Optimum solution of time fractional coupled system of partial differential equations

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Abstract. In the present article, the Optimal Homotopy Asymptotic Method (OHAM) has been extended for the first time to derive the approximate solution of coupled system of fractional order partial differential equations. The fractional Whitham-Broer-Kaup system has been solved as test example. Numerical results obtained by the proposed method are compared with that of Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM). The fractional derivatives are described in the Caputo sense. Numerical results show that the proposed method is reliable and efficient for solution of fractional order coupled system partial differential equations. The accuracy of the method increases by taking higher order approximations.

Keywords: Optimal Homotopy Asymptotic Method (OHAM), time fractional Whitham-Broer-Kaup (WBK) equation, fractional order coupled system of partial differential equations.

1. Introduction

Differential equations of fractional order have been the center of attention of many studies due to their frequent applications in the areas of electromagnetic,

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electrochemistry, acoustics, material science, physics, viscoelasticity and engineering [1, 2]. These kinds of problems are more complex than the differential equations of integer order. Due to the complexities of fractional calculus, most fractional order differential equations do not have the exact solutions, and as alternative approximation methods are used extensively for solution of these types of equations. Some of the recent approximate methods for fractional order differential equations are Homotopy Perturbation Method (HPM), Homotopy Perturbation Transform Method (HPTM), Homotopy Analysis Method (HAM), Reduced Differential Transform Method (RDTM) and Adomian decomposition method (ADM) [3, 4, 5, 6, 7].

Marinca and Herisanu introduced (OHAM) for solving Non-linear Differential equations which made the perturbation methods independent of the assumption of small parameters and huge computational work [8, 9, 10, 11]. The method has been recently extended by Sarwar et al. for solution of differential equations of fractional order [12, 13, 14, 15].

In this presentation, we have extended (OHAM) for the first time for finding the approximate solution of coupled system of fractional order partial differential equations. Particularly, the extended formulation is demonstrated by illustrative examples of the following fractional version of the Whitham-Broer-Kaup (WBK) system.

The fractional order (WBK) system describes the propagation of shallow water waves with different dispersion relations. It takes the following general form:

$$\begin{cases} D_t^{\alpha} u + u u_x + v_x + b u_{xx} = 0, \\ D_t^{\alpha} v + (uv)_x + a u_{xxx} - b v_{xx} = 0, \end{cases} \tag{1}$$

where $u(x,t)$ denotes the horizontal velocity, $v(x,t)$ is the height that deviates from the equilibrium position, a, b are constants that are represented in different diffusion powers, $0 < \alpha \leq 1$ for $\alpha = 1$ we get the standard (WBK) system. The fractional order modified Bouissnesq (MB) system and fractional order approximate long wave (ALW) system are the special cases of and (WBK) System for $a = 1, b = 0$ and $a = 0, b = 0.5$ respectively. These equations have become the subject of active and broad research area in recent decades [16, 17, 18].

The present paper is divided into 5 sections. In Section 2 the basic definitions and properties of fractional calculus are given. Section 3 is devoted to analysis of (OHAM) to coupled system of partial differential equations of fractional order. In Section 4, the 1st order approximate solutions of Time-fractional (WBK) and time-fractional (ALW) equation are given, in which the time fractional derivatives are described in the Caputo sense. In section 5 comparisons are made between the results of 1st order approximate solution by the proposed method with 3rd order (ADM) and (VIM) solutions [19, 20]. In all cases the proposed method yields better results.

2. Preliminaries and notations

In this section we state some definitions and results from the literature which are relevant to our work. Riemann-Liouville, Welyl, Reize, Compos, Caputo proposed many definitions.

Definition 2.1. *A real function $f(r), r > 0$ is said to be in space $C_\mu, \mu \in R$ if there a real number $p > \mu$ such that $f(r) = r^p f_1(r)$, where $f_1(r) \in C(0,\infty)$, and it is said to be in the space C_μ^m if only if $f^{(m)} \in C_\mu, m \in N$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as*

$$J_a^\alpha f(r) = \frac{1}{\Gamma(\alpha)} \int_a^r (r-\mu)^{\alpha-1} f(\mu) d\mu, \alpha r > 0, J_a^0 f(r) = f(r).$$

When we formulate the model of real world problems with fractional calculus, the Riemann-Liouville have certain disadvantages. Caputo proposed a modified fractional differential operator D_a^α in his work on the theory of viscoelasticity.

Definition 2.3. *The fractional derivative of $f(r)$ in Caputo sense is defined as*

$$D_a^\alpha f(r) = J_a^{m-\alpha} D^m f(r) = \frac{1}{\Gamma(m-a)} \int_a^r (r-\eta)^{m-\alpha-1} f^{(m)}(\eta) d\eta,$$

$m-1 < \alpha \leq m, m \in N, r > 0, f \in C_{-1}^m$.

Definition 2.4. *If $m-1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m, \mu \geq -1$, then $D_a^\alpha J_a^\alpha f(r) = f(r)$ and $J_a^\alpha D_a^\alpha f(r) = f(r) - \sum_{k=0}^{m-1} f^{(k)}(r) \frac{(r-a)^k}{k!}, r > 0$*

The operator J^αsatisfies the following properties: for $f \in C_\mu^m, \alpha, \beta > 0, \mu \geq -1, \gamma \geq -1$,

(a) $J_a^\alpha f(r)$ exists for almost every $r \in [a,b]$.

(b) $J_a^\alpha J_a^\beta f(r) = J_a^{\alpha+\beta} f(r)$.

(c) $J_a^\alpha J_a^\beta f(r) = J_a^\beta J_a^\alpha f(r)$.

(d) $J_a^\alpha (r-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (r-a)^{\alpha+\gamma}$.

