Weakly ω-continuous functions in bitopological spaces

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Abstract. In this paper, as a generalization of u-ω-continuous functions, we introduce the notion of weakly ω-continuous functions in bitopological spaces and obtain several characterizations and some of its properties.
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1. Introduction

The concept of bitopological spaces was first introduced by Kelly [13]. After the introduction of the definition of a bitopological space by Kelly, a large number of

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topologists have turned their attention to the generalization of different concepts used in topological space. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [11]. Several characterizations and properties of ω-closed sets were provided in [4, 2, 3, 11, 12, 16]. In this paper, we introduce and study (i, j)-weakly ω-continuous functions on bitopological space and obtain several characterizations and its relation with another (i, j)-ω-continuous functions-

2. Preliminaries

Throughout this paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) always mean bitopological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ), $i\,\mathrm{Cl}(A)$ and $i\,\mathrm{Int}(A)$ denote the closure of A with respect to τ_i and the interior of A with respect to τ_i, respectively. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [11] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ. The intersection of all ω-closed sets containing A is called the ω-closure [11] of A and is denoted by $\omega\,\mathrm{Cl}(A)$. The family of all ω-open sets of X is denoted by $\omega(\tau)$.

Definition 2.1. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

1. *(i, j)-preopen [14] if $A \subset i\,\mathrm{Int}(j\,\mathrm{Cl}(A))$;*
2. *(i, j)-regular open [8] if $A \subset j\,\mathrm{Cl}(i\,\mathrm{Int}(A))$;*

On each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i, j)-preopen (resp. (i, j)-regular open) set is called an (i, j)-preclosed (resp. (i, j)-regular closed) set. The family of all (i, j)-preopen (resp. (i, j)-regular open, (i, j)-preclosed, (i, j)-regular closed) sets of (X, τ_1, τ_2) is denote by (i, j)-$PO(X)$ (resp. (i, j)-$RO(X)$, (i, j)-$PC(X)$, (i, j)-$RC(X)$).

Definition 2.2 ([1])**.** *Let (X, τ_1, τ_2) be a bitopological space and let $A \subset X$. Then*

1. *A is said to be u-ω-open in (X, τ_1, τ_2) if $A \in \omega(\tau_1) \cup \omega(\tau_2)$,*

2. *A is said to be u-ω-closed in* (X, τ_1, τ_2) *if* $X - A$ *is u-ω-open in* (X, τ_1, τ_2).

The family of all u-ω-open sets in (X, τ_1, τ_2) *is denoted by* $\omega(\tau_1, \tau_2)$, *and the family of all u-ω-closed sets in* (X, τ_1, τ_2) *is denoted by* $\omega C(\tau_1, \tau_2)$.

Definition 2.3. 1. *The u-ω-closure of A in* (X, τ_1, τ_2) *is denoted by* (τ_1, τ_2)-$\omega \operatorname{Cl}(A)$ *and defined as follows:* (τ_1, τ_2)-$\omega \operatorname{Cl}(A) = \omega \operatorname{Cl}_{\tau_1}(A) \cap \omega \operatorname{Cl}_{\tau_2}(A)$.

2. *The u-ω-interior of A in* (X, τ_1, τ_2) *is denoted by* (τ_1, τ_2)-$\omega \operatorname{Int}(A)$ *and defined as follows:* (τ_1, τ_2)-$\omega \operatorname{Int}(A) = \omega \operatorname{Int}_{\tau_1}(A) \cup \omega \operatorname{Int}_{\tau_2}(A)$.

Lemma 2.4. *Let* (X, τ_1, τ_2) *be a bitopological space and A a subset of X. Then*

1. (τ_1, τ_2)-$\omega \operatorname{Int}(A)$ *is u-ω-open;*
2. (τ_1, τ_2)-$\omega \operatorname{Cl}(A)$ *is u-ω-closed;*
3. *A is u-ω-open if and only if* $A = (\tau_1, \tau_2)$-$\omega \operatorname{Int}(A)$;
4. *A is u-ω-closed if and only if* $A = (\tau_1, \tau_2)$-$\omega \operatorname{Cl}(A)$;
5. (τ_1, τ_2)-$\omega \operatorname{Int}(X \backslash A) = X \backslash (\tau_1, \tau_2)$-$\omega \operatorname{Cl}(A)$;
6. (τ_1, τ_2)-$\omega \operatorname{Cl}(X \backslash A) = X \backslash (\tau_1, \tau_2)$-$\omega \operatorname{Int}(A)$.

