,0}^2 + \frac{1}{4}\|\tilde{P}_{N,1,0,0,0}^1 \psi - \psi\|_{1,1,0,0,0}^2 \\
&\leq cN^{2-2r}(\frac{2n}{S} n\|\psi\|_{r,0,0,*}^2 + \|\psi\|_{r,1,0,*}^2).
\end{aligned} \tag{3.12}
$$

According to [3], we obtain the following relations for $r \geq 0$

$$
\begin{aligned}
A_{r,0,0}^{(1)}(\psi) &\leq (\frac{S}{2n})^{2r-1} \sum_{k=r-[\frac{r}{2}]+1}^{r} \|\partial_x^k v\|^2, \\
A_{r,0,0}^{(2)}(\psi) &\leq (\frac{S}{2n})^{2r-1} \sum_{k=r}^{[\frac{r+1}{2}]} \|\partial_x^k v\|^2, \\
A_{r,1,0}^{(1)}(\psi) &\leq c \sum_{k=r-[\frac{r}{2}]+1}^{r} \sum_{j=1}^{k} \|(x-S+1)^{\frac{r}{2}+j-\frac{1}{2}} \partial_x^j v\|_{w_{n+1}}^2, \\
A_{r,1,0}^{(2)}(\psi) &\leq c \sum_{k=1}^{[\frac{r+1}{2}]} \sum_{j=1}^{k} \|(x-S+1)^{k+j-1} \partial_x^j v\|_{w_{n+1}}^2.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\|\hat{P}_N^1 v - v\|_w^2 &\leq cN^{-2r}(n\|\psi\|_{r,0,0,*}^2 + \|\psi\|_{r,1,0,*}^2) \\
&\leq cN^{-2r}(nA_{r,0,0}^{(1)}(\psi) + A_{r,1,0}^{(1)}(\psi) + nA_{r,0,0}^{(2)}(\psi) + A_{r,1,0}^{(2)}(\psi)) \\
&\leq cN^{-2r}\|v\|_{r,w,A}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|w^{\frac{-3}{4}}(x)\partial x(\hat{P}_N^1 v - v)\|_w^2 &\leq cN^{2-2r}(\frac{2n}{S} n\|\psi\|_{r,0,0,*}^2 + \|\psi\|_{r,1,0,*}^2) \\
&\leq cN^{2-2r}\|v\|_{r,w,A}^2.
\end{aligned}
$$

This proves the first and second result. □

4. Legendre hybrid interpolation approximation

4.1 Legendre-Gauss-Radau hybrid interpolation approximation

To get the high accuracy of the approximated solutions, we use Legendre-Gauss-Radau hybrid interpolation. Let

$$\int_{\Lambda_{S,n}} \phi(x)w(x)dx = \sum_{i=1}^{n+1}\sum_{j=0}^{N} \phi(\alpha_{N,i,j})w_{N,i,j} \quad \forall\phi \in \mathcal{H}_{2N,n} \tag{4.1}$$

such that for $j = 0, \dots, N-1$

$$\alpha_{N,i,j} = \begin{cases} \dfrac{S}{2n}(\delta_{N,i,j} + 2i - 1), & i = 1, \dots, n, \\ \dfrac{\delta_{N,i,j}(S-1) - S - 1}{\delta_{N,i,j} - 1}, & i = n+1, \end{cases} \tag{4.2}$$

$$\alpha_{N,n+1,N} = S, \quad \alpha_{N,i,N} = \frac{S(i-1)}{n} \; (i = 1, \dots, n),$$

where $\alpha_{N,i,j}$ are the zeros of $F_{N,i}(x) + F_{N+1,i}(x)$, $\delta_{N,i,j}$ are the zeros of $L_N(x) + L_{N+1}(x)$ and $w_{N,i,j}$ are the corresponding Christoffel numbers. By using the weights of the Legendre-Gauss-Radau quadrature for $(j = 0, \dots, N-1)$, we obtain

$$w_{N,i,j} = \begin{cases} \dfrac{1}{(N+1)^2}\left(\dfrac{2i - \frac{2n}{S}\alpha_{N,i,j}}{[F_{N,i}(\alpha_{N,i,j})]^2}\right), & i = 1, \dots, n, \\ \dfrac{1}{(N+1)^2}\left(\dfrac{2}{(\alpha_{N,i,j} - S + 1)(F_{N,i}(\alpha_{N,i,j}))^2}\right), & i = n+1, \end{cases} \tag{4.3}$$

