Coupled fixed point theorem in quasi metric space

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Abstract. In the present paper a unique common coupled fixed point theorem has been proven for quasi metric space with modified - ω distance function. This result is improvement, modification and extension in the study of quasi metric space. An example has been given to illustrate the work.

Keywords: fixed point, quasi-metric space, coupled fixed point, modified-ω distance.

1. Introduction and preliminaries

The fixed point theory is one of the important topics of functional analysis. The whole metric fixed point theory based on a very powerful theorem Banach Contraction principle [5]. Since then many researchers worked on it and develop the results in different directions.

Bhaskar and Lakshmikantham [6] initiated the concept of coupled fixed point in the following way:

Definition 1.1 ([6]). An element $(a, b) \in X \times X$ is called a coupled fixed point of mapping $J : X \times X \to X$ if $J(a, b) = a, J(b, a) = b$.

Further several researchers Lakshmikantham and Ciric [10], Choudhary and Kundu [8], Luong et al. [11], Razani and Parvaneh [12], Alsulami [3], Samet et al. [13], Alotaibi [2] extended the coupled fixed point theorems in partial order metric space under different constraints but it is not found in quasi metric space, so the authors are motivated towards the present work.

Remark 1. Every metric space is quasi metric space but the converse is not true.

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Remark 2. The results which are true for quasi metric space need not be true for metric space.
Wilson [16] introduced the concept of quasi metric space as here under:

Definition 1.2 ([16]). The function $q : X \times X \to [0, \infty)$ is a quasi-metric if it satisfies

(i) $q(a, b) = 0 \Leftrightarrow a = b$;

(ii) $q(a, b) \leq q(a, c) + q(c, b)$, for all $a, b, c \in X$.

The pair (X, q) is called quasi metric space.

The study of fixed point theorems on quasi metric space further continued by Aydi, et al. [4], Bilgili, et al. [7], Shatanawi, et al. [14], Shatanawi, et al. [15], Alegre, et al. [1].

Definition 1.3 ([4,9]). A sequence $\{a_l\}$ converges to $a \in X$if $\lim_{l\to\infty} q(a_l, a) = \lim_{l\to\infty} q(a, a_l) = 0$.

Definition 1.4 ([9]). Let $\{a_l\}$ be a sequence in X. Then:

(i) $\{a_l\}$ is called left Cauchy if for any $\delta > 0, \exists N_0 \in N$, such that $q(a_l, a_m) < \delta \forall l \geq m > N_0$

(ii) $\{a_l\}$ is called right Cauchy if for any $\delta > 0, \exists N_0 \in N$,such that $q(a_l, a_m) < \delta \forall m \geq l > N_0$.

Definition 1.5 ([4,9]). $\{a_l\}$ is called Cauchy sequence if for any $\delta > 0, \exists N_0 \in N$ such that $q(a_l, a_m) \leq \delta \forall l, m > N_0$ or $\{a_l\}$ is right and left Cauchy.

Definition 1.6 ([1]). The modified -ω distance on (X, q) is a function $p : X \times X \to [0, \infty)$ which satisfies the following

(Q1) $p(a_1, a_2) \leq p(a_1, a_3) + p(a_3, a_2), \forall a_1, a_2, a_3 \in X$.

(Q2) $p : X \to [0, \infty)$is lower semi continuous$\forall a \in X$.

(Q3) $\forall \varepsilon > 0 \exists \delta > 0$ such that $p(a_1, a_2) \leq \delta, p(a_2, a_3) \leq \delta \Rightarrow q(a_1, a_3) \leq \varepsilon \forall a_1, a_2, a_3 \in X$.

2. Main result

The aim of this paper is to establish coupled fixed point theorems in quasi metric space.

Theorem 2.1. *Let (X, q) be a complete quasi metric space equipped with an mw distance mapping p, also J and K: $X \times X \to X$ be two continuous functions such that the pair (J, K) satisfy*

$$p\{J(a,b),K(c,d)\} \leq h\max\{p(a,J(a,b)),p(c,K(c,d))\}, \tag{2.1}$$
$$p\{K(a,b),J(c,d)\} \leq h\max\{p(a,K(a,b)),p(c,J(c,d))\}. \tag{2.2}$$

here $h \in [0,1)$. Then J and K have unique common coupled fixed point.

