ω^*-topology

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Abstract. For any space $(X,\, \tau)$, we introduced the concept of ω^*-open sets, then we found and discussed a new topology on X, namely $\tau_{\omega}*$ which is strictly closed between τ_θ (θ-topology) and τ_ω (ω-topology) on X and it is independent with the original topology τ. Furthermore, the relationship of ω^*-open sets with some other types of sets are given.
Keywords: topology, ω-topology, θ-topology, ω^*-topology and ω^*-open sets.

1. Introduction

In the twentieth century until now, many generalizations of open sets were introduced, such as in (Stone [18], Levien [12], Njastad [15], Veličo [19], Mashhour et. al. [14], Abd El-Monsef [1], Hdeib [11], Andrijevič [5], Park et. al. [16], Al-Hawary et. al. [3], Ekici et. al. [10], Darwesh [6], Darwesh et.al. [9], Darwesh [7], Darwesh [8], Al Ghour et. al. [2]) introduced the notion of (regular open, semi-open, α-open, θ-open and δ-open, preopen, β-open, ω-open, b-open, δ-semiopen, ω^o-open, ω_θ-open, δsc-open, $k\omega$-open, ω_δ-open, ω_p-open, θ_ω-open) sets in topological spaces. Throughout the present paper, R (resp., Q, Ir and N) deonte the set of all real (resp., rational, irrational and natural) numbers and (X, τ) denotes a topological space (briefly, a space) on which no separation axioms are assumed unless otherwise explicitly stated. For any space (X, τ), the family of all the above types of sets denote by $(RO(X), SO(X)$, τ_α, τ_θ, τ_δ, $PO(X)$, $\beta O(X)$, τ_ω, $bO(X)$, $\delta SO(X)$, τ_{ω^o}, τ_{ω_θ}, $\delta SCO(X)$, $k\omega O(X)$, τ_{ω_δ}, $\omega_p O(X)$, $\tau_{\theta_\omega})$, respectively. Stone [18] showed $RO(X)$ forms a base of some topology on X, named semi-regularization topology on X and its denoted by

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τ_s. Each of the families (τ_α, τ_θ, $\tau_\delta = \tau_s$, τ_ω, τ_{ω^o}, τ_{ω_θ}, τ_{ω_δ} and τ_{θ_ω}) forms a topology on X. While, each of $SO(X)$, $PO(X)$, $\beta O(X)$, $bO(X)$, $\delta SO(X)$, $\delta SCO(X)$, $\omega_p O(X)$ forms a supra topology on X, but $RO(X)$ and $k\omega O(X)$ form neither a topology nor a supra topology on X.

In the present paper, we construct a new topology τ_{ω^*} on a space (X, τ), by using a new notion of ω^*-open sets. Then, we show that τ_{ω^*} is strictly finer than τ_θ and strictly coarser than τ_ω. Also, we obtain some basic properties of the space (X, τ_{ω^*}). For any subset A of a space X, we denote by ClA (resp., $Cl_\omega A$, $Cl_\delta A$, $Cl_\theta A$, $IntA$, $Int_\delta A$ and $Int_\theta A$) the closure (resp., ω-closure, δ-closure, θ-closure, interior, δ-interior and θ-interior) of A.

Definition 1.1. *A subset A of a space (X, τ) is said to be regular open [18] (resp., semi-open [12], preopen [14], α-open [15], β-open [1], b-open [5], δ-semiopen [16], ω_p-open [8]) if $A = IntClA$ (resp., $A \subseteq ClIntA$, $A \subseteq IntClA$, $A \subseteq IntClIntA$, $A \subseteq ClIntClA$, $A \subseteq IntClA \bigcup ClIntA$, $A \subseteq Clint_\delta A$, $A \subseteq IntCl_\omega A$).*

Definition 1.2. *A subset A of a space (X, τ) is said to be θ-open [19] (resp., δ-open [19], θ_ω-open), if for each $x \in A$, there exists an open set G such that $x \in G \subseteq ClG \subseteq A$ (resp., $x \in G \subseteq IntClG \subseteq A$, $x \in G \subseteq Cl_\omega G \subseteq A$).*