3. Analysis of (OHAM) for system of fractional order partial differential equations

Consider the general fractional order differential system:

$$\begin{cases} \dfrac{\partial^\alpha u(r,t)}{\partial t} = A(u(r,t)) + f_1(r,t), \alpha > 0, \\ \dfrac{\partial^\alpha v(r,t)}{\partial t} = B(v(r,t)) + f_2(r,t), \alpha > 0. \end{cases} \tag{2}$$

Subject to the initial conditions

$$
\begin{cases}
D_0^{\alpha-k_1}u(r,0)=h_{k_1}(r), k_1=0,1,...,n-1,\\
D_0^{\alpha-n}u(r,0)=0, n=[\alpha],\\
D_0^{k_1}u(r,0)=g_{k_1}(r), k_1=0,1,...,n-1,\\
D_0^{n}u(r,0)=0, n=[\alpha],\\
D_0^{\beta-k_2}v(r,0)=h_{k_2}(r), k_2=0,1,...,n-1,\\
D_0^{\beta-n}v(r,0)=0, n=[\beta],\\
D_0^{k_2}v(r,0)=g_{k_2}(r), k_2=0,1,...,n-1,\\
D_0^{n}v(r,0)=0, n=[\beta],
\end{cases}
\tag{3}
$$

where $\frac{\partial^\alpha}{\partial t^\alpha}, \frac{\partial^\beta}{\partial t^\beta}$ denote the Caputo or Riemann-Liouville fraction derivative operator, A and B are differential operators, $u(r,t)$ and $v(r,t)$ are unknown functions and $f_1(r,t), f_2(r,t)$ are known analytic functions, r and t denote spatial and temporal independent variables respectively.

Construct an optimal Homotopy for fractional order system of partial differential equation, $\phi(r,t;p):\Omega\times[0.1]\to R$ which is

$$
\begin{cases}
(1-p)\Big(\dfrac{\partial^\alpha u(r,t)}{\partial t^\alpha}-f_1(r,t)\Big)\\
\qquad -H_1(r,p)\Big(\dfrac{\partial^\alpha u(r,t)}{\partial t^\alpha}-A(u(r,t))+f_1(r,t)\Big)=0,\\
(1-p)\Big(\dfrac{\partial^\beta v(r,t)}{\partial t^\beta}-f_2(r,t)\Big)\\
\qquad -H_2(r,p)\Big(\dfrac{\partial^\beta v(r,t)}{\partial t^\beta}-B(v(r,t))+f_2(r,t)\Big)=0,
\end{cases}
\tag{4}
$$

where $r\in\Omega$ and $p\in[0,1]$ is an embedding parameter, $H_1(r,p)$ and $H_2(r,p)$ are non-zero auxiliary functions for $p\neq 0$ and $H_1(r,0)=H_2(r,0)=0$ when p increase in the interval $[0,1]$ the solution $\phi(r,t)$ ensures a fast convergence to the exact solution. The efficiency of (OHAM), which does not need a small parameter in the equation, is based on the construction and determination of the auxiliary function combined with a convenient way to optimally control the convergence of the solution. The auxiliary function provides us with a simple way to adjust and control the convergence also increase the accuracy of the results and effectiveness of the method.

The auxiliary functions $H_1(r,p)$ and $H_2(r,p)$ can be written in the form:

$$
\begin{cases}
H_1(r,p)=pk_1(r,C_{1i})+p^2k_2(r,C_{1i})+...+p^mk_m(r,C_{1i}),\\
H_2(r,p)=pk_1(r,C_{2i})+p^2k_2(r,C_{2i})+...+p^mk_m(r,C_{2i}),
\end{cases}
\tag{5}
$$

where C_{1i} and $C_{2i}, i=1,2,...$ are auxiliary convergence control parameters and $k_i(r), i=1,2,...$ can be function on the variable r.

By expanding $\phi(r,t,p,C_i)$ in Taylor's series about p, to get approximate solutions as:

$$\left\{\begin{aligned} u(r,t,C_i) &= u_0(r,t) + \sum_{k_1=1}^{m} u_k(r,t,C_{1i})p^{k_1}, i=1,2,... \\ v(r,t,C_i) &= v_0(r,t) + \sum_{k_2=1}^{m} v_k(r,t,C_{2i})p^{k_2}, i=1,2,... \end{aligned}\right. \tag{6}$$

It is clear if it convergence at $p=1$, one has:

$$\left\{\begin{aligned} u(r,t,C_i) &= u_0(r,t) + \sum_{k_1=1}^{m} u_k(r,t,C_{1i}) \\ v(r,t,C_i) &= v_0(r,t) + \sum_{k_2=1}^{m} v_k(r,t,C_{2i}). \end{aligned}\right. \tag{7}$$

Equating the coefficients of like powers of p after substituting Eq. (6) in Eq.(4), we get zero order, first order, second order and high order problems

$$P^0 : \left\{ \frac{\partial^\alpha u_0(r,t)}{\partial t^\alpha} - f_1 = 0, \frac{\partial^\beta v_0(r,t)}{\partial t^\beta} - f_2 = 0 \right.$$

$$P^1 : \left\{\begin{aligned} &\frac{\partial^\alpha u_1(r,t,C_{11})}{\partial t^\alpha} \\ &\quad -(1+C_{11})\frac{\partial^\alpha u_0(r,t)}{\partial t^\alpha} + (1+C_{11})f_2 + C_{11}A(u_0(r,t)) = 0, \\ &\frac{\partial^\beta v_1(r,t,C_{21})}{\partial t^\beta} \\ &\quad -(1+C_{21})\frac{\partial^\beta v_0(r,t)}{\partial t^\beta} + (1+C_{21})f_2 + C_{21}B(v_0(r,t)) = 0, \\ &\cdots\cdots\cdots\cdots \end{aligned}\right. \tag{8}$$