Lemma 2.5. *Let* (X, τ_1, τ_2) *be a bitopological space and* $A \subset X$. *A point* $x \in (\tau_1, \tau_2)$-$\omega \operatorname{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every u-ω-open set U of* (X, τ_1, τ_2) *containing x.*

Definition 2.6 ([9])**.** *A subset A of X is said to be* (i, j)-θ-*closed if* $A = (i, j)$-$\operatorname{Cl}_\theta(A)$. *A subset A of X is said to be* (i, j)-θ-*open if* $X \backslash A$ *is* (i, j)-θ-*closed. The* (i, j)-θ-*interior of A, denoted by* (i, j)-$\operatorname{Int}_\theta(A)$, *is defined as the union of all* (i, j)-θ-*open sets contained in A. Hence* $x \in (i, j)$-$\operatorname{Int}_\theta(A)$ *if and only if there exists a* τ_i-*open set U containing x such that* $x \in U \subset j \operatorname{Cl}(U) \subset A$.

Lemma 2.7 ([9])**.** *Let* (X, τ_1, τ_2) *be a bitopological space and A a subset of X. Then*

1. (i, j)-$\operatorname{Int}_\theta(X \backslash A) = X \backslash (i, j)$-$\operatorname{Cl}_\theta(A)$;
2. (i, j)-$\operatorname{Cl}_\theta(X \backslash A) = X \backslash (i, j)$-$\operatorname{Int}_\theta(A)$.

Lemma 2.8 ([9])**.** *Let* (X, τ_1, τ_2) *be a bitopological space. If U is a* τ_j-*open set of X, then* (i, j)-$\operatorname{Cl}_\theta(U) = i \operatorname{Cl}(U)$.

Definition 2.9 ([1])**.** *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be u-ω-continuous if for each* $x \in X$ *and each* $V \in \sigma_1 \cup \sigma_2$ *containing* $f(x)$, *there exists an u-ω-open set U containing x such that* $f(U) \subset V$.

3. Properties of (i,j)-weakly ω-continuous functions

Definition 3.1. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-weakly ω-continuous if for each $x\in X$ and each σ_i-open set V of Y containing $f(x)$, there exists an u-ω-open set U containing x such that $f(U)\subset j\operatorname{Cl}(V)$.*

Proposition 3.2. *Every u-ω-continuous function is (i,j)-weakly ω-continuous.*

Proof. Straightforward. □

The following example shows that the converse of Proposition 3.2 is not true in general.

Example 3.3. Take $X=\mathbb{R}$ and two topologies $\tau_1=\tau_2=\{\emptyset,\mathbb{R},\mathbb{R}\backslash\mathbb{Q}\}$ and $Y=\{a,b,c\}$ with topologies $\sigma_1=\sigma_2=\{\emptyset,Y,\{a\},\{a,b\}\}$. Define $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ as

$$f(x)=\begin{cases}a, & \text{if } x\in\mathbb{Q},\\ b, & \text{if } x\notin\mathbb{Q}\end{cases}$$

It is easy to see that f is (i,j)-weakly ω-continuous but does not is *u-ω*-continuous function.