Theorem 1.1 ([13])**.** *A space (X, τ) is regular if and only if $\tau_\theta = \tau$.*

Definition 1.3 ([9])**.** *A subset A of a space (X, τ) is said to be $k\omega$-open, if its both open and ω-compact subset of (X, τ).*

Definition 1.4 ([6])**.** *A δ-semiopen subset A of a space (X, τ) is said to be δsc-open if, for each point $x \in A$ there is a closed set F in X such that $x \in F \subseteq A$.*

Definition 1.5. *A subset A of a space (X, τ) is said to be ω-open [11] (resp., ω^o-open [3], ω_θ-open [10], and ω_δ-open [7]), if for each $x \in A$, there exists an open set G containing x such that $G \setminus A$ (resp., $G \setminus IntA$, $G \setminus Int_\theta A$, $G \setminus Int_\delta A$) is a countable subset of X.*

Definition 1.6 ([4])**.** *A space (X, τ) is said to be:*

1. *locally countable, if for each point $x \in X$ there exists a countable open subset G of X containing x,*

2. *anti-locally countable, if each nonempty open subset of X is uncountable.*

Theorem 1.2 ([11, 4])**.** *A space (X, τ) is lindelof if and only if (X, τ_ω) is lindelof.*

2. ω^*-open sets and ω^*-topology

We begin this section with the definition of ω^*-open sets, and then we use it to define a new topology on the given space. Also, we give some of its basic properties.

Definition 2.1. *A subset A of a space (X,τ) is said to be an ω^*-open set, if for each $x \in A$, there exists an open set G such that $x \in G$ and $ClG \setminus A$ is a countable set. Also, A is said to be ω^*-closed, if $X \setminus A$ is ω^*-open.*

Theorem 2.1. *A subset A of a space (X,τ) is ω^*-open if and only if for each $x \in A$, there exist an open set G_x containing x and a countable set C_x such that $ClG_x \setminus C_x \subseteq A$.*

Proof. Let A be an ω^*-open subset of X and $x \in A$. There exists an open set G_x such that $x \in G$ and $ClG_x \setminus A$ is a countable set. Hence, the set $C_x = ClG_x \setminus A$ is the required countable set.

Conversely, suppose $x \in A$, so by our hypothesis, there exist an open set G_x containing x and a countable set C_x such that $ClG_x \setminus C_x \subseteq A$. This implies that, $ClG_x \setminus A \subseteq C_x$. Thus, A is an ω^*-open subset of X. □

Theorem 2.2. *If A is an ω^*-closed subset of (X,τ), then $A \subseteq IntF \cup C$ for some closed set F and a countable set C.*

Proof. If $A = X$, we put $F = A$ and $C = \emptyset$, to get $A \subseteq IntF \cup C$. Otherwise, we choose a point $x \in X$ which is not in A, and since $X \setminus A$ is ω^*-open, then by Theorem 2.1, there is an open set G which contains x and a countable set C such that $ClG \setminus C \subseteq X \setminus A$. Hence, $F = X \setminus G$ and C are the required sets. □

Theorem 2.3. *A subset A of a space (X,τ) is ω^*-closed if and only if $A = X$ or for each point x not in A, there is a closed set F not containing x and a countable set C such that $A \subseteq IntF \cup C$.*

Proof. Let A be an ω^*-closed subset of X. Then, either $A = X$ or A is a proper subset of X If $A = X$, then there is nothing to prove, otherwise A is a proper ω^*-closed subset of X, then by Theorem 2.2 there exists a closed set F not containing x and a countable set C such that $A \subseteq IntF \cup C$.