and

$$w_{N,i,N} = \frac{1}{(N+1)^2} \quad (i = 1, \dots, n+1).$$

According to (15.3.10) in Szegö [11],

$$w_{N,i,j} = \begin{cases} \dfrac{2\pi}{N}\sqrt{\dfrac{8ni}{S}\alpha_{N,i,j} + 4i - \dfrac{4n}{S}\alpha_{N,i,j} - \dfrac{4n^2}{S^2}\alpha_{N,i,j}^2 - 4i^2}, \\ \qquad i = 1, \dots, n\,, j = 0, \dots, N-1, \\ \dfrac{4\pi(\alpha_{N,i,j} - S)^{1/2}}{N(\alpha_{N,i,j} - S + 1)}, \\ \qquad i = n+1. \end{cases} \tag{4.4}$$

The discrete norm and discrete inner product related to the Legendre-Gauss-Radau hybrid interpolation points from continuous functions f, g are,

$$\begin{aligned} (f,g)_{w,N} &= \sum_{i=1}^{n+1}\sum_{j=0}^{N} f(\alpha_{N,i,j})g(\alpha_{N,i,j})w_{N,i,j}, \\ \|f\|_{w,N} &= (f,f)_{w,N}^{\frac{1}{2}}. \end{aligned}$$

Thanks to (4.1), we have

$$(\phi,\psi)_{w,N} = (\phi,\psi)_w \quad \forall \phi,\psi \in \mathcal{H}_{2N,n}. \tag{4.5}$$

For any $f \in C(\Lambda_{S,n})$, the Legendre-Gauss-Radau hybrid interpolant $I_N f(x) \in \mathcal{H}_{N,n}$, satisfying

$$\begin{aligned} &I_N f(\alpha_{N,i,j}) = f(\alpha_{N,i,j}) \quad (j=0,\ldots,N; \quad i=1,\ldots,n+1),\\ &(I_N f - f, \phi)_{w,N} = 0,; \quad \forall \phi \in \mathcal{H}_{N,n}. \end{aligned}$$

Theorem 4.1. *The following relation is obtained for any $v \in H^1_{w,A}(\Lambda_{S,n})$,*

$$\|I_N v\|_w \le c(\|v\|_w + N^{-1}\|w(x)^{\frac{-1}{4}}\partial x v\|)$$

Proof. By using (4.4) and (4.5), we obtain

$$\begin{aligned} \|I_N v\|_w^2 = \|I_N v\|^2_{w,N} &= \sum_{i=1}^{n+1}\sum_{j=0}^{N} v^2(\xi_{N,i,j}) w_{N,i,j}\\ &= \sum_{i=1}^{n+1}\sum_{j=0}^{N-1} v^2(\alpha_{N,i,j}) w_{N,i,j} + \sum_{i=1}^{n+1} v^2(\alpha_{N,i,N}) w_{N,i,N}\\ &= \sum_{i=1}^{n}\sum_{j=0}^{N-1} v^2(\alpha_{N,i,j}) w_{N,i,j} + \sum_{j=0}^{N-1} v^2(\alpha_{N,n+1,j}) w_{N,n+1,j}\\ &+ \sum_{i=1}^{n} v^2(\alpha_{N,i,N}) w_{N,i,N} + v^2(\alpha_{N,n+1,N}) w_{N,n+1,N}\\ &\le cN^{-1}\left\{\sum_{i=1}^{n}\sum_{j=0}^{N-1} v^2(\alpha_{N,i,j})\left[\frac{8ni}{S}\alpha_{N,i,j} + 4i - \frac{4n}{S}\alpha_{N,i,j} - \frac{4n^2}{S^2}\alpha_{N,i,j} - 4i^2\right]^{\frac{1}{2}}\right.\\ &\left.+ \sum_{j=0}^{N-1} v^2(\alpha_{N,n+1,j})(\alpha_{N,n+1,j} - S)^{\frac{1}{2}}(\alpha_{N,n+1,j} - S + 1)^{-1}\right\}\\ &+ \sum_{i=1}^{n} v^2(\frac{S(i-1)}{n})(N+1)^{-2} + v^2(S)(N+1)^{-2}. \end{aligned}$$