These problems contain the time fractional derivatives. Therefore we apply the J^α, J^β operators on the above problems, and obtain a series of solutions as follow:

$$\left\{\begin{aligned} &u_0(r,t) = J^\alpha[f_1], \quad v_0(r,t) = J^\beta[f_2], \\ &u_1(r,t;C_{11}) = J^\alpha\left[(1+C_{11})\frac{\partial^\alpha u_0(r,t)}{\partial t^\alpha} - (1+C_{11})f_1 - C_{11}A(u_0(r,t))\right], \\ &v_1(r,t;C_{21}) = J^\beta\left[(1+C_{21})\frac{\partial^\beta v_0(r,t)}{\partial t^\beta} - (1+C_{21})f_2 - C_{21}B(v_0(r,t))\right], \\ &\cdots\cdots\cdots\cdots \end{aligned}\right. \tag{9}$$

By putting the above solutions in Eq. (7), one can get the approximate solution $u(r,t,C_{1i})$, and $v(r,t,C_{2i})$. The auxiliary convergence control parameters

$C_{11}, C_{12}, ...$ and $C_{21}, C_{22}, ...$ can be found by using either: Ritz method, least square method, collocation method, or Galerkins's method. In this paper, least square method is used to find the optimal values of auxiliary convergence control parameters in which we first construct the residuals

$$R_1^2(r,t;C_{in}) = \frac{\partial^\alpha u(r,t)}{\partial t^\alpha} - A(u(r,t)) + f_1(r,t), \alpha > 0,$$
$$R_2^2(r,t;C_{ni}) = \frac{\partial^\beta v(r,t)}{\partial t^\beta} - B(v(r,t)) + f_2(r,t), \beta > 0,$$

minimize the residuals by following relation

$$(10)\qquad \begin{cases} \chi(C_{in}) = \int_0^t \int_\Omega R_1^2(r,t;C_{in})drdt, \\ \chi(C_{ni}) = \int_0^t \int_\Omega R_2^2(r,t;C_{ni})drdt \end{cases}$$

and then calculate the optimum values of auxiliary constants C_{1i} and C_{2i} by solving the following system of equations.

$$(11)\qquad \begin{cases} \frac{\partial \chi}{\partial C_{11}} = \frac{\partial \chi}{\partial C_{12}} = ... = \frac{\partial \chi}{\partial C_{1m}} = 0, \\ \frac{\partial \gamma}{\partial C_{21}} = \frac{\partial \gamma}{\partial C_{22}} = ... = \frac{\partial \gamma}{\partial C_{2m}} = 0. \end{cases}$$

The convergence of (OHAM) [10] is based on the construction and determination of auxiliary functions $H_1(r,p)$ and $H_2(r,p)$. The solution $\phi(r,t)$ ensures a fast convergence to the exact solution as p increase in the interval $[0,1]$. The auxiliary functions are combined with a convenient way to optimally control the convergence of the solution. They provide us with a simple way to adjust and control the convergence and also increase the accuracy of the results and effectiveness of the method.

Theorem 3.1. *If the series (6) convergence to $u(x,t), v(x,t)$, where $u_k(x,t)$, $v_k(x,t) \in L(R^+)$ is produced by zero order problem and the K-order deformation, then $u(x,t), v(x,t)$ is the exact solution of (1).*

Proof. Since the series

$$(12)\qquad \begin{cases} \sum_{k=1}^{\infty} u_{i,k}(r,t,C_1,c_2,...), \quad \sum_{k=1}^{\infty} v_{i,k}(r,t,C_1,c_2,...) \end{cases}$$

converges, it can be written as

$$(13)\qquad \begin{cases} \psi_{1i}(r,t) = \sum_{k=1}^{\infty} u_{1i,k}(r,t,C_1,c_2,...), \\ \psi_{2i}(r,t) = \sum_{k=1}^{\infty} v_{2i,k}(r,t,C_1,c_2,...) \end{cases}$$

and it holds that

$$\begin{cases} \lim_{k\to\infty} u_{1i,k}(r,t,C_1,c_2,...) = 0, \\ \lim_{k\to\infty} v_{2i,k}(r,t,C_1,c_2,...) = 0. \end{cases} \tag{14}$$

In fact the following equation is satisfied

$$\begin{cases} u_{i,1}(r,t,C_1) + \sum k = 2^n u_{i,k}(r,t,\overrightarrow{C}_k) - \sum_{k=2}^{n} u_{i,k-1}(r,t,\overrightarrow{C}_{k-1}), \\ = u_{i,2}(r,t,\overrightarrow{C}_2) - u_{i,1}(r,t,C_1) + ... + u_{i,n}(r,t,\overrightarrow{C}_n) - u_{i,n-1}(r,t,\overrightarrow{C}_{n-1}) \\ = u_{i,n}(r,t,\overrightarrow{C}_n), \\ v_{i,1}(r,t,C_1) + \sum k = 2^n v_{i,k}(r,t,\overrightarrow{C}_k) - \sum_{k=2}^{n} v_{i,k-1}(r,t,\overrightarrow{C}_{k-1}), \\ = v_{i,2}(r,t,\overrightarrow{C}_2) - v_{i,1}(r,t,C_1) + ... + v_{i,n}(r,t,\overrightarrow{C}_n) - v_{i,n-1}(r,t,\overrightarrow{C}_{n-1}) \\ = v_{i,n}(r,t,\overrightarrow{C}_n). \end{cases} \tag{15}$$