Conversely, if $A = X$, then its ω^*-closed. Otherewise, suppose that for each $x \in X \setminus A$, there is a closed set F not contain x and a countable set C such that $A \subseteq IntF \cup C$. The set $G = X \setminus F$ is an open subset of X which contains x and $ClG \setminus C = X \setminus (IntF \cup C) \subseteq X \setminus A$. By Theorem 2.1, we obtaining that $X \setminus A$ is ω^*-open, whence A is ω^*-closed. □

Theorem 2.4. *The intersection (union) of two ω^*-open (ω^*-closed) sets is ω^*-open (ω^*-closed).*

Proof. Let A and B be two ω^*-open sets. If $A \cap B = \emptyset$, there is nothing to prove. Otherwise, for $x \in A \bigcap B$, there are two open sets G and U contining x such that $ClG \setminus A$ and $ClU \setminus B$ are countable sets. Since, $Cl(G \cap U) \setminus (A \cap B) \subseteq (ClG \setminus A) \cup (ClU \setminus B)$, so $A \cap B$ is ω^*-open. □

Theorem 2.5. *The union (intersection) of any family of ω^*-open sets is ω^*-open (ω^*-closed).*

Proof. Let $\{A_\lambda; \lambda \in \Lambda\}$ be any family of ω^*-open sets and $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$.Then, there is $\lambda_0 \in \Lambda$ and an open set G such that $x \in G \cap A_{\lambda_0}$ and $ClG \setminus A_{\lambda_0}$ is a countable set. Since, $ClG \setminus (\bigcup_{\lambda\in\Lambda} A_\lambda) \subseteq ClG \setminus A_{\lambda_0}$. Hence, $\bigcup_{\lambda\in\Lambda} A_\lambda$ is ω^*-open. □

Theorem 2.6. *For any space (X, τ). The family τ_{ω^*} of all ω^*-open subsets of X form a topology on X.*

Proof. Since $\emptyset$ has no any point, so $\emptyset \in \tau_{\omega^*}$. Since $ClX \setminus X = \emptyset$, so $X \in \tau_{\omega^*}$ and hence, the result is followed by using Theorem 2.4 and Throrem2.5. □

we call the toplogy of Theorem 2.6 is the ω^*-topology.

Definition 2.2. *A point $x \in X$ is said to be a $*$-condensation point of a subst A of a space X, if $ClG \bigcap A$ is an uncountable set. The set of all $*$-condensation points of a set A is denoted by $cond^*(A)$.*

Theorem 2.7. *A subset A of a space X is ω^*-closed if and only if $cond^*(A) \subseteq A$.*

Proof. Let A be an ω^*-closed subset of X and $x \in cond^*(A)$. On contrary, we suppose that $x \notin A$. there exists an open set G containing x such that $ClG \setminus (X \setminus A)$ is countable. This implies that, $ClG \bigcap A$ is countable. So, $x \notin cond^*(A)$ which is a contradiction. Hence, $cond^*(A) \subseteq A$.

Conversely, suppose that $cond^*(A) \subseteq A$ and $x \in X \setminus A$. There exists an open set G containing x such that $ClG \bigcap A$ is countable. Hence, $ClG \setminus (X \setminus A)$ is countable, so $X \setminus A$ is ω^*-open. Therefore, A is ω^*-closed. □

Corollary 2.1. *Each countable subset of any space is ω^*-closed.*

Proof. If A is countable, then $cond^*(A) = \emptyset$. So, by Theorem 2.7 A is ω^*-closed. □

Corollary 2.2. *If (X, τ) is any space, then the co countable topology T_{coc} on X is coarser than τ_{ω^*}.*

Proof. If $A \in T_{coc}$, then $X \setminus A$ is a countable subset of X. By Corollary 2.1 $X \setminus A$ is ω^*-closed. Hence, $A \in \tau_{\omega^*}$. □

Corollary 2.3. *For any space (X, τ), the space (X, τ_{ω^*}) is a T_1-space.*

Proof. It follows from Corollary 2.2. □

Corollary 2.4. *If (X, τ) is a countable space, then (X, τ_{ω^*}) is a discrete space.*

Proof. It follows from Corollary 2.2. □

Recalling that a space (X, τ) is said be a hyperconnected space [17], if each nonempty open subsets of X is dense in X.