Proof. Let $a_{2l+1} = J(a_{2l}, b_{2l}), b_{2l+1} = J(b_{2l}, a_{2l})$,

$$a_{2l+2} = K(a_{2l+1}, b_{2l+1}), b_{2l+2} = K(b_{2l+1}, a_{2l+1}).$$

Now, to prove that $p(a_l, a_{l+1}) = 0$ or $p(a_{l+1}, a_l) = 0$ and $p(b_l, b_{l+1}) = 0$ or $p(b_{l+1}, b_l) = 0$.
Case I. Consider $p(a_l, a_{l+1}) = 0$, for $l = 2n$,

$$\begin{aligned} p(a_{2n+1}, a_{2n+2}) &= p\{J(a_{2n}, b_{2n}), K(a_{2n+1}, b_{2n+1})\}, \\ &\leq h\max\{p(a_{2n}, J(a_{2n}, b_{2n})), p(a_{2n+1}, K(a_{2n+1}, b_{2n+1}))\} \\ &= h\max\{p(a_{2n}, a_{2n+1}), p(a_{2n+1}, a_{2n+2})\}. \end{aligned}$$

(i) If $p(a_{2n}, a_{2n+1})$ is maximum, then $p(a_{2n+1}, a_{2n+2}) = 0$.

(ii) If $p(a_{2n+1}, a_{2n+2})$ is maximum, then $p(a_{2n+1}, a_{2n+2}) = 0$,

$$p(a_{2n+1}, a_{2n+2}) = 0. \tag{2.3}$$

Now,

$$\begin{aligned} p(b_{2n+1}, b_{2n+2}) &= p\{J(b_{2n}, a_{2n}), K(b_{2n+1}, a_{2n+1})\} \\ &\leq h\max\{p(b_{2n}, J(b_{2n}, a_{2n})), p(b_{2n+1}, K(b_{2n+1}, a_{2n+1}))\} \\ &= h\max\{p(b_{2n}, b_{2n+1}), p(b_{2n+1}, b_{2n+2})\}. \end{aligned}$$

(i) If $p(b_{2n}, b_{2n+1})$ is maximum then $p(b_{2n+1}, b_{2n+2}) = 0$.
(ii) If $p(b_{2n+1}, b_{2n+2})$ is maximum then $p(b_{2n+1}, b_{2n+2}) = 0$,

$$p(b_{2n+1}, b_{2n+2}) = 0. \tag{2.4}$$

Now,

$$\begin{aligned} p(a_{2n+2}, a_{2n+1}) &= p\{K(a_{2n+1}, b_{2n+1}), J(a_{2n}, b_{2n})\} \\ &\leq h\max\{p(a_{2n+1}, K(a_{2n+1}, b_{2n+1})), p(a_{2n}, J(a_{2n}, b_{2n}))\} \\ &= h\max\{p(a_{2n+1}, a_{2n+2}), p(a_{2n}, a_{2n+1})\}. \end{aligned}$$

(i) $p(a_{2n+1}, a_{2n+2})$ is maximum then $p(a_{2n+2}, a_{2n+1}) = 0$.

(ii) If $p(a_{2n}, a_{2n+1})$ is maximum then $p(a_{2n+2}, a_{2n+1}) = 0$,

$$p(a_{2n+2}, a_{2n+1}) = 0. \tag{2.5}$$

Similarly one can show that

$$p(a_{2n}, a_{2n+2}) = 0, p(b_{2n}, b_{2n+2}) = 0 \tag{2.6}$$

using Q1. From (2.1.5), (2.1.6) and Q3,

$$q(a_{2n}, a_{2n+1}) = 0 \begin{cases} \therefore q(a_l, a_{l+1}) = 0 \Rightarrow a_l = a_{l+1} = J(a_l, b_l), \\ q(b_l, b_{l+1}) = 0 \Rightarrow b_l = b_{l+1} = K(b_l, a_l). \end{cases} \tag{2.7}$$

(a_l, b_l) is coupled fixed point.