Now, we have

$$\begin{cases} L_{i,1}(u_{i,1}(r,t,C_1)) + \sum_{k=2}^{\infty} L_1(u_{i,k}(r,t,\overrightarrow{C}_k)) - \sum_{k=2}^{\infty} L_i(u_{i,k-1}(r,t,\overrightarrow{C}_{k-1})) \\ = L_i(u_{i,1}(r,t,C_1)) + \sum_{k=2}^{\infty} L_i(u_{i,k}(r,t,\overrightarrow{C}_k)) - \sum_{k=2}^{\infty} L_i(u_{i,k-1}(r,t,\overrightarrow{C}_{k-1})) = 0, \\ L_{i,1}(v_{i,1}(r,t,C_1)) + \sum_{k=2}^{\infty} L_1(v_{i,k}(r,t,\overrightarrow{C}_k)) - \sum_{k=2}^{\infty} L_i(v_{i,k-1}(r,t,\overrightarrow{C}_{k-1})) \\ = L_i(v_{i,1}(r,t,C_1)) + \sum_{k=2}^{\infty} L_i(v_{i,k}(r,t,\overrightarrow{C}_k)) - \sum_{k=2}^{\infty} L_i(v_{i,k-1}(r,t,\overrightarrow{C}_{k-1})) = 0 \end{cases} \tag{16}$$

which satisfies

$$\begin{cases} L_{i,1}(u_{i,1}(r,t,C_1)) + L_i(\sum_{k=2}^{\infty} u_{i,k}(r,t,\overrightarrow{C}_k)) - L_i(\sum_{k=2}^{\infty} u_{i,k-1}(r,t,\overrightarrow{C}_{k-1})) \\ = \sum_{k=2}^{\infty} C_m[L_i(u_{i,k-m}(r,t,\overrightarrow{C}_{k-m})) + N_{i,k-m}(u_{i,k-1}(r,t,C_{k-1}))] + f_{1i}(r,t) = 0, \\ L_{i,1}(v_{i,1}(r,t,C_1)) + L_i(\sum_{k=2}^{\infty} v_{i,k}(r,t,\overrightarrow{C}_k)) - L_i(\sum_{k=2}^{\infty} v_{i,k-1}(r,t,\overrightarrow{C}_{k-1})) \\ = \sum_{k=2}^{\infty} C_m[L_i(v_{i,k-m}(r,t,\overrightarrow{C}_{k-m})) + N_{i,k-m}(v_{i,k-1}(r,t,C_{k-1}))] + f_{2i}(r,t) = 0. \end{cases} \tag{17}$$

Now, if $C_m, m = 1, 2, ...$ is properly chosen, then equation leads to

$$\begin{cases} L_i(u_i(r,t)) + A = 0, \\ L_i(v_i(r,t)) + B = 0, \end{cases} \tag{18}$$

which is the exact solution. □

4. Applications

4.1 Time fractional Whitam-Broker-Kaup equation

Consider the time fractional WBK system (1) subject to initial conditions

$$u(x,0) = \lambda - 2\beta k \coth(k\zeta), v(x,0) = -2\beta(\beta + b)k^2 csch^2(k\zeta),$$

where $\beta = \sqrt{a + b^2}, \zeta = x + c$, and λ, c, k are arbitrary constants. For $\alpha = 1$, the exact solution of the given system is

$$\begin{cases} u(x,t) = \lambda - 2\beta k \coth(k(\zeta - \lambda t)), \\ v(x,t) = -2\beta(\beta + b)k^2 csch^2(k(\zeta - \lambda t)). \end{cases} \tag{19}$$

According the formulation discussed in section 3, we have the following zero and first order problems with initial conditions are:

Zero order problem:

$$\begin{cases} \dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0, u_0(x,0) = \lambda - 2\beta k \coth(k\zeta), \\ \dfrac{\partial^\alpha v_0(x,t)}{\partial t^\alpha} = 0, v_0(x,0) = -2\beta(\beta + b)k^2 csch^2(k\zeta). \end{cases} \tag{20}$$

First order problem:

$$\begin{cases} -\dfrac{\partial u_1(x,t)}{\partial t} - C_{12}\dfrac{\partial u_0(x,t)}{\partial t} - C_{12}u_0(x,t)\dfrac{\partial u_0(x,t)}{\partial x} \\ + C_{12}\dfrac{\partial v_0(x,t)}{\partial x} - bC_{12}\dfrac{\partial^2 u_0(x,t)}{\partial x^2} = 0,\ u_1(x,0) = 0 \\ -\dfrac{\partial v_1(x,t)}{\partial t} - C_{11}\dfrac{\partial v_0(x,t)}{\partial t} \\ - C_{11}v_0(x,t)\dfrac{\partial v_0(x,t)}{\partial x} - C_{11}u_0(x,t)\dfrac{\partial v_0(x,t)}{\partial x} \\ - bC_{11}\dfrac{\partial^2 v_0(x,t)}{\partial x^2} - \&C_{11}a\dfrac{\partial^3 u_0(x,t)}{\partial x^3} = 0, \\ v_1(x,0) = 0. \end{cases} \tag{21}$$

Using J^α both sides of eq. (20-21), then we have,

$$u_0(x,t) = \lambda - 2\beta k \coth(k\zeta), \tag{22}$$

$$v_0(x,t) = -2\beta(\beta+b)k^2 csch^2(k\zeta), \tag{23}$$

$$u_1(x,t,C_{12}) = \frac{2C_{12}\beta k^2 t^\alpha \lambda csch^2(k(x+c))}{\Gamma(1+\alpha)}, \tag{24}$$

$$v_1(x,t,C_{11}) = \frac{1}{\Gamma(1+\alpha)}\Bigg(2C_{11}\beta k^3 t^\alpha csch^4(k(x+c))\Big(2k(b^2-\beta^2+a)$$
$$(2+\cosh(2k(x+c))) + (b+\beta)\lambda\sinh(2k(x+c))\Big)\Bigg). \tag{25}$$

For numerical calculation, we take $k = 0.1, \lambda = 0.005, a = b = 1.5, c = 10$ in the above problems.