Theorem 2.8. *If (X,τ) is a hyperconnected space, then τ_{ω^*} is the co countable topology T_{coc} on X.*

Proof. Let $A \in T_{\omega^*}$. If $A = \emptyset$, then $A \in \tau_{coc}$. Otherwise, we choose a fixed point x in A, and an open sets G containing x such that $ClG \setminus A = C$, for some countable set C. Since X is hyperconnected, so $ClG = R$ and $A = R \setminus C$. Hence, $A \in T_{coc}$. Thus, $\tau_{\omega^*} \subseteq T_{coc}$. The incusion $T_{coc} \subseteq \tau_{\omega^*}$ is followed from Corollary 2.2. Therefore, $\tau_{\omega^*} = T_{coc}$. □

3. Some properties and relationships of τ_{ω^*}

In this section, we give more properties of the topology τ_{ω^*} and investigate its relationships with other classes of sets.

Theorem 3.1. *If τ and $\Im$ are two topologies on X such that $\tau \subseteq \Im$, then $\tau_{\omega^*} \subseteq \Im_{\omega^*}$.*

Proof. Let $A \in \tau_{\omega^*}$. If $A = \emptyset$, then $A \in \Im_{\omega^*}$. Let $A \neq \emptyset$. Then by Theorem 2.1, for each $x \in A$, there is $G \in \tau$ (hence $G \in \Im$) and a countable subset C of X such that $Cl_{\Im}G \setminus C \subseteq Cl_{\tau}G \setminus C \subseteq A$. Thus, by Theorem 2.1 we obtaining that $A \in \Im_{\omega^*}$. □

For a space (X,τ), we denote by $(\tau_\theta)_\omega$ the ω-topology of (X,τ_θ), so we have:

Theorem 3.2. *$\tau_\theta \subseteq (\tau_\theta)_\omega \subseteq \tau_{\omega^*} \subseteq \tau_\omega$, for any space (X,τ).*

Proof. By [4, Theorem 2.5], we have $\tau_\theta \subseteq (\tau_\theta)_\omega$. Let A be any member of $(\tau_\theta)_\omega$. If $A = \emptyset$, then $A \in \tau_{\omega^*}$. Otherwise, for an arbirary point x in A, there exists $V \in \tau_\theta$ such that $x \in V$ and $V \setminus A$ is a countable set. Since $V \in \tau_\theta$, there is $G \in \tau$ in which $x \in G \subseteq ClG \subseteq V$. Therefore, $ClG \setminus A$ is countable. Hence, $A \in \tau_{\omega^*}$,so $(\tau_\theta)_\omega \subseteq \tau_{\omega^*}$.

Now, since $G \setminus A \subseteq ClG \setminus A$ for any open subset G of X, this implies that for any $A \in \tau_{\omega^*}$, we have $A \in \tau_\omega$. Hence, $\tau_{\omega^*} \subseteq \tau_\omega$. □

The following examples show that the inclusions in the above result cannot replace with equality in general:

Example 3.1. Consider the set $A = \{1\}$ and $B = R \setminus N$ in the one point excluded topological space(R,τ) [17, Example 13, P. 47], where $\tau = \{G \subseteq R; 0 \notin G\} \cup \{R\}$. Then, $A = \{1\} \in \tau_{\omega^*}$ but $A \notin (\tau_\theta)_\omega$. However, the set $B \in (\tau_\theta)_\omega$ but $B \notin \tau_\theta$.

Example 3.2. Consider the set $A = \{0,1\}$ in the one point included topological space (R,τ) [17, Example 8, P. 44], where $\tau = \{G \subseteq R; 0 \in G\} \cup \{\emptyset\}$. Then, $A = \{0,1\} \in \tau$ (and hence, $A \in \tau_\omega$) but $A \notin \tau_{\omega^*}$.

Proposition 3.1. *If (X,τ_θ) is a locally countable space, then (X,τ_{ω^*}) is a discrete space.*

Proof. Let(X, τ_θ) be a locally countable space. By [4, Theorem 3.2], $(X, (\tau_\theta)_\omega)$ is a discrete space. Then, by Theorem 3.2 (X, τ_{ω^*}) is a discrete space. □

Remark 3.1. .

1. Since the space of Example 3.2 is locally counatble but (R, τ_{ω^*}) is neither a T_2-space nor a discrete space, this means that; the countablity of X in Corollary 2.4 unable to replace with locally countable, and the T_1 in Corollary 2.3 can not replace with any higher separation axioms.