According to trace theorem, we have

$$\begin{aligned} &\|v(S)\|^2 \le c\|v\|^2_{1,w},\\ &\|v(\frac{S(i-1)}{n})\|^2 \le c\|v\|^2_{1,w} \quad (i=1,\ldots,n). \end{aligned}$$

Let

$$\alpha_{N,i,j} = \begin{cases} \dfrac{S}{2n}(\cos\theta_{N,i,j} + 2i - 1), & i = 1,\ldots,n\\ \dfrac{(S-1)\cos\theta_{N,i,j} - S - 1}{\cos\theta_{N,i,j} - 1}, & i = n+1, \end{cases}$$

and

$$\hat{v}(\theta_{N,i,j}) = \begin{cases} v(\frac{S}{2n}(\cos\theta_{N,i,j} + 2i - 1)) & i = 1, \dots, n \\ v(\frac{(S-1)\cos\theta_{N,i,j} - S - 1}{\cos\theta_{N,i,j} - 1}) & i = n+1. \end{cases}$$

By using (4.2) and Theorem 8.9.1 [11],

$$\theta_{N,i,j} = \frac{1}{N}(j\pi + O(1)); \quad j = 0, \dots, N-1.$$

where $O(1)$ is bounded uniformly.

Now, let $a_0 = \frac{O(1)}{N+1}$ and $a_1 = \frac{N\pi + O(1)}{N+1}$. Then $\theta_{N,i,j} \in K_j \subset [a_0, a_1]$, K_j being of size $\dfrac{c}{N+1}$. Consequently,

$$\begin{aligned}
\|I_N v\|_w^2 \le{}& cN^{-1}\left\{\sum_{i=1}^{n}\sum_{j=0}^{N-1}\hat{v}^2(\theta_{N,i,j})\left[\frac{8ni}{S}(\frac{S}{2n}(\cos\theta_{N,i,j} + 2i - 1))\right.\right. \\
&\left.\left. -\frac{4n}{S}(\frac{S}{2n}(\cos\theta_{N,i,j} + 2i - 1)) - \frac{4n^2}{S^2}(\frac{S}{2n}(\cos\theta_{N,i,j} + 2i - 1))^2 - 4i^2 + 4i\right]^{\frac{1}{2}}\right\} \\
&+ cN^{-1}\left\{\sum_{j=0}^{N-1}\hat{v}^2(\theta_{N,n+1,j})\left(\frac{-\cos\theta_{N,n+1,j} - 1}{\cos\theta_{N,n+1,j} - 1}\right)^{\frac{1}{2}}\left(\frac{-2}{\cos\theta_{N,n+1,j} - 1}\right)^{-1}\right\} \\
&+ N^{-2}\sum_{i=1}^{n}\hat{v}^2(\frac{S(i-1)}{n}) + \hat{v}^2(S) \\
\le{}& cN^{-1}\left\{\sum_{i=1}^{n}\sum_{j=0}^{N}\hat{v}^2(\hat{\theta}_{N,i,j})\sin\hat{\theta}_{N,i,j} + \sum_{j=0}^{N}\hat{v}^2(\hat{\theta}_{N,n+1,j})\sin\hat{\theta}_{N,n+1,j}\right\} \\
&+ cN^{-2}(n+1)\|v(x)\|_{1,w}^2.
\end{aligned}$$

We apply inequality (13.7) in [7] for interval K_j. Thus,

$$\begin{aligned}
\|I_N v\|_w^2 \le{}& c\sum_{i=1}^{n+1}\sum_{j=0}^{N-1}\left(\|\hat{v}(\theta)\sin^{\frac{1}{2}}\theta\|_{L^2(K_j)}^2 + N^{-2}\|\partial_\theta\hat{v}(\theta)\sin^{\frac{1}{2}}\theta\|_{L^2(K_j)}^2\right) \\
&+ c(n+1)N^{-2}\|v(x)\|_{1,w}^2 \\
\le{}& c(\|v(x)\|_{L^2_w(\Lambda_{a,n})}^2 + N^{-2}\|w(x)^{\frac{-1}{4}}\partial_x v\|_{L^2_w(\Lambda_{a,n})}^2 + N^{-2}\|\partial_x v\|_{L^2_w(\Lambda_{a,n})}^2).
\end{aligned}$$