Theorem 3.4. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following properties are equivalent;*

1. *f is (i,j)-weakly ω-continuous;*
2. *(τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(i\operatorname{Cl}(B))))\subset f^{-1}(i\operatorname{Cl}(B))$ for every subset B of Y;*
3. *(τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(F)))\subset f^{-1}(F)$ for every (i,j)-regular closed set F of Y;*
4. *(τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(V))\subset f^{-1}(i\operatorname{Cl}(V))$ for every σ_j-open set V of Y;*
5. *$f^{-1}(V)\subset(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$ for every σ_i-open set V of Y.*

Proof. (1) ⇒ (2): Let B be any subset of Y and $x\in X\backslash f^{-1}(i\operatorname{Cl}(B))$. Then, $f(x)\in Y\backslash i\operatorname{Cl}(B)$ and there exists a σ_i-open set V of Y containing $f(x)$ such that $V\cap B=\emptyset$. Therefore, $V\cap j\operatorname{Int}(i\operatorname{Cl}(B))=\emptyset$ and hence $j\operatorname{Cl}(V)\cap j\operatorname{Int}(i\operatorname{Cl}(B))=\emptyset$. Therefore, there exists an u-ω-open set U containing x such that $f(U)\subset j\operatorname{Cl}(V)$. Hence $U\cap f^{-1}(j\operatorname{Int}(i\operatorname{Cl}(B)))=\emptyset$ and $x\in X\backslash(\tau_1,\tau_2)$-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(i\operatorname{Cl}(B))))$. Thus, $(\tau_1,\tau_2)-\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(i\operatorname{Cl}(B))))\subset f^{-1}(i\operatorname{Cl}(B))$.

(2) ⇒ (3): Let F be an (i,j)-regular closed set of Y.

Then, we have $F=i\operatorname{Cl}(j\operatorname{Int}(F))$ and (τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(F)))=(\tau_1,\tau_2)$-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(i\operatorname{Cl}(j\operatorname{Int}(F)))))\subset f^{-1}(i\operatorname{Cl}(j\operatorname{Int}(F)))=f^{-1}(F)$.

(3) ⇒ (4): Let V be a σ_j-open set of Y. Then $i\operatorname{Cl}(V)$ is (i,j)-regular closed. Then (τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(V))\subset(\tau_1,\tau_2)$-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(i\operatorname{Cl}(V))))\subset f^{-1}(i\operatorname{Cl}(V))$.

(4) $\Rightarrow$ (5): Let V be a σ_i-open set of Y. Then $Y\backslash j\operatorname{Cl}(V)$ is σ_j-open and we have (τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(Y\backslash j\operatorname{Cl}(V)))\subset f^{-1}(i\operatorname{Cl}(Y\backslash j\operatorname{Cl}(V)))$ and $X\backslash(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))\subset X\backslash f^{-1}(i\operatorname{Int}(j\operatorname{Cl}(V)))\subset X\backslash f^{-1}(V)$. Therefore, we obtain $f^{-1}(V)\subset(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$.

(5) $\Rightarrow$ (1): Let $x\in X$ and V be a σ_i-open set containing $f(x)$. We have $x\in f^{-1}(V)\subset(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$. Put $U=(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$. Then U is an u-ω-open set containing x and $f(U)\subset j\operatorname{Cl}(V)$. This shows that f is (i,j)-weakly ω-continuous. □