The values of C_{11} and C_{12}, are found by using least square method, which are given below. $C_{12} = -1.000316679785577005$ and $C_{11} = -0.9985613944569582$.

4.2 Time fractional (ALW) equation

Consider the time fractional (WBK)

$$\Big\{ D_t^\alpha u + uu_x + \frac{1}{2}u_{xx} + v_x = 0, \;\; D_t^\alpha v + (uv)_x - \frac{1}{2}v_{xx} = 0. \tag{26}$$

Subject to initial conditions:

$$u(x,0) = \lambda - k\coth(k\zeta), v(x,0) = -k^2 csch^2(k\zeta),$$

where $\zeta = x + c$, and λ, c, k, are arbitrary constants. For $\alpha = 1$, the exact solution of the given system is:

$$\big\{ u(x,t) = \lambda - k\coth(k(\zeta - \lambda t)), \;\; v(x,t) = -k^2 csch^2(k(\zeta - \lambda t)). \tag{27}$$

According the formulation discussed in section 2 we have the following zero and 1st order problems with initial conditions are given as:

Zero order problem:

$$\begin{cases} \dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0, u_0(x,0) = \lambda - k\coth(k\zeta), \\ \dfrac{\partial^\alpha v_0(x,t)}{\partial t^\alpha} = 0, v_0(x,0) = -k^2 csch^2(k\zeta) \end{cases} \tag{28}$$

First order problem:

$$\begin{cases} \dfrac{-\partial u_0(x,t)}{\partial t} - C_{11}\dfrac{\partial u_0(x,t)}{\partial t} + \dfrac{\partial u_1(x,t)}{\partial t} - C_{11}u_0(x,t)\dfrac{\partial u_0(x,t)}{\partial x} \\ \qquad - C_{11}\dfrac{\partial v_0(x,t)}{\partial x} - \dfrac{1}{2}C_{11}\dfrac{\partial^2 u_0(x,t)}{\partial x^2} = 0, \\ u_1(x,0) = 0 \end{cases} \tag{29}$$

$$\begin{cases} \dfrac{-\partial v_0(x,t)}{\partial t} - C_{12}\dfrac{\partial v_0(x,t)}{\partial t} + \dfrac{\partial v_1(x,t)}{\partial t} - C_1 v_0(x,t)\dfrac{\partial u_0(x,t)}{\partial x} \\ \qquad - C_1 u_0(x,t)\dfrac{\partial v_0(x,t)}{\partial x} + \dfrac{1}{2}C_1\dfrac{\partial^2 v_0(x,t)}{\partial x^2} = 0, \\ v_1(x,0) = 0 \end{cases} \tag{30}$$

Using the same procedure we have zero and 1st order problems solution are

$$u_0(x,t) = \lambda - k\coth(k\zeta), \tag{31}$$

$$v_0(x,t) = -k^2 csch^2(k\zeta), \tag{32}$$

$$u_1(x,t,C_{11}) = \frac{C_{11}k^2t^\alpha \lambda csch^2(k(x+c))}{\Gamma(1+\alpha)}, \tag{33}$$

$$\begin{aligned} v_1(x,t,C_1) = \frac{-1}{\Gamma(1+\alpha)}\Bigg(&C_1k^3t^\alpha csch^4(k(x+c))\Big(2k \\ &+\Big(k\cosh(2k(x+c)) - \lambda\sinh(2k(x+c))\Big)\Big)\Bigg) \end{aligned} \tag{34}$$

We take $k = 0.1, \lambda = 0.005, a = b = 1.5, c = 10$. To find optimum values we used collection method. $C_1 = 0.026058512796348854, C_{11} = -1.0008938281135809$

5. Results and discussion

(OHAM) formulation is tested upon the fractional (WBK) and (ALW) equations. We have used mathematica 7 for most of our computational work.

Tables 1-2 shows comparison of absolute errors of 1st order (OHAM) solution with (ADM) and (VIM) solutions of $u(x,t)$ and $v(x,t)$ for (WBK) system at $\alpha = 1$. In Table 3 and 4 the results obtained by 1st order approximation of proposed method for the (ALW) equation are compared with (ADM) and (VIM).

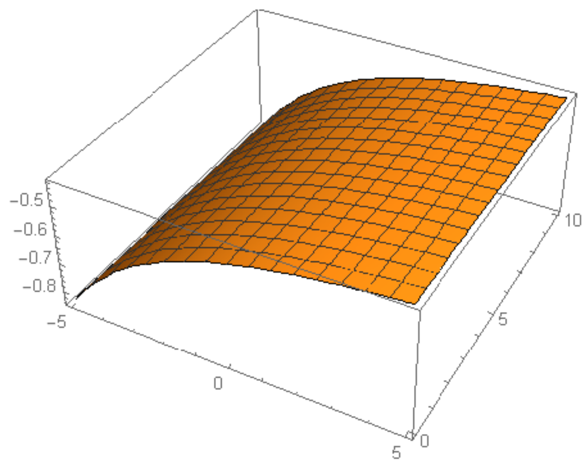
Figures 1-2 show the 3D plots of exact verses approximate solutions of $u(x,t)$ and $v(x,t)$ for (WBK) equation. Figures 3 and 4 show the 2D plots of 1st order approximate solution by proposed method for $u(x,t)$ and $v(x,t)$ of (WBK) equation. Figures 5-6 show the 3D plots of exact verses approximate solutions of $u(x,t)$ and $v(x,t)$ for (ALW) equation. Figures 7 and 8 show the 2D plots of 1st order approximate solution by proposed method for $u(x,t)$ and $v(x,t)$ of (ALW) equation.