Which implies the desired result. □

Theorem 4.2. *For any $v \in H^r_{w,A}(\Lambda_{S,n})$ and $0 \le \beta \le 1 \le r$,*

$$\|I_N v - v\|_{\beta,w} \le cN^{2\beta - r}\|v\|_{r,w,A}.$$

Proof. By using Theorems 3.4 and 4.1 and Cauchy-Schwartz inequality and $I_N(\hat{P}_N^1 v) = \hat{P}_N^1 v$, we can conclude

$$\|I_N v - \hat{P}_N^1 v\|_w \leq c(\|\hat{P}_N^1 v - v\|_w + N^{-1}\|w(x)^{\frac{-1}{4}}\partial_x(\hat{P}_N^1 v - v)\|),$$
$$\|I_N v - \hat{P}_N^1 v\|_w \leq cN^{-r}\|v\|_{r,w,A} \tag{4.6}$$

and

$$\|\hat{P}_N^1 v - v\|_w \leq cN^{-r}\|v\|_{r,w,A}. \tag{4.7}$$

Moreover, by (4.6) and (4.7) and Theorem 2.2, we get

$$\begin{aligned} \|I_N v - v\|_{1,w} &\leq \|\hat{P}_N^1 v - v\|_{1,w} + \|I_N v - \hat{P}_N^1 v\|_{1,w} \\ &\leq cN^2\|\hat{P}_N^1 v - v\|_w + cN^2\|I_N v - \hat{P}_N^1 v\|_w \\ &\leq cN^{2-r}\|v\|_{r,w,A}. \end{aligned} \tag{4.8}$$

Finally, the result follows from (4.6), (4.7), (4.8) and space interpolation. □

5. Error estimation

We consider the following problem ($v > 0$)

$$\begin{cases} -\partial_x^2 V(x) + vV(x) = g(x), & 0 < x < \infty \\ V(0) = 0, & i = 1, \ldots, n, n+1 \\ (x - S + 1)^{-\frac{3}{2}} V(x) \to 0, & \text{as} \quad x \to \infty. \end{cases} \tag{5.1}$$

To solve this problem with $v > \dfrac{3}{4}$, we find $V \in H^1_{0,w}(\Lambda_{S,n})$ such that

$$a_w^v(V, v) = (g, v)_w, \quad \forall v \in H^1_{0,w}(\Lambda_{S,n}). \tag{5.2}$$

By considering $g \in (H^1_{0,w}(\Lambda_{S,n}))'$ and the Lax-Milgram Lemma and Lemma 3.1, (5.2) has a unique solution in $H^1_{0,w}(\Lambda_{S,n})'$.

The Legendre hybrid spectral method for (5.1) is to obtain $v_N \in \mathcal{H}^0_{N,n}$ such that

$$a_w^v(v_N, \phi) = (g, \phi)_w, \ \ \forall \phi \in \mathcal{H}^0_{N,n}. \tag{5.3}$$

Theorem 5.1. *If* $V \in H^r_{w,B}(\Lambda_{S,n}) \cap H^1_{0,w}(\Lambda_{S,n}), v > \frac{3}{4}$ *and* $r \geq 1$, *then* $\|v_N - V\|_{1,w} \leq cN^{1-r}\|V\|_{r,w,B}$.

Proof. Let $V_N = P_N^{1,0} V$ and $\hat{V}_N = v_N - V_N$. By using equations (5.2) and (5.3), we have

$$a_w^v(V_N, \phi) = (g, \phi)_w, \ \ \forall \phi \in \mathcal{H}^0_{N,n} \tag{5.4}$$

and

$$a_w^v(\hat{V}_N, \phi) = 0, \ \ \forall \phi \in \mathcal{H}^0_{N,n}. \tag{5.5}$$

Therefore, $v_N = V_N$ and the result is obtained from Theorem 3.3. □

We use the Legendre-Gauss-Radau hybrid pseudospectral method for (5.1). Thus, we have $a^v_{w,N}(u,v) = n(\partial_x u, \partial_x v)_{w_i,N} + (\partial_x u, \partial_x v - 2v(x-S+1)^{-1})_{w_{n+1},N} + v(u,v)_{w,N}, i{=}1,\ldots,n$. We know from (4.5) that for any $\phi, \psi \in \mathcal{H}_{N-1,n}$, we have

$$a^v_{w,N}(\phi,\psi) = a^v_w(\phi,\psi). \tag{5.6}$$

A Legendre hybrid pseudospectral method for (5.1) is to obtain $v_N \in \mathcal{H}^0_{N-1,n}$ such that

$$a^v_{w,N}(v_N,\phi) = (g,\phi)_{w,N}\ , \forall \phi \in \mathcal{H}^0_{N-1,n}. \tag{5.7}$$