Theorem 3.5. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

1. *f is (i,j)-weakly ω-continuous;*
2. *$f((\tau_1,\tau_2)$-$\omega\operatorname{Cl}(A))\subset(i,j)$-$\operatorname{Cl}_\theta(f(A))$ for every subset A of X;*
3. *(τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(B))\subset(f^{-1}(i,j)$-$\operatorname{Cl}_\theta(B))$ for every subset B of Y;*
4. *(τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(i,j)$-$\operatorname{Cl}_\theta(B))))\subset f^{-1}((i,j)$-$\operatorname{Cl}_\theta(B))$ for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Assume that f is (i,j)-weakly ω-continuous. Let A be any subset of X, $x\in(\tau_1,\tau_2)$-$\omega\operatorname{Cl}(A)$ and V be a σ_i-open set of Y containing $f(x)$. Then, there exists an u-ω-open set U containing x such that $f(U)\subset j\operatorname{Cl}(V)$. Since $x\in(\tau_1,\tau_2)$-$\omega\operatorname{Cl}(A)$, $U\cap A\neq\emptyset$ and hence $\emptyset\neq f(U)\cap f(A)\subset j\operatorname{Cl}(V)\cap f(A)$. Therefore, we obtain $f(x)\in(i,j)$-$\operatorname{Cl}_\theta(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then $f((\tau_1,\tau_2)$-$\omega\operatorname{Cl}f^{-1}(B)))\subset(i,j)$-$\operatorname{Cl}_\theta(f(f^{-1}(B)))\subset(i,j)$-$\operatorname{Cl}_\theta(B)$ and hence (τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(B))\subset f^{-1}((i,j)$-$\operatorname{Cl}_\theta(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Since (i,j)-$\operatorname{Cl}_\theta(B)$ is σ_i-closed in Y, by Lemma 2.8, (τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}((i,j)$-$\operatorname{Cl}_\theta(B))))\subset f^{-1}((i,j)$-$\operatorname{Cl}_\theta(j\operatorname{Int}((i,j)$-$\operatorname{Cl}_\theta(B))))=f^{-1}(i\operatorname{Cl}(j\operatorname{Int}((i,j)$-$\operatorname{Cl}_\theta(B))))\subset f^{-1}(i\operatorname{Cl}(i,j)$-$\operatorname{Cl}_\theta(B)))=f^{-1}((i,j)$-$\operatorname{Cl}_\theta(B))$.

(4) $\Rightarrow$ (1): Let V be any σ_j-open set of Y. Then by Lemma 2.8, $V\subset j\operatorname{Int}(i\operatorname{Cl}(V))=j\operatorname{Int}((i,j)$-$\operatorname{Cl}_\theta(V))$ and we have (τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(V))\subset(\tau_1,\tau_2)$-$\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}((i,j)$-$\operatorname{Cl}_\theta(V))))\subset f^{-1}((i,j)$-$\operatorname{Cl}_\theta(V))=f^{-1}(i\operatorname{Cl}(V))$. Thus we obtain (τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(V))\subset f^{-1}(i\operatorname{Cl}(V))$. It follows from Theorem 3.4, that f is (i,j)-weakly ω-continuous. □

Theorem 3.6. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

1. *f is (i,j)-weakly ω-continuous;*
2. *(τ_1,τ_2)-$\omega\operatorname{Cl}(f^{-1}(V))\subset f^{-1}(i\operatorname{Cl}(V))$ for every (j,i)-preopen set V of Y;*
3. *$f^{-1}(V)\subset(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$ for every (i,j)-preopen set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any (j,i)-preopen set of Y. Suppose that $x \notin f^{-1}(i\,\mathrm{Cl}(V))$. Then there exists a σ_i-open set W containing $f(x)$ such that $W \cap V = \emptyset$. Hence we have $i\,\mathrm{Cl}(W \cap V) = \emptyset$. Since V is (j,i)-preopen, we have $V \cap j\,\mathrm{Cl}(W) \subset j\,\mathrm{Int}(i\,\mathrm{Cl}(V)) \cap j\,\mathrm{Cl}(W) \subset j\,\mathrm{Cl}(j\,\mathrm{Int}(i\,\mathrm{Cl}(V)) \cap W) \subset j\,\mathrm{Cl}(i\,\mathrm{Cl}(V)) \cap W) \subset j\,\mathrm{Cl}(i\,\mathrm{Cl}(V \cap W)) = \emptyset$. Since f is (i,j)-weakly ω-continuous and W is a σ_i-open set containing $f(x)$, there exists an u-ω-open set U of X containing x such that $f(U) \subset j\,\mathrm{Cl}(W)$. Then $f(U) \cap V = \emptyset$ and hence $U \cap f^{-1}(V) = \emptyset$. This shows that $x \notin (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(f^{-1}(V))$. Therefore, we obtain (τ_1,τ_2)-$\omega\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(i\,\mathrm{Cl}(V))$.