Table 1: Comparison of 1st order (OHAM) solution of $u(x,t)$ at $\alpha = 1$, with (ADM) and (VIM)

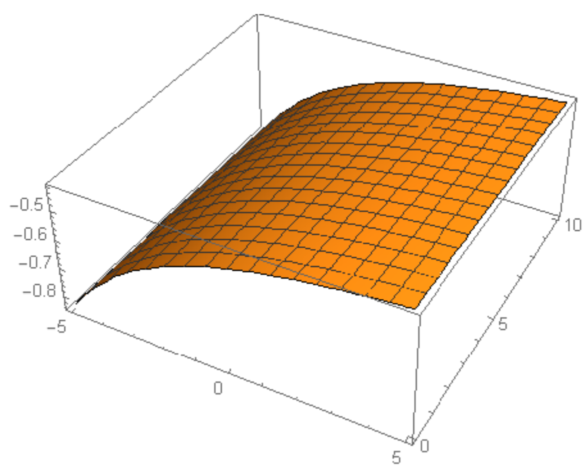
(x,t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	1.04892×10^{-4}	1.23033×10^{-4}	3.43361×10^{-9}
(0.1,0.3)	9.64474×10^{-5}	3.69597×10^{-4}	4.9484×10^{-9}
(0.1,0.5)	8.88312×10^{-5}	6.16873×10^{-4}	6.7427×10^{-10}
(0.2,0.1)	4.25408×10^{-4}	1.19869×10^{-4}	3.34996×10^{-9}
(0.2,0.3)	3.91098×10^{-4}	3.60098×10^{-4}	4.86307×10^{-9}
(0.2,0.5)	3.60161×10^{-4}	6.01006×10^{-4}	5.41391×10^{-10}
(0.3,0.1)	9.71922×10^{-4}	1.16789×10^{-4}	3.26868×10^{-9}
(0.3,0.3)	8.93309×10^{-4}	3.50866×10^{-4}	4.77828×10^{-9}
(0.3,0.5)	8.22452×10^{-4}	5.85610×10^{-4}	4.17512×10^{-10}
(0.4,0.1)	1.75596×10^{-3}	1.13829×10^{-4}	3.18968×10^{-9}
(0.4,0.3)	1.61430×10^{-3}	3.41948×10^{-4}	4.6943×10^{-9}
(0.4,0.5)	1.48578×10^{-3}	5.70710×10^{-4}	3.02393×10^{-10}
(0.5,0.1)	2.79519×10^{-3}	1.10936×10^{-4}	3.11288×10^{-9}
(0.5,0.3)	2.56714×10^{-3}	3.33274×10^{-4}	4.61117×10^{-9}
(0.5,0.5)	2.36148×10^{-3}	5.56235×10^{-4}	1.95473×10^{-10}

Table 2: Comparison of 1st order (OHAM) solution of $v(x,t)$ at $\alpha = 1$, with (ADM) and (VIM)

(x,t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	6.41419×10^{-3}	1.10430×10^{-4}	1.75426×10^{-8}
(0.1,0.3)	5.99783×10^{-3}	3.31865×10^{-4}	4.68301×10^{-8}
(0.1,0.5)	5.61507×10^{-3}	5.54071×10^{-4}	6.8385×10^{-8}
(0.2,0.1)	1.33181×10^{-2}	1.07016×10^{-4}	1.70078×10^{-8}
(0.2,0.3)	1.24441×10^{-2}	3.21601×10^{-4}	4.54496×10^{-8}
(0.2,0.5)	1.16416×10^{-2}	5.36927×10^{-4}	6.64572×10^{-8}
(0.3,0.1)	2.07641×10^{-2}	1.03737×10^{-4}	1.64933×10^{-8}
(0.3,0.3)	1.93852×10^{-2}	3.11737×10^{-4}	4.41191×10^{-8}
(0.3,0.5)	1.81209×10^{-2}	5.20447×10^{-4}	6.45949×10^{-8}
(0.4,0.1)	2.88100×10^{-2}	1.00579×10^{-4}	1.5998×10^{-8}
(0.4,0.3)	2.68724×10^{-2}	3.02245×10^{-4}	4.28363×10^{-8}
(0.4,0.5)	2.50985×10^{-2}	5.04593×10^{-4}	6.27954×10^{-8}
(0.5,0.1)	3.75193×10^{-2}	9.75385×10^{-5}	1.55211×10^{-8}
(0.5,0.3)	3.49617×10^{-2}	2.93107×10^{-4}	4.15991×10^{-8}
(0.5,0.5)	3.26339×10^{-2}	4.89335×10^{-4}	6.10561×10^{-8}

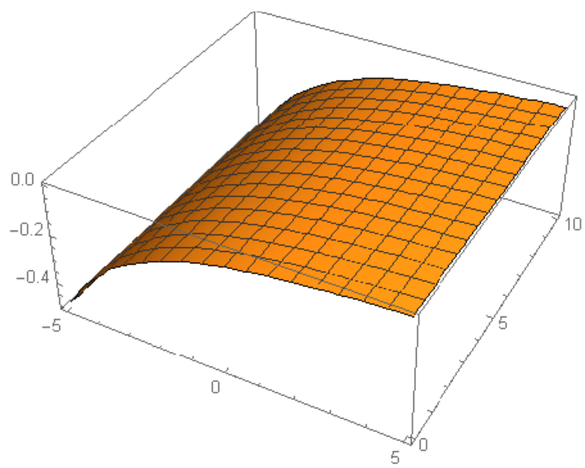


(a) 1st order approximate solution $u(x,t)$ obtained by (OHAM) for eq. (10)