2. If $\tau = \{G \subseteq R; 0 \in G\} \cup \{\emptyset\}$ and $\Im = \{G \subseteq R; 1 \in G\} \cup \{\emptyset\}$ are two different one point included topologies on R. Then, neither $\tau \subseteq \Im$ nor $\Im \subseteq \tau$, but $\tau_{\omega^*} = T_{coc} = \Im_{\omega^*}$. This means that, the converse of Theorem 3.1 is not true in general.

3. The set $A = \{0, 1\}$ in Example 3.2 is open, α-open, preopen, semi-open, b-open, β-open, ω_p-open and $k\omega$-open but not ω^*-open. While, the set $B = \{2\}$ in the sierpinski space (X, τ) [17, Example 11, P. 44], where $X = \{1, 2\}$ and $\tau = \{\emptyset, \{1\}, X\}$ is ω^*-open but it is neither open, α-open, preopen, semi-open, b-open, β-open, ω_p-open nor $k\omega$-open. This means that:

   (a) τ_{ω^*} is independent with each of the topologies τ and τ_α.

   (b) τ_{ω^*} is independent with each of the classes $PO(X)$, $SO(X)$, $bO(X)$, $\beta O(X)$, $\omega_p O(X)$ and $k\omega O(X)$.

4. The set $A = Q$ in the space (R, τ), where $\tau = \{\emptyset, Q, R\}$ is ω_θ-open, ω_δ-open, ω^o-open and θ_ω-open but not ω^*-open. However, the set $B = R\backslash\{0\}$ in the indiscrete space (R, T_{ind}) is ω^*-open but it is neither ω_θ-open, ω_δ-open, ω^o-open nor θ_ω-open. This means that; τ_{ω^*} is independent with each of the topologies τ_{ω_θ}, τ_{ω_δ}, τ_{ω^o} and τ_{θ_ω}.

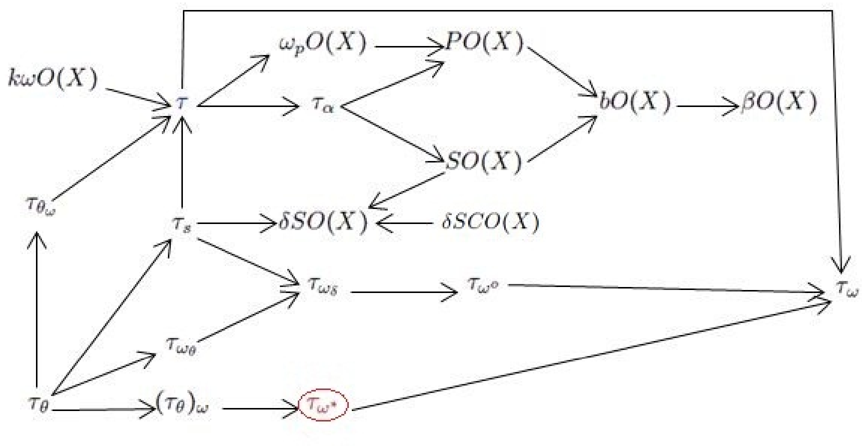
5. Since the set $A = R \setminus \{0\}$ in the indiscrete space (R, T_{ind}) is ω^*-open but it is neither δ-open, δ-semiopen nor δsc-open, while the set $A = Q$ in the space (R, τ), where $\tau = \{\emptyset, Q, Ir \cap (0, 1), Q \cup (0, 1), R\}$ is δ-open and δ-semiopen but not ω^*-open. However, the set $A = Q$ in the space (R, σ), where $\sigma = \tau \cap T_{coc}$, is δsc-open but not ω^*-open. Then, we obtain:

   (a) τ_{ω^*} is independent with the semi-regularization topology τ_s.

   (b) τ_{ω^*} is independet with each of the classes $\delta SO(X)$ and $\delta SCO(X)$.