Theorem 5.2. *If* $V \in H^r_{w,B}(\Lambda_{S,n}) \cap H^1_{0,w}(\Lambda_{S,n}), g \in H^{r-1}_{w,A}(\Lambda_{S,n}), v > \frac{3}{4}$ *and* $r \geq 1$*, then*

$$\|v_N - V_N\|_{1,w} \leq cN^{1-r}(\|V\|_{r,w,B} + \|g\|_{r-1,w,A}) \tag{5.8}$$

Proof. By using (4.5) and Theorem 2.3, we obtain

$$\begin{aligned} |(g,\phi)_{w,N}| &= |(I_N(w^{\frac{-1}{2}}(x)g), w^{\frac{1}{2}}(x)\phi(x))_{w,N}| = |(I_N(w^{\frac{-1}{2}}(x)g), w^{\frac{1}{2}}(x)\phi(x))_w| \\ &\leq \|I_N(w^{\frac{-1}{2}}(x)g)\|_{w^2}\|\phi\|_{w^2} \\ &\leq \|I_N(w^{\frac{-1}{2}}(x)g)\|_{1,w}\|\phi\|_{1,w}(\forall\phi \in \mathcal{H}_{N-1,n}). \end{aligned} \tag{5.9}$$

Hence, by Lemma 3.1 and the Lax-Milgram Lemma, (5.7) has a unique solution such that $\|v_N\|_{1,w} \leq c\|I_N(w^{\frac{-1}{2}}(x)g)\|_{1,w}$ Let $V_N = P^{1,0}_{N-1}V$. Then by (5.6) and (5.7), we have for any $\phi \in \mathcal{H}^0_{N-1,n}$,

$$a^v_w(V_N,\phi) = (g,\phi)_w, \ \ a^v_w(v_N,\phi) = (I_N g,\phi)_w. \tag{5.10}$$

Therefore, $a^v_w(V_N - v_n, \phi) = (g - I_N g, \phi)_w, \phi \in \mathcal{H}^0_{N-1,n}$. Let $\varepsilon = v - \frac{3}{4} > 0$. By taking $\phi = V_N - v_N$ and using (3.8), we obtain

$$\begin{aligned} &\frac{\varepsilon}{2}\|v_N - V_N\|^2_w + \frac{2\varepsilon}{9}|v_N - V_N|^2_{1,w} \leq a^v_w(v_N - V_N, v_N - V_N) \\ &\leq (g - I_N g, V_N - v_N)_w \leq \|g - I_N g\|_w \|V_N - v_N\|_w \\ &\leq \varepsilon\|g - I_N g\|^2_w + \frac{\varepsilon}{2}\|V_N - v_N\|^2_w. \end{aligned} \tag{5.11}$$

Therefore, by using Theorems 3.3 and 4.2, we arrive at $\|v_N - V\|_{1,w} \leq \|V_N - V\|_{1,w} + \|v_N - V_N\|_{1,w} \leq cN^{1-r}(\|V\|_{r,w,B} + \|g\|_{r-1,w,A})$. □

6. Numerical results

We present an efficient algorithm for solving (5.1) and compare and report some numerical experiments by using the hybrid pseudospectral scheme (5.7). The combinations of hybrid Legendre polynomials should be used as basis functions like [16] (see [15]). Indeed, setting

$$\psi_j(x) = \begin{cases} F_{j,i}(x) + F_{j+1,i}(x), & x \in \Lambda_i, i = 1, \dots, n \\ F_{j,n+1}(x) + F_{j+1,n+1}(x), & x \in \Lambda_{n+1}, \end{cases}$$

we have

$$\mathcal{H}^0_{N,n} = \left\{ \psi \in \mathcal{H}_{N,n} : \psi(0) = 0, \psi(Si/n) = 0, i = 1, \dots, n \right\},$$

$$\mathcal{H}^0_{N-1,n} = span \left\{ \psi_j : j = 0, 1, \dots, N-2 \right\}. \tag{6.1}$$