(2) $\Rightarrow$ (3): Let V be any (i,j)-preopen set of Y. By (2), we have $f^{-1}(V) \subset f^{-1}(i\,\mathrm{Int}(j\,\mathrm{Cl}(V))) = X \setminus f^{-1}(i\,\mathrm{Cl}(Y \setminus j\,\mathrm{Cl}(V))) \subset X \setminus (\tau_1,\tau_2) - \omega\,\mathrm{Cl}(f^{-1}(Y \setminus j\,\mathrm{Cl}(V))) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(j\,\mathrm{Cl}(V)))$.

(3) $\Rightarrow$ (1): Let V be any σ_i-open set of Y. Then V is (i,j)-preopen set in Y and $f^{-1}(V) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(j\,\mathrm{Cl}(V)))$. By Theorem 3.4, f is (i,j)-weakly ω-continuous. $\square$

Lemma 3.7. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is (i,j)-weakly ω-continuous and $g : (Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ is pairwise continuous, then the composition $g \circ f : (X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is (i,j)-weakly ω-continuous.*

Proof. Let $x \in X$ and W be an η_i-open set of Z containing $g(f(x))$. Then, $g^{-1}(W)$ is a σ_i-open set of Y containing $f(x)$ and there exists an u-ω-open set U of X containing x such that $f(U) \subset j\,\mathrm{Cl}(g^{-1}(W))$. Since g is pairwise continuous, we obtain $(g \circ f)(U) \subset g(j\,\mathrm{Cl}(g^{-1}(W))) \subset g(g^{-1}(j\,\mathrm{Cl}(W))) \subset j\,\mathrm{Cl}(W)$. $\square$

Definition 3.8. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-regular [13] if for each $x \in X$ and each τ_i-open set U containing x, there exists a τ_i-open set V such that $x \in V \subset j\,\mathrm{Cl}(V) \subset U$.*

Lemma 3.9 ([15]). *If a bitopological space (X,τ_1,τ_2) is (i,j)-regular, then (i,j)-$\mathrm{Cl}_\theta(F) = F$ for every τ_i-closed set F.*

Theorem 3.10. *Let (Y,σ_1,σ_2) be an (i,j)-regular bitopological space. For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

1. *f is u-ω-continuous;*
2. *$f^{-1}((i,j)\text{-}\mathrm{Cl}_\theta(B))$ is u-ω-closed in X for every subset B of Y;*
3. *f is (i,j)-weakly ω-continuous;*
4. *$f^{-1}(F)$ is u-ω-closed in X for every (i,j)-θ-closed set F of Y;*
5. *$f^{-1}(V)$ is u-ω-open in X for every (i,j)-θ-closed set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Since (i,j)-$\mathrm{Cl}_\theta(B)$ is σ_i-closed in Y, $f^{-1}((i,j)\text{-}\mathrm{Cl}_\theta(B))$ is u-ω-closed in X.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then we have (τ_1,τ_2)-$\omega\,\mathrm{Cl}(f^{-1}(B)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(f^{-1}((i,j)\text{-}\mathrm{Cl}_\theta(B))) = f^{-1}((i,j)\text{-}\mathrm{Cl}_\theta(B))$. By Theorem 3.5, f is (i,j)-weakly ω-continuous.

(3) $\Rightarrow$ (4): Let F be any (i,j)-θ-closed set of Y. Then by Theorem 3.5, (τ_1,τ_2)-$\omega\,\mathrm{Cl}(f^{-1}(F)) \subset f^{-1}((i,j)\text{-}\mathrm{Cl}_\theta(F)) = f^{-1}(F)$. Therefore, by Lemma 2.4, $f^{-1}(F)$ is u-ω-closed in X.

(4) $\Rightarrow$ (5): Let V be any (i,j)-θ-open set of Y. By (4) $f^{-1}(Y-V) = X\backslash f^{-1}(V)$ is u-ω-closed in X and hence $f^{-1}(V)$ is u-ω-open in X.