6. Thus, we have the following diagram:

Theorem 3.3. *If (X, τ) is a regular space, then $\tau \subseteq \tau_{\omega^*}$ and $\tau_\omega = \tau_{\omega^*}$.*



Proof. It follows from Theorem 1.1 and Theorem 3.2 that $\tau \subseteq \tau_{\omega^*}$ and $\tau_{\omega^*} \subseteq \tau_\omega$. So, it remains only to show $\tau_\omega \subseteq \tau_{\omega^*}$. Suppose that $A \in \tau_\omega$, there is an open set G such that $x \in G$ and $G \setminus A$ is countable. Since X is regular space and $x \in G$, so there exists an open set U such that $x \in U \subseteq ClU \subseteq G$. Hence, $ClU \setminus A$ is countable, so $A \in \tau_{\omega^*}$. Thus, $\tau_\omega = \tau_{\omega^*}$. □

Since the sierpinski space (X, τ) in part 3 of Remark 3.1 (also, see [17, Example 8, P. 44]) is not regular but $\tau \subseteq \tau_{\omega^*} = \tau_\omega$. This means that, the converse of Theorem 3.3 is not true in general.

Theorem 3.4. *$(\tau_{\omega*})_{\omega*} \subseteq \tau_{\omega*}$, for any space (X, τ).*

Proof. Let $x \in A \in (\tau_{\omega*})_{\omega*}$. By Theorem 2.1, there is $U_x \in \tau_{\omega*}$ and a countable set K_x such that $Cl_{\tau_{\omega*}} U_x \setminus K_x \subseteq A$. Again, by Theorem 2.1 there is $G_x \in \tau$ and a countable set L_x in which $Cl_\tau G_x \setminus L_x \subseteq U_x$. Since $K_x \cup L_x$ is countable and $Cl_\tau G_x \setminus (K_x \cup L_x) \subseteq U_x \setminus K_x \subseteq Cl_{\tau_{\omega*}} U_x \setminus K_x \subseteq A$. Hence, by Theorem 2.1 $A \in \tau_{\omega*}$. □

Corollary 3.1. *$(\tau_{\omega^*})_{\omega^*} = \tau_{\omega^*}$, if (X, τ) is a regular space.*

Proof. It followes from Theorem 3.1, Theorem 3.3 and Theorem 3.4 □

Since the sierpinski space (X, τ) in part 3 of Remark 3.1 is not regular but $(\tau_{\omega^*})_{\omega^*} = \tau_{\omega^*}$. This means that, the converse of Corollary 3.1 is not true in general.

Theorem 3.5. *If Y is a subset of a space (X, τ), then $(\tau_{\omega^*})_Y \subseteq (\tau_Y)_{\omega^*}$.*

Proof. Let $A \in (\tau_{\omega^*})_Y$. Then, there is $G \in \tau_{\omega^*}$ such that $A = G \cap Y$. For $x \in A$, there is $V \in \tau$ such that $x \in V$ and $ClV \setminus G$ is countable. Since $U = V \cap Y \in \tau_Y$, $x \in U$ and $Cl_Y U \subseteq ClV$, then $Cl_Y U \setminus A = Cl_Y U \setminus (G \cap Y) = Cl_Y U \setminus G \subseteq ClV \setminus G$ which is countable. Hence, $A \in (\tau_Y)_{\omega^*}$. Thsu, $(\tau_{\omega^*})_Y \subseteq (\tau_Y)_{\omega^*}$. □

The following examples show that the converse inclusion in Theorem 3.5 is not true even Y is open (or Y is closed).

Example 3.3. Consider the open subset $Y = Ir$ in the space (R, τ), where $\tau = \{\emptyset, R\} \cup \{A; A \subseteq Ir\}$. Then τ_Y is the discrete topology on Y, therefore $(\tau_Y)_{\omega^*}$ is also the discrete topology on Y. But, Since (R, τ) is hyperconnected space,so by Theorem 2.8, we get $\tau_{\omega^*} = T_{coc}$, so that $(\tau_{\omega^*})_Y = T_{coc}$. Hence, $(\tau_Y)_{\omega^*} \nsubseteq (\tau_{\omega^*})_Y$.