Hence, setting

$$b_{kj} = (\psi_j, \psi_k)_{w,N} = (\psi_j, \psi_k)_w, a_{kj} = a^v_{w,N}(\psi_j, \psi_k) = -(\psi''_j, \psi_k)_w, \tag{6.2}$$

$$v_N = \sum_{j=0}^{N-2} x_j \psi_j(x), \overline{x} = (x_0, x_1, \dots, x_{N-2})^t,$$

$$\overline{g} = (g_0, g_1, \dots, g_{N-2})^t \text{ with } g_j = (g, \psi_j)_w,$$

the Legendre hybrid pseudospectral relation (5.7) is changed to

$$(vB + A)\overline{x} = \overline{g}. \tag{6.3}$$

Now, with replacing hybrid functions, we have

$$\begin{aligned} b_{kj} &= \sum_{i=1}^{n+1} \int_{\Lambda_i} \psi_j(x)\psi_k(x) w_i(x) dx \\ &= \sum_{i=1}^{n} \int_{\Lambda_i} (F_{j,i}(x) + F_{j+1,i}(x))(F_{k,i}(x) + F_{k+1,i}(x)) w_i(x) dx \\ &+ \int_{\Lambda_{n+1}} (F_{j,n+1}(x) + F_{j+1,n+1}(x))(F_{k,n+1}(x) + F_{k+1,n+1}(x)) w_{n+1}(x) dx \end{aligned}$$

and

$$\begin{aligned} a_{kj} &= -\sum_{i=1}^{n+1} \int_{\Lambda_i} \partial^2_x \psi_j(x)\psi_k(x) w_i(x) dx \\ &= -\sum_{i=1}^{n} \int_{\Lambda_i} (\partial^2_x F_{j,i}(x) + \partial^2_x F_{j+1,i}(x))(F_{k,i}(x) + F_{k+1,i}(x)) w_i(x) dx \\ &- \int_{\Lambda_{n+1}} (\partial^2_x F_{j,n+1}(x) + \partial^2_x F_{j+1,n+1}(x))(F_{k,n+1}(x) + F_{k+1,n+1}(x)) w_{n+1}(x) dx. \end{aligned}$$

Setting

$$x = \begin{cases} \dfrac{S}{2n}(\gamma + 2i - 1), & (i = 1, \dots, n) \\ \dfrac{\gamma(S-1) - S - 1}{\gamma - 1}, & \end{cases}$$

one verify that

$$b_{kj} = \frac{S}{2n}(\sum_{i=1}^{n}\int_{-1}^{1}(L_j(\gamma)+L_{j+1}(\gamma))(L_k(\gamma)+L_{k+1}(\gamma))d\gamma) \\ -\frac{1}{4}\int_{-1}^{1}(L_j(\gamma)+L_{j+1}(\gamma))(L_k(\gamma)+L_{k+1}(\gamma))d\gamma$$

and

$$a_{kj} = -\frac{2n}{S}(\sum_{i=1}^{n}\int_{-1}^{1}(\partial_\gamma^2 L_j(\gamma)+\partial_\gamma^2 L_{j+1}(\gamma))(L_k(\gamma)+L_{k+1}(\gamma))d\gamma) \\ -\int_{-1}^{1}(\frac{2}{(1-\gamma)^2}(\partial_\gamma L_j(\gamma)+\partial_\gamma L_{j+1}(\gamma))+\partial_\gamma^2 L_j(\gamma) \\ +\partial_\gamma^2 L_{j+1}(\gamma))(L_k(\gamma)+L_{k+1}(\gamma))(1-\gamma)d\gamma.$$

By using Orthogonality of Legendre polynomials, the equation is solved. We test the proposed method with numerical examples for solving (5.1) with v=2 and different values of S. We solve them and compare the results with rational method [5].

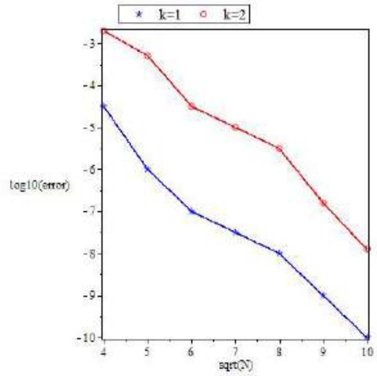
Example 1. $V(x) = \sin kxe^{-x}$.