Also, (2.1.8) $p(a_{2n}, a_{2n+1}) \leq p(a_{2n}, a_{2n+2}) + p(a_{2n+2}, a_{2n+1}) = 0$. From (2.1.3), (2.1.8) and Q3

$$\begin{aligned} &q(a_{2n}, a_{2n+2}) = 0, \\ &q(a_{2n+1}, a_{2n+2}) \leq q(a_{2n+1}, a_{2n}) + q(a_{2n}, a_{2n+2}) = 0. \end{aligned} \tag{2.8}$$

Similarly, $q(b_{2n+1}, b_{2n+2}) \leq q(b_{2n+1}, b_{2n}) + q(b_{2n}, b_{2n+2}) = 0$. Hence,

$$\begin{aligned} &a_{2n} = a_{2n+1} = a_{2n+2} \Rightarrow a_l = a_{l+1} = a_{l+2} \\ &\Rightarrow a_l = J(a_l, b_l) = K(a_{l+1}, b_{l+1}), \end{aligned}$$

when $l \to \infty$, as J and K are continuous $a = J(a, b) = K(a, b)$.

Also, $b_{2n} = b_{2n+1} = b_{2n+2} \Rightarrow b_l = b_{l+1} = b_{l+2} \Rightarrow b_l = J(b_l, a_l) = K(b_{l+1}, a_{l+1})$, when $l \to \infty$, as J and K are continuous $\Rightarrow b = J(b, a) = K(b, a)$. Thus, (a, b) is common coupled fixed point of J and K.

Case II: When l is odd, $l = 2n + 1$,

$$\begin{aligned} &p(a_{2n+1}, a_{2n+2}) = 0, \\ &p(a_{2n+2}, a_{2n+3}) = p\{K(a_{2n+1}, b_{2n+1}), J(a_{2n+2}, b_{2n+2})\} \\ &\leq h \max\{p(a_{2n+1}, K(a_{2n+1}, b_{2n+1})), p(a_{2n+2}, J(a_{2n+2}, b_{2n+2}))\} \\ &= h \max\{p(a_{2n+1}, a_{2n+2}), p(a_{2n+2}, a_{2n+3}))\}. \end{aligned} \tag{2.9}$$

This implies

$$p(a_{2n+2}, a_{2n+3}) = 0. \tag{2.10}$$

Also, it can be proved that

$$p(a_{2n+3}, a_{2n+2}) = 0, p(b_{2n+2}, b_{2n+3}) = 0, p(b_{2n+3}, b_{2n+2}) = 0. \tag{2.11}$$

Now, by Q1

$$p(a_{2n+1}, a_{2n+3}) \leq p(a_{2n+1}, a_{2n+2}) + p(a_{2n+2}, a_{2n+3}) = 0. \tag{2.12}$$

From (2.1.12), (2.1.13) and Q_3

$$q(a_{2n+1}, a_{2n+2}) = 0. \tag{2.13}$$

Also, $q(a_{2n+1}, a_{2n+3}) = 0$. Similarly, $q(b_{2n+1}, b_{2n+2}) = 0$ and $q(b_{2n+1}, b_{2n+3}) = 0$. From (2.1.10), (2.1.11) and Q_3

$$\begin{aligned} &q(a_{2n+1}, a_{2n+3}) = 0, a_{2n+1} = a_{2n+2} = a_{2n+3} \Rightarrow a_l = a_{l+1} = a_{l+2} \\ &\Rightarrow a_l = J(a_l, b_l) = K(a_{l+1}, b_{l+1}). \end{aligned} \tag{2.14}$$

Similarly, one can prove that $\Rightarrow b = J(b, a) = K(b, a)$. Thus, (a, b) are common coupled fixed point of J and K.

Similarly, one can show that if $p(a_{l+1}, a_l) = 0$, (a, b) are common coupled fixed point of J and K. Now, assume that $p(a_l, a_{l+1}) \neq 0, p(a_{l+1}, a_l) \neq 0$. Then,

$$\begin{aligned} &p(a_{2n+1}, a_{2n}) = p((Ja_{2n}, b_{2n}), K(a_{2n+1}, b_{2n+1})) \\ &\leq h \max\{p(a_{2n}, J(a_{2n}, b_{2n})), p(a_{2n+1}, K(a_{2n+1}, b_{2n+1}))\} \\ &= h \max\{p(a_{2n}, a_{2n+1}), p(a_{2n+1}, a_{2n+2})\}. \end{aligned} \tag{2.15}$$

(i) If $p(a_{2n}, a_{2n+1})$is maximum, then $p(a_{2n+1}, a_{2n+2}) \leq hp(a_{2n}, a_{2n+1})$.

(ii) If $p(a_{2n+1}, a_{2n+2})$ is maximum, then $p(a_{2n+1}, a_{2n+2}) = 0$, which is contradiction.