Example 3.4. Consider the closed subset $Y = R \setminus \{0\}$ in the space of Example 3.2. Then, τ_Y is the discrete topology on Y, and so $(\tau_Y)_{\omega^*}$ is the discrete topology on Y.Now, Since (R, τ) is hyperconnected space, then Theorem 2.8, we get $\tau_{\omega^*} = T_{coc}$. Thus, $(\tau_{\omega^*})_Y = T_{coc}$. This means that $(\tau_Y)_{\omega^*} \nsubseteq (\tau_{\omega^*})_Y$.

Theorem 3.6. *If Y is a clopen subset of (X, τ), then $(\tau_Y)_{\omega^*} \subseteq \tau_{\omega^*}$ and $(\tau_Y)_{\omega^*} = (\tau_{\omega^*})_Y$.*

Proof. Let $A \in (\tau_Y)_{\omega^*}$. Then for each $x \in A$, there is $G \in \tau_Y$ such that $x \in G$ and $Cl_Y G \setminus A$ is countable. Since Y is clopen, so $G \in \tau$ and $Cl_Y G = ClG$. Hence, $ClG \setminus A$ is countable. This means that $A \in \tau_{\omega^*}$. Thus, $(\tau_Y)_{\omega^*} \subseteq \tau_{\omega^*}$. Since, $A = A \cap Y \in (\tau_{\omega^*})_Y$, then $(\tau_Y)_{\omega^*} \subseteq (\tau_{\omega^*})_Y$, and from by Theorem 3.5 $(\tau_{\omega^*})_Y \subseteq (\tau_Y)_{\omega^*}$, then $(\tau_Y)_{\omega^*} = (\tau_{\omega^*})_Y$. □

We say that a space (X, τ) is ω^*-lindelof if (X, τ_{ω^*}) is a lindelof space.

Theorem 3.7. *If (X, τ) is lindelof, then (X, τ_{ω^*}) is lindelof.*

Proof. Let (X, τ) be a lindelof sapce. By Theorem 1.2, we obtaining that (X, τ_ω) is lindelof. Since $\tau_{\omega^*} \subseteq \tau_\omega$, then (X, τ_{ω^*}) is lindelof. □

The following example shows that the converse of above result is not true in general:

Example 3.5. Consider the left ray topological sapce (R, τ), $\tau = \{(-\infty, r); r \in R\} \cup \{\emptyset, R\}$. It is clear that, this space a non-lindelof hyperconnected space, then by Theorem 2.8, we get $\tau_{\omega^*} = T_{coc}$, whence (R, τ_{ω^*}) is lindelof.

Corollary 3.2. *If (X, τ_{ω^*}) is lindelof, then (X, τ_θ) is lindelof.*

Proof. It followes from Theorem 3.4. □

Theorem 3.8. *If (X, τ) is almost lindelof space, then (X, τ_{ω^*}) is lindelof.*

Proof. Let $\{V_\lambda; \lambda \in \Lambda\}$ be an ω^*-open cover of X. Then, for each $x \in X$, there is $\lambda_x \in \Lambda$ and an open set G_{λ_x} containing x such that $ClG_{\lambda_x} \setminus V_{\lambda_x} = C_{\lambda_x}$ for some countable set C_{λ_x}. Since $\{G_{\lambda_x}; x \in X\}$ is an open cover of X, so there exists a countable subset X_0 of X such that $X = \bigcup_{x \in X_0} ClG_{\lambda_x} = (\bigcup_{x \in X_0} ClG_{\lambda_x} \setminus V_{\lambda_x}) \cup \bigcup_{x \in X_0} V_{\lambda_x} = \bigcup_{x \in X_0} C_{\lambda_x} \cup \bigcup_{x \in X_0} V_{\lambda_x}$. Since $\bigcup_{x \in X_0} C_{\lambda_x}$ is countable, then there exists a countable subset Λ_0 of Λ such that $\bigcup_{x \in X_0} C_{\lambda_x} \subseteq \bigcup_{\lambda \in \Lambda_0} V_\lambda$. Thus, $X = \bigcup_{\lambda \in \Lambda_0} V_\lambda \cap \bigcup_{x \in X_0} V_{\lambda_x}$. Hence, (X, τ_{ω^*}) is a lindelof space. □

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