(5) $\Rightarrow$ (1): Since Y is (i,j)-regular, by Lemma 3.9, (i,j)-$\mathrm{Cl}_\theta(B) = B$ for every σ_i-closed set B of Y and hence every σ_i-open set is (i,j)-θ-open. Therefore, $f^{-1}(V)$ is u-ω-open for every σ_i-open set V of Y. Hence f is u-ω-continuous. □

Definition 3.11. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be weakly (i,j)-* quasicontinuous [15] if for every σ_i-open set V of Y, $f^{-1}(j\,\mathrm{Cl}(V)\backslash V)$ is biclosed in X.*

Theorem 3.12. *If a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)-weakly ω-continuous and weakly (i,j)-*quasicontinuous, then f is u-ω-continuous.*

Proof. Let $x\in X$ and V be any σ_i-open set of Y containing $f(x)$. Since f is (i,j)-weakly ω-continuous, there exists an u-ω-open set U of X containing x such that $f(U)\subset j\,\mathrm{Cl}(V)$. Hence $x\notin f^{-1}(j\,\mathrm{Cl}(V)\backslash V)$. Then $x\in U\backslash f^{-1}(j\,\mathrm{Cl}(V)\backslash V) = U\cap(X\backslash f^{-1}(j\,\mathrm{Cl}(V)\backslash V)$.

Since U is u-ω-open and $X\backslash f^{-1}(j\,\mathrm{Cl}(V)\backslash V)$ is biopen,

$$G = U\cap(X\backslash f^{-1}(j\,\mathrm{Cl}(V)\backslash V)$$

is u-ω-open. Then $x\in G$ and $f(G)\subset V$. For, if $y\in G$, then $f(y)\notin j\,\mathrm{Cl}(V)\backslash V$ and hence $f(y)\in V$. Therefore, f is u-ω-continuous. □

Definition 3.13. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to have u-ω-interiority condition if (τ_1,τ_2)-$\omega\,\mathrm{Int}(f^{-1}(j\,\mathrm{Cl}(V))\subset f^{-1}(V)$ for every σ_i-open set V of Y.*

Example 3.14. Take $X=\mathbb{R}$ and two topologies $\tau_1=\tau_2=\{\emptyset,\mathbb{R},\mathbb{Q}\}$ and $Y=\{a,b,c\}$ with topologies $\sigma_1=\sigma_2=\{\emptyset,Y,\{a\},\{a,b\}\}$. Define $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ as

$$f(x)=\begin{cases} a, & \text{if } x\in\mathbb{Q},\\ b, & \text{if } x\notin\mathbb{Q}\end{cases}$$

It is easy to see that the function f, satisfies the u-ω-interiority condition.

Example 3.15. in Example3.3, we can see easily that f does not satisfies the u-ω-interiority condition.

Theorem 3.16. *If a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)-weakly ω-continuous and satisfies the u-ω-interiority condition, then f is u-ω-continuous.*

Proof. Let V be any σ_i-open set of Y. Since f is (i,j)-weakly ω-continuous, by Theorem 3.4, $f^{-1}(V) \subset (\tau_1,\tau_2)\text{-}\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$. By the u-ω-interiority condition of f, we have $(\tau_1,\tau_2)\text{-}\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V))) \subset f^{-1}(V)$ and hence $f^{-1}(V$ $= (\tau_1,\tau_2)\text{-}\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$. By Lemma 2.4, $f^{-1}(V)$ is u-ω-open in X and so f is u-ω-continuous. □

Definition 3.17. *Let (X,τ_1,τ_2) be a bitopological space and A be a subset of X. The (τ_1,τ_2)-ω-frontier of A is defined as $(\tau_1,\tau_2)\text{-}\omega Fr(A) = (\tau_1,\tau_2)\text{-}\omega\operatorname{Cl}(A)\cup$ $(\tau_1,\tau_2)\text{-}\omega\operatorname{Cl}(X\backslash A) = (\tau_1,\tau_2)\text{-}\omega\operatorname{Cl}(A)\backslash(\tau_1,\tau_2)\text{-}\omega\operatorname{Int}(A)$.*

Theorem 3.18. *The set of all points x of X at which a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is not (i,j)-weakly ω-continuous is identical with the union of the (τ_1,τ_2)-ω-frontiers of the inverse images of the σ_i-closure of σ_i-open sets of Y containing $f(x)$.*

Proof. Let x be a point of X at which $f(x)$ is not (i,j)-weakly ω-continuous.