Hence, $p(a_{2n+1}, a_{2n+2}) \leq hp(a_{2n}, a_{2n+1})$. By similar process we can prove that

$$p(a_{2n}, a_{2n+1}) \leq hp(a_{2n-1}, a_{2n}).$$

Also, we can prove

$$p(b_{2n+1}, b_{2n+2}) \leq hp(b_{2n}, b_{2n+1})$$

and

$$p(b_{2n}, b_{2n+1}) \leq hp(b_{2n-1}, b_{2n}).$$

Thus,

$$p(a_l, a_{l+1}) \leq hp(a_{l-1}, a_l) \text{ and } p(b_l, b_{l+1}) \leq hp(b_{l-1}, b_l). \tag{2.16}$$

Now,

$$\begin{aligned} &p(a_{l+1}, a_l) = p(J(a_l, b_l), K(a_{l-1}, b_{l-1})) \\ &\leq h \max\{p(a_l, J(a_l, b_l), p(a_{l-1}, K(a_{l-1}, b_{l-1}))\} \\ &= h \max\{p(a_l, a_{l+1}), p(a_{l-1}, a_l)\}, \\ &p(a_{l+1}, a_l) \leq hp(a_{l-1}, a_l). \end{aligned} \tag{2.17}$$

Repeating l times

$$p(a_{l+1}, a_l) \leq h^l p(a_1, a_0). \tag{2.18}$$

Also, one can prove

$$p(a_l, a_{l+1}) \leq h^l p(a_0, a_1). \tag{2.19}$$

Similarly, we can prove that $p(b_{l+1}, b_l) \leq h^l p(b_1, b_0)$and $p(b_l, b_{l+1}) \leq h^l p(b_0, b_1)$. Thus, as $l \to \infty, p(a_l, a_{l+1}) = 0, p(a_{l+1}, a_l) = 0, p(b_l, b_{l+1}) = 0, p(b_{l+1}, b_l) = 0$. Now, to prove $\{a_l\}$ and $\{b_l\}$ are Cauchy sequences, we need to prove $\lim_{s,t\to\infty} p(a_s, a_t) = 0$, for each $s, t \in N$.

Case I: If s is odd and t is even with $s < t$, then

$$\begin{aligned}
p(a_s, a_t) &= p(Ja_{s-1}, b_{s-1}), K(a_{t-1}, b_{t-1}) \\
&\leq h \max\{p(a_{s-1}, J(a_{s-1}, a_{s-1})), p(a_{t-1}, K(a_{t-1}, a_{t-1}))\} \\
&= h \max\{p(a_{s-1}, a_s), p(a_{t-1}, a_t)\} \\
&= hp(a_{s-1}, a_s).
\end{aligned}$$

Thus, we have $p(a_s, a_t) \leq h^s p(a_0, a_1)$.

Let $s, t \to \infty$. Then,

$$\lim_{s,t\to\infty} p(a_s, a_t) = 0, \tag{2.20}$$

with $s < t$. Similarly $\lim_{s,t\to\infty} p(b_s, b_t) = 0$, with $s < t$.

Case II: If s is odd and t is even with $s > t$

$$\begin{aligned}
p(a_s, a_t) &= p(J(a_{s-1}, b_{s-1}), K(a_{t-1}, b_{t-1})) \\
&\leq h \max\{p(a_{s-1}, J(a_{s-1}, b_{s-1})), p(a_{t-1}, K(a_{t-1}, b_{t-1}))\} \\
&= h \max\{p(a_{s-1}, a_s), p(a_{t-1}, a_t)\} \\
&= hp(a_{t-1}, a_t).
\end{aligned}$$

Thus, $p(a_s, a_t) \leq h^t p(a_0, a_1)$.

Let $s, t \to \infty$. Then

$$\lim_{s,t\to\infty} p(a_s, a_t) = 0, \tag{2.21}$$

with $s > t$. Similarly $\lim_{s,t\to\infty} p(b_s, b_t) = 0$ with $s > t$.

By similar argument, one can show for s even and t odd, for $s < t$ and $s > t$ $\lim_{s,t\to\infty} p(a_s, a_t) = 0$ and $\lim_{s,t\to\infty} p(b_s, b_t) = 0$.

Now, we show that $\{x_l\}$ is right Cauchy sequence. Consider the following cases:

(i) If $l, r \in N$ such that l is odd and r is even with $r > l$, then we have $\lim_{l,r\to\infty} p(a_l, a_r) = 0$ from (2.1.19).