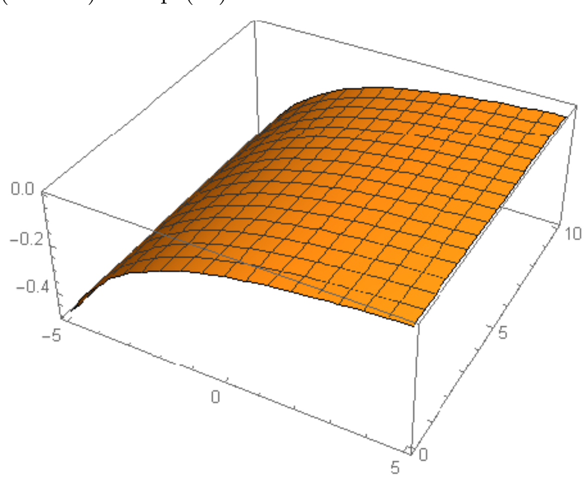


(b) Exact solution of $u(x,t)$ for eq. (10)

Figure 1: The exact and approximate solution for $u(x,t)$

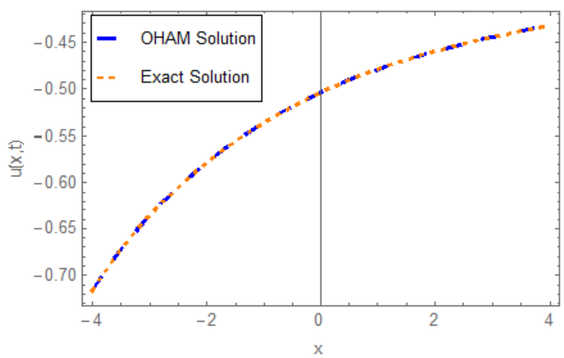


(a) 1st order approximate solution $v(x,t)$ obtained by (OHAM) for eq. (10)

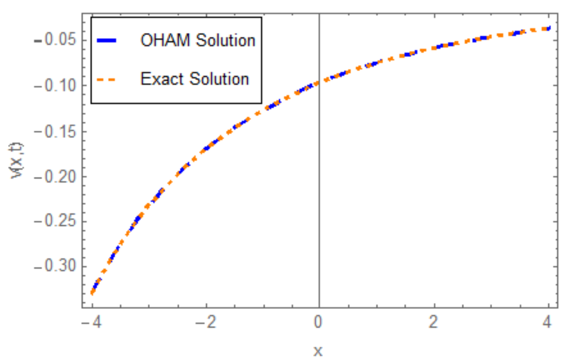


(b) Exact solution of $v(x,t)$ for eq. (10)

Figure 2: The Exact and Approximate solution for $v(x,t)$



(a) 2D plots of 1st order (OHAM) verses exact solution for $u(x,t)$ when $-4 \leq x \leq 4$ at $t = 0.1$



(b) 2D plots of 1st order (OHAM) verses exact solution for $v(x,t)$ when $-4 \leq x \leq 4$ at $t = 0.1$

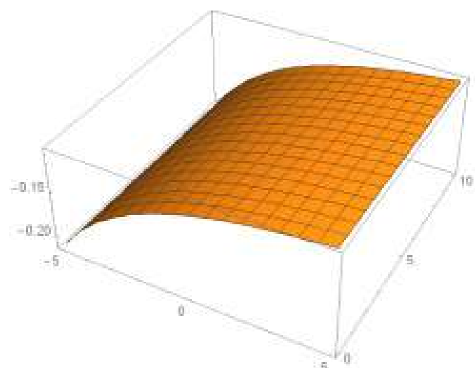
Figure 3: 2D plots of 1st order (OHAM) verses exact solution for $u(x,t), v(x,t)$

Table 3: Comparison of 1st order (OHAM) solution of $u(x,t)$ at $\alpha = 1$, with (ADM) and (VIM)

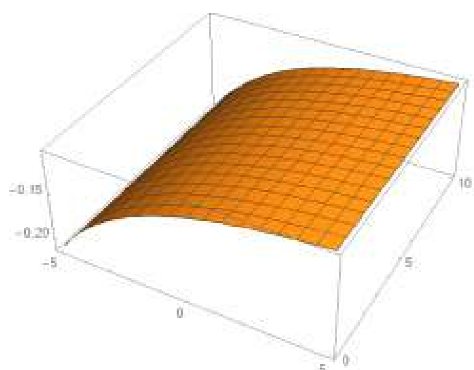
(x,t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	8.02989×10^{-6}	3.17634×10^{-5}	3.43361×10^{-9}
(0.1,0.3)	7.38281×10^{-6}	9.54273×10^{-5}	4.9484×10^{-9}
(0.1,0.5)	6.79923×10^{-6}	1.59274×10^{-4}	6.7427×10^{-10}
(0.2,0.1)	3.23228×10^{-5}	3.09466×10^{-5}	3.34996×10^{-9}
(0.2,0.3)	2.97172×10^{-5}	9.29725×10^{-5}	4.86307×10^{-9}
(0.2,0.5)	2.73673×10^{-5}	1.55176×10^{-4}	5.41391×10^{-10}
(0.3,0.1)	7.32051×10^{-5}	3.01549×10^{-5}	3.26868×10^{-9}
(0.3,0.3)	6.73006×10^{-5}	9.05935×10^{-5}	4.77828×10^{-9}
(0.3,0.5)	6.19760×10^{-5}	1.51204×10^{-4}	4.17512×10^{-10}
(0.4,0.1)	1.31032×10^{-4}	2.93874×10^{-5}	3.18968×10^{-9}
(0.4,0.3)	1.20455×10^{-4}	8.82871×10^{-5}	4.6943×10^{-9}
(0.4,0.5)	1.10919×10^{-4}	1.47354×10^{-4}	3.02393×10^{-10}
(0.5,0.1)	2.06186×10^{-4}	2.86433×10^{-5}	3.11288×10^{-9}
(0.5,0.3)	1.89528×10^{-4}	8.60509×10^{-5}	4.61117×10^{-9}
(0.5,0.5)	1.74510×10^{-4}	1.43620×10^{-4}	1.95473×10^{-10}