Then, there exists a σ_i-open set V of Y containing $f(x)$ such that $U\cap(X\backslash f^{-1}(j\operatorname{Cl}(V)))\neq\emptyset$ for every u-ω-open set U of X containing x. By Lemma 2.5, $x\in(\tau_1,\tau_2)\text{-}\omega\operatorname{Cl}(X\backslash f^{-1}(j\operatorname{Cl}(V)))$. Since $x\in f^{-1}(j\operatorname{Cl}(V)))$, we have $x\in(\tau_1,\tau_2)\text{-}\omega\operatorname{Cl}(f^{-1}(j\operatorname{Cl}(V)))$ and hence $x\in(\tau_1,\tau_2)\text{-}\omega Fr(f^{-1}(j\operatorname{Cl}(V)))$. Conversely, if f is (i,j)-weakly ω-continuous at x, then for each σ_i-open set V of Y containing $f(x)$, there exists an u-ω-open set U containing x such that $f(U)\subset j\operatorname{Cl}(V)$ and hence $x\in U\subset f^{-1}(j\operatorname{Cl}(V))$. Hence $x\in(\tau_1,\tau_2)\text{-}\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))$. This contradicts that $x\in(\tau_1,\tau_2)\text{-}\omega Fr(f^{-1}(j\operatorname{Cl}(V)))$. □

Definition 3.19. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-almost ω-continuous [6] if for each $x\in X$ and each σ_i-open set V containing $f(x)$, there exists an u-ω-open set U of X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{Cl}(V))$.*

Lemma 3.20. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is almost u-ω-continuous if and only if $f^{-1}(V)$ is u-ω-open for each (i,j)-regular open set V of Y .*

Definition 3.21. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-almost regular [17] if for each $x\in X$ and each (i,j)-regular open set U containing x, there exists an (i,j)-regular open set V of X such that $x\in V\subset j\operatorname{Cl}(V)\subset U$.*

Theorem 3.22. *Let a bitopological space $((Y,\sigma_1,\sigma_2)$ be (i,j)-almost regular. Then a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)-almost ω-continuous if and only if it is (i,j)-weakly ω-continuous.*

Proof. *Necessity.* This is obvious. Sufficiency. Suppose that f is (i,j)-weakly ω-continuous. Let V be any (i,j)-regular open set of Y and $x\in f^{-1}(V)$. Then we have $f(x)\in V$. By the almost (i,j)-regularity of Y, there exists an (i,j)-regular open set V_0 of Y such that $f(x)\in V_0\subset j\operatorname{Cl}(V_0)\subset V$. Since f is (i,j)-weakly ω-continuous, there exists an u-ω-open set U of X containing x such that $f(U)\subset j\operatorname{Cl}(V_0)\subset V$. This implies that $x\in U\subset f^{-1}(V)$. Therefore, we have

$f^{-1}(V) \subset (\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(V))$ and hence $f^{-1}(V) = (\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(V))$. By Lemma 2.4, $f^{-1}(V)$ is u-ω-open and by Lemma 3.20, f is (i,j)-almost ω-continuous. □

Definition 3.23. *A subset K of a bitopological space (X,τ_1,τ_2) is said to be (i,j)-quasi H-closed relative to X [7] if for each cover $\{U_\alpha : \alpha \in \Omega\}$ of K by τ_i-open sets of X, there exists a finite subset Ω_0 of Ω such that $K \subset \cup\{j\operatorname{Cl}(U_\alpha) : \alpha \in \Omega_0\}$.*

Definition 3.24. *A subset K of a bitopological space (X,τ_1,τ_2) is said to be u-ω-compact relative to X if every cover of K by u-ω-open sets of X has a finite subcover.*

Theorem 3.25. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is (i,j)-weakly ω-continuous and K is u-ω-compact relative to X, then $f(K)$ is (i,j)-quasi H-closed relative to Y .*