This function decreases exponentially at infinity. Theorem 5.2 predicts that errors will decrease faster than any algebraic rate. We solve (5.1) for this function by using the proposed method with v=2 and $S = 1, 2, 4$. In Fig. 1, we plot the $\log_{10}$ of errors for $k = 1, 2$. The results show that errors have been further reduced with (S=4). In Fig. 2, we plot the $\log_{10}$ of errors and compare the present method(S=4) with rational method [5].

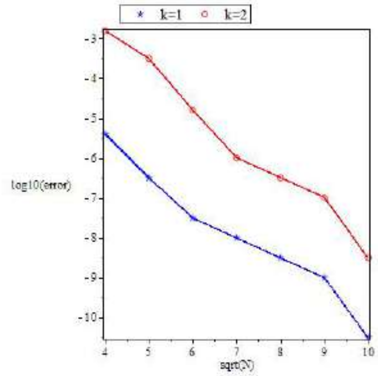
Example 2. $V(x) = x/(1+x)^h$. The second example decays algebraically at infinity without essential singularity. We solve (5.1) for this function by using the proposed method with v=2 and $S = 1, 2, 3$. In Fig. 3, we plot the $\log_{10}$ of errors for $h = 1.5, 2.5$. The results show that errors have been further reduced with (S=3). In Fig. 4, we plot the $\log_{10}$ of errors and compare the present method(S=3) with rational method [5].

Example 3. $V(x) = (\sin 2x)/(1+x)^h$.

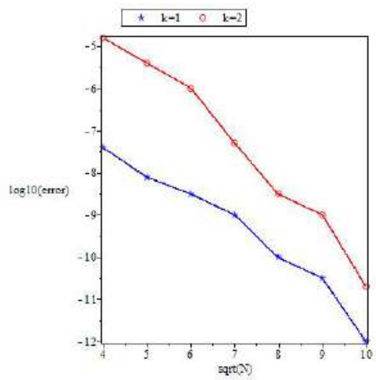
This function decays algebraically at infinity but it also has an essential singularity at infinity. We solve (5.1) for this function by using the proposed method with $v = 2$ and $S = 2, 4, 6$. In Fig. 5, we plot the $\log_{10}$ of errors for $h = 3.5, 5$. The results show that errors have been further reduced with (S=6). In Fig. 6, we plot the $\log_{10}$ of errors and compare the present method ($S = 6$) with rational method [5]. The observed convergence rate plotted in Fig. 6 agrees well with the theoretical result.



(a) S=1



(b) S=2



(c) S=4

Figure 1: Log-errors with different values of S and k

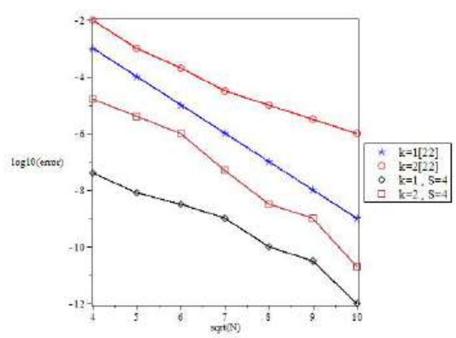
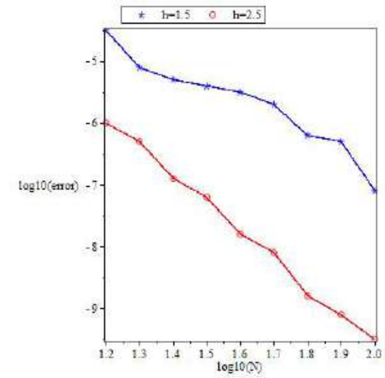
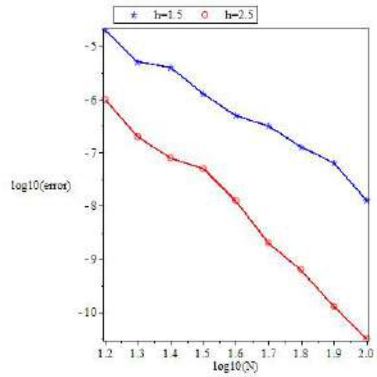


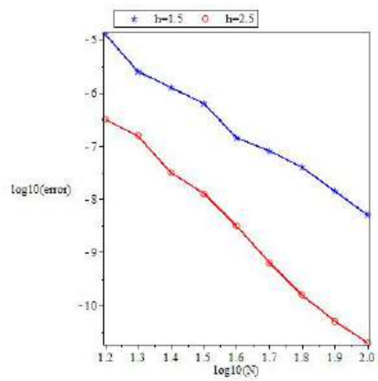
Figure 2: Log-errors and Comparison of the present method (S=4) with rational method [5].