Table 4: Comparison of 1st order (OHAM) solution of $v(x,t)$ at $\alpha = 1$, with (ADM) and (VIM)

(x,t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	4.81902×10^{-4}	8.29712×10^{-6}	1.75426×10^{-8}
(0.1,0.3)	4.50818×10^{-4}	2.49346×10^{-5}	4.68301×10^{-8}
(0.1,0.5)	4.22221×10^{-4}	4.16299×10^{-5}	6.8385×10^{-8}
(0.2,0.1)	9.76644×10^{-4}	8.04063×10^{-6}	1.70078×10^{-8}
(0.2,0.3)	9.13502×10^{-4}	2.41634×10^{-5}	4.5496×10^{-8}
(0.2,0.5)	8.55426×10^{-4}	4.03419×10^{-5}	6.64572×10^{-8}
(0.3,0.1)	1.48482×10^{-3}	7.79401×10^{-6}	1.64933×10^{-8}
(0.3,0.3)	1.38858×10^{-3}	2.34220×10^{-5}	4.41191×10^{-8}
(0.3,0.5)	1.30009×10^{-3}	3.91034×10^{-5}	6.45949×10^{-8}
(0.4,0.1)	2.00705×10^{-3}	7.55675×10^{-6}	1.5998×10^{-8}
(0.4,0.3)	1.87661×10^{-3}	2.27087×10^{-5}	4.28363×10^{-8}
(0.4,0.5)	1.75670×10^{-3}	3.79121×10^{-5}	6.27954×10^{-8}
(0.5,0.1)	2.54396×10^{-3}	7.32847×10^{-6}	1.55211×10^{-8}
(0.5,0.3)	2.37815×10^{-3}	2.20224×10^{-5}	4.15991×10^{-8}
(0.5,0.5)	2.22578×10^{-3}	3.67658×10^{-5}	6.10561×10^{-8}

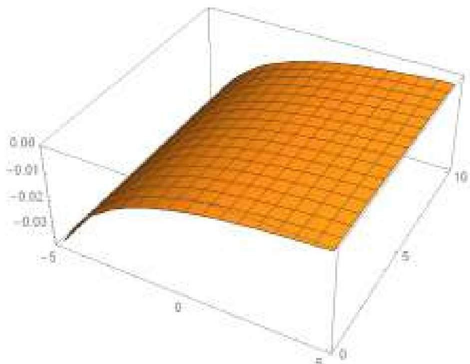


(a) 1st order approximate solution $u(x,t)$ obtained by (OHAM) for eq. (26)

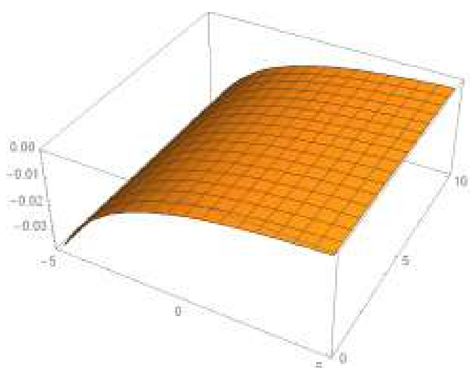


(b) Exact solution of $u(x,t)$ for eq. (26)

Figure 4: The Exact and Approximate solution for $u(x,t)$

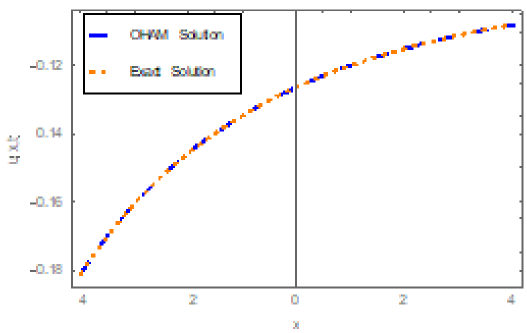


(a) 1st order approximate solution $v(x,t)$ obtained by (OHAM) for eq. (26)

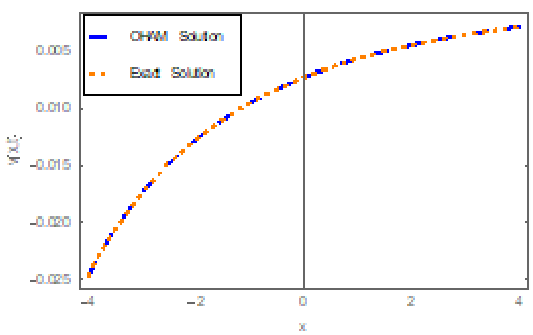


(b) Exact solution of $v(x,t)$ for eq. (26)

Figure 5: The Exact and Approximate solution for $v(x,t)$



(a) 2D plots of 1st order (OHAM) verses exact solution for $u(x,t)$ when $-4 \leq x \leq 4$ at $t = 0.1$



(b) 2D plots of 1st order (OHAM) verses exact solution for $v(x,t)$ when $-4 \leq x \leq 4$ at $t = 0.1$

Figure 6: 2D plots of 1st order (OHAM) verses exact solution for $u(x,t), v(x,t)$

6. Conclusion

(OHAM) converges rapidly to the exact solution at lower order of approximations for (WBK) system. The results obtained by proposed method are very encouraging in comparison with (ADM) and (VIM). As a result it would be more appealing for researchers to apply this method for solving systems of nonlinear fractional order partial differential equations in different fields of science especially in fluid dynamics and physics. The accuracy of method can further be increased by taking higher order of approximations of the proposed method.

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