Proof. Let K be u-ω-compact relative to X and $\{V_\alpha : \alpha \in \Omega\}$ any cover of $f(K)$ by σ_i-open sets of (Y,σ_1,σ_2). Then $f(K) \subset \cup\{V_\alpha : \alpha \in \Omega\}$ and so $K \subset \cup\{f^{-1}(V_\alpha : \alpha \in \Omega\}$. Since f is (i,j)-weakly ω-continuous, by Theorem 3.4 we have $f^{-1}(V_\alpha) \subset (\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V_\alpha)))$ for each $\alpha \in \Omega$. Therefore, $K \subset \cup\{(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V_\alpha))) : \alpha \in \Omega\}$. Since K is u-ω-compact relative to X and (τ_1,τ_2)-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V_\alpha)))$ is u-ω-open for each $\alpha \in \Omega$, there exists a finite subset Ω_0 of Ω such that $K \subset \cup\{(\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V_\alpha))) : \alpha \in \Omega_0\}$. This implies that $f(K) \subset \cup\{f((\tau_1,\tau_2)$-$\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V_\alpha)))) : \alpha \in \Omega_0\}$ $\subset \cup\{f(f^{-1}(j\operatorname{Cl}(V_\alpha))) : \alpha \in \Omega_0\} \subset \cup\{j\operatorname{Cl}(V_\alpha) : \alpha \in \Omega_0\}$. Hence $f(K)$ is (i,j)-quasi H-closed relative to Y. □

For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, we define $D_{w\omega}(f)$ as follows: $D_{w\omega}(f) = \{x \in X : f$ is not (i,j)-weakly ω-continuous at $x\}$.

Theorem 3.26. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties hold:*

$$\begin{aligned}
D_{w\omega}(f) &= \bigcup_{B\in P(Y)} \{(\tau_1,\tau_2)-\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(i\operatorname{Cl}(B))))\backslash f^{-1}(i\operatorname{Cl}(B))\}\\
&= \bigcup_{F\in(i,j)-RC(Y)} \{(\tau_1,\tau_2)-\omega\operatorname{Cl}(f^{-1}(j\operatorname{Int}(F)))\backslash f^{-1}(F)\}\\
&= \bigcup_{V\in\sigma_j} \{(\tau_1,\tau_2)-\omega\operatorname{Cl}(f^{-1}(V))\backslash f^{-1}(i\operatorname{Cl}(V))\}\\
&= \bigcup_{V\in\sigma_i} \{f^{-1}(V)\backslash(\tau_1,\tau_2)-\omega\operatorname{Int}(f^{-1}(j\operatorname{Cl}(V)))\}\\
&= \bigcup_{A\in P(X)} \{f((\tau_1,\tau_2)-\omega\operatorname{Cl}(A))\backslash(i,j)-\operatorname{Cl}_\theta(f(A))\}\\
&= \bigcup_{B\in P(Y)} \{(\tau_1,\tau_2)-\omega\operatorname{Cl}(f^{-1}(B))\backslash(f^{-1}(i,j)-\operatorname{Cl}_\theta(B))\}
\end{aligned}$$

$$= \bigcup_{B \in P(Y)} \{(\tau_1, \tau_2) - \omega \operatorname{Cl}(f^{-1}(j \operatorname{Int}(i, j) - \operatorname{Cl}_\theta(B)))) \backslash f^{-1}((i, j) - \operatorname{Cl}_\theta(B))\}$$

$$= \bigcup_{V \in (j,i)-PO(Y)} \{(\tau_1, \tau_2) - \omega \operatorname{Cl}(f^{-1}(V)) \backslash f^{-1}(i \operatorname{Cl}(V))\}$$

$$= \bigcup_{V \in (i,j)-PO(Y)} \{f^{-1}(V) \backslash (\tau_1, \tau_2) - \omega \operatorname{Int}(f^{-1}(j \operatorname{Cl}(V)))\}.$$

Proof. The proof follows from Theorems 3.4, 3.5 and 3.6. □

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