(ii) If $l, r \in N$ such that l is even and r is odd with $r > l$, then we have $\lim_{l,r\to\infty} p(a_l, a_r) = 0$ from (2.1.21).

(iii) $l, r \in N$ such that l and r both are even with $r > l$,then we have $p(a_l, a_r) \leq p(a_l, a_{l+1}) + p(a_{l+1}, a_r)$. Thus, $\lim_{l,r\to\infty} p(a_l, a_r) = 0, l > r$.

(iv) If $l, r \in N$ such that l and r both are even with $r > l$ then, we have $p(a_l, a_r) \leq p(a_l, a_{r+1}) + p(a_{r+1}, a_r)$. Thus, $\lim_{l,r\to\infty} p(a_l, a_r) = 0, l > r$.

Hence, $\{a_l\}$ is right Cauchy sequence. With similar argument $\{b_l\}$ is right Cauchy sequence and also $\{a_l\}$,$\{b_l\}$ are left Cauchy sequences. Hence $\{a_l\}$ and $\{b_l\}$ both are Cauchy sequences.

Since (X, q) is complete quasi metric space, therefore $\lim_{l\to\infty} p(a_{2l}, a) = 0 = \lim_{l\to\infty} p(a, a_{2l})$

Now, $a = \lim_{l\to\infty} a_{2l+1} = \lim_{l\to\infty} J(a_{2l}, b_{2l}) = J(a, b)$ and $b = \lim_{l\to\infty} b_{2l+1} = \lim_{l\to\infty} J(b_{2l}, a_{2l}) = J(b, a)$. Thus, (a, b) is coupled fixed point of J and with similar argument it can be shown that (a, b) is coupled fixed point of K.

Hence, (a, b) is common coupled fixed point of J and K. To prove uniqueness first we have to prove $p(v, v) = 0$,

$$p(v, v) = p\{J(v, v), K(v, v)\} \leq h \max\{p(v, J(v, v)), p(v, K(v, v))\} = hp(v, v).$$

Thus, $p(v, v) = 0$.

Let (a, b) and (a', b') be two coupled fixed points of J and K, $J(a, b) = K(a, b) = a$ and $J(b, a) = K(b, a) = b$, $J(a', b') = K(a', b') = a'$ and $J(b', a') = K(b', a') = b'$.

Now,

$$\begin{aligned} p(a, a') &= p\{J(a, b), K(a', b')\}, \leq h \max\{p(a, J(a, b)), p(a', K(a', b'))\} \\ &= h \max\{p(a, a), p(a', a')\} = 0. \end{aligned}$$

Thus, $a = a'$, similarly we can show that $b = b'$. This implies $(a, b) \equiv (a', b')$. □

Hence, (a, b) is unique common coupled fixed point of J and K.

Corollary 2.2. *Let (X, q) be a complete quasi metric space equipped with an mw distance mapping p and J:$X \times X \to X$ be continuous function such that:*

(i) $p\{J(a, b), J(c, d)\} \leq h \max\{p(a, J(a, b)), p(c, J(c, d))\}$, *where* $h \in [0, 1)$. *Then J has unique common coupled fixed point.*

Proof. Consider $K = I$ (Identity map), we get the result. The following example validate our result. □

Example 2.3. Let $X = (0, 3] p(a, b) = \frac{a+2b}{8}, q(a, b) = \frac{a+3b-2}{4}, J(a, b) = 4a + 3b - 15, K(a, b) = 3a + 2b - 10$. Here, q is quasi metric space,p is modified-ω distance function.

L.H.S. $= p\{J(a, b), K(c, d)\} = p(4a + 3b - 15, 3c + 2d - 10) = \frac{1}{8}(4a + 3b + 6c + 4d - 35)$

R.H.S. $=h\max\{p(a,J(a,b)),p(c,K(c,d))\}=h,\max\{p(a,4a+3b-15),p(c,3c+2d-10)\}=h\max\{\frac{9a+6b-30}{8},\frac{7c+4d-20}{8}\}=\frac{h}{8}\max\{9a+6b-30,7c+4d-20\}$. Condition (i) and (ii) of theorem are satisfied. Also, $J(2,3)=2=K(2,3),J(3,2)=3=K(3,2)$. Thus, $(2,3)$ is coupled common fixed point of J and K.

Conclusion. A unique common coupled fixed point theorem proved for quasi metric space.

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