(a) S=1



(b) S=2



(c) S=3

Figure 3: Log-errors with different values of S and h

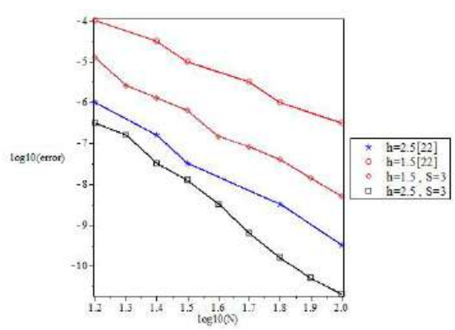
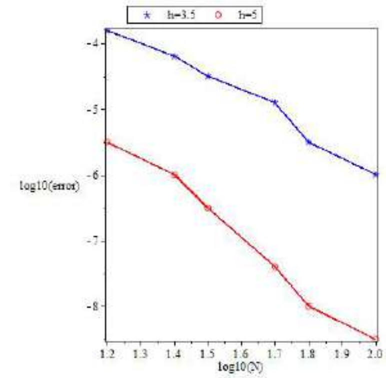
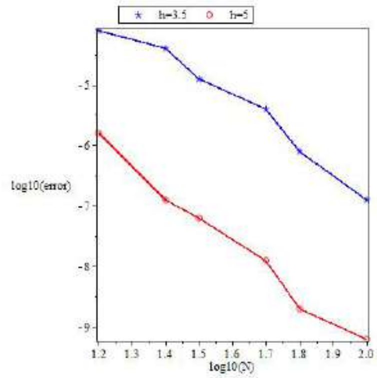


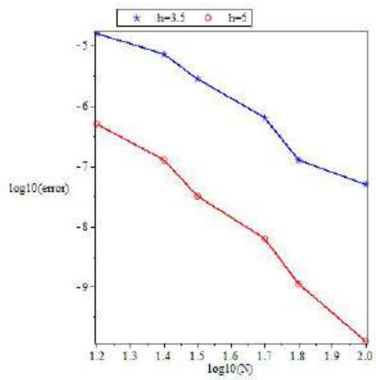
Figure 4: Log-errors and Comparison of the present method ($S = 3$) with rational method [5].



(a) S=2



(b) S=4



(c) S=6

Figure 5: Log-errors with different values of S and h

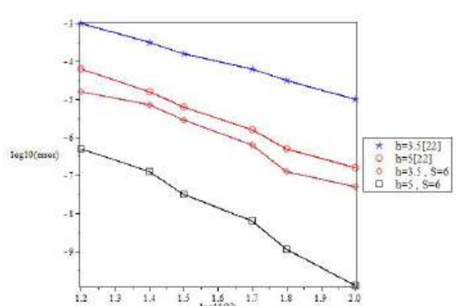


Figure 6: Log-errors and Comparison of the present method ($S = 6$) with rational method [5].

7. Conclusion

This paper presented spectral method and pseudospectral method on semi-infinite interval $[0, \infty)$ by using hybrid Legendre functions and decomposition of the half-line into $[0, S]$ and $[S, \infty)$ depending on the problem. Basic results on the proposed hybrid approximation of several projections and interpolation were established. The obtained results indicate that the errors decay more rapidly as compared to usual rational approximation. Moreover, it was shows that the proposed hybrid scheme is more suited for problems with oscillatory behaviour.

8. Suggestion

The results of this research can be successfully used in engineering, physics and mechanics and other sciences in infinite domains. In solving the differential equations in semi-infinite interval $[0, \infty)$, where the desired interval is decomposed into $[0, S]$ and $[S, \infty)$, it is suggested that the best value for S would be chosen in future research. In physical models for functions, there is damping from a point onwards and the best value for S can be chosen. To select the value of S, the difference between the values of the function at consecutive points would be obtained. Whenever this difference is less than a certain value of ϵ, the desired point is the best value for S.

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