On some class of degenerate elliptic equations with $L^1(\Omega)$ coefficients in $P(X)$-Sobolev spaces

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Abstract. This paper deals with a class of nonlinear degenerate elliptic problem with coefficients in $L^1(\Omega)$ of the form $-\mathrm{div}\, a(x,u,\nabla u) = f$, where $a(x,u,\nabla u)$ is allowed to be degenerate with the unknown u. We prove the existence of weak and entropy solutions under some hypothesis on f. We also study the same problem with a lower order term.

Keywords: $L^1(\Omega)$ coefficients, weak and entropy solutions, nonlinear degenerate elliptic equations, Sobolev spaces with variable exponent.

1. Introduction

Let Ω be a bounded subset of $\mathbb{R}^N, N \geq 2$.

$$\begin{cases} -\mathrm{div}(g(u)|\nabla u|^{p-2}\nabla u) = h(u)|\nabla u|^p + f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

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with $1 < p < \infty$, g is a continuous positive function. Also under the additional hypothesis

$$\lim_{s\to\infty} \frac{h(s)}{g(s)} = 0,$$

they obtain L^∞-estimate. This kind of problems has been widely studied. we refer the reader to [7, 11, 12, 13, 15, 16].

In this paper we study the existence of weak and entropy solutions for the following nonlinear degenerate elliptic problem:

$$(1) \qquad \begin{cases} -\operatorname{div} a(x,u,\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

in the setting of the Sobolev space with variable exponent $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and

$$f \in L^m(\Omega), \quad \text{where} \;\; m = 1 \text{ or } m = (p^*)'.$$

Moreover we assume that $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function such that for every s in $\mathbb{R}$, $\xi,\ \xi'$ in $\mathbb{R}^N$ $(\xi \neq \xi')$, and for almost every $x \in \Omega$,

$$(2) \qquad |a(x,s,\xi)| \leq \gamma\alpha(x)(1+g(s))|\xi|^{p(x)-1},$$

$$(3) \qquad a(x,s,\xi)\xi \geq \alpha(x)g(s)|\xi|^{p(x)},$$

$$(4) \qquad [a(x,s,\xi) - a(x,s,\xi')][\xi - \xi'] > 0,$$

where γ is a positive real constant.

The function $\alpha(x)$ satisfies, for $\nu > 0$

$$(5) \qquad \alpha \in L^1(\Omega), \quad \alpha(x) > \nu,$$

and

$$(6) \qquad g : \;\; \mathbb{R} \mapsto \mathbb{R}^+, \;\; \text{is a positive continuous function}, \;\; g(0) = 0 \text{ and}$$

$$(7) \qquad g^{\frac{1}{p^\pm - 1}} \notin L^1([0.+\infty[) \cup L^1(]-\infty.0]).$$

Then we study the perturbated problem

$$(8) \qquad \begin{cases} -\operatorname{div} a(x,u,\nabla u) + H(x,u,\nabla u) = f, & \text{in } \Omega, \\ \;u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $H(x,s,\xi)$ is a Carathéodory function satisfying

$$(9) \qquad \mid H(x,s,\xi) \mid \leq \beta(x)h(s) \mid \xi \mid^{p(x)},$$

for almost every $x \in \Omega$, for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $\gamma > 0$, where the functions $\beta(.)$ satisfies, for $\theta > 0$

$$\beta \in L^1(\Omega), \qquad \beta(x) \geq 0 \text{ and } \beta(x) \leq \theta\alpha(x) \text{ a.e. in } \Omega, \text{ and} \tag{10}$$

$h, g: \quad \mathbb{R} \mapsto \mathbb{R}^+$, are positive continuous functions, $g > 0$,

$$\frac{h}{g} \in L^1(\mathbb{R}). \tag{11}$$

Furthermore, we will study the case when $f \in L^1(\Omega)$ and $f \in L^{(p^*(x))'}(\Omega)$.

To deal with this kind of problem, it is natural to work under the framework of Sobolev spaces with variable exponents. The study of differential equations with variable exponents has been a very active field in recent years, with applications in electro-rheological fluids and image processing, and so on. We refer the readers to [13] and references therein.

Some remarks concerning the difficulties in dealing with this problem are in order. First of all, the nonlinearity of the operator and the singularity of the coefficient will be solved by coupling energy estimates of type $\int_\Omega \alpha(x)|\nabla u|^{p(x)} < \infty$, on the one hand, we suppose no growth limitation on function g so that it can happen $a(x, u, \nabla u) \notin W^{-1,p'(\cdot)}(\Omega)$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and, on the other hand, it is not coercive. To deal with that equation we will obtain the solution by approximation, getting a sequence of approximated solutions $(u_n)_n$ which converge to the solution u. To avoid some troubles in this convergence process, we will consider the convergence of another sequence $(A(u_n))_n$, function A being the primitive of $g^{\frac{1}{p^{\pm}-1}}$ such that $A(0) = 0$. The other remark is about the lower order term: it does not satisfy the sign condition, since h is positive. Thus, it appears the problem of getting the a priori estimates. This hindrance is overcome by considering test functions of exponential type.

With a modified version of the classical Minty Lemma (see [5]). We stress that in our treatment we avoid the use of weighted Sobolev spaces.

The content of this paper is organized as follows. We start by presenting preliminary results, some notations and definitions in Section 2. Section 3 is devoted to stating our main result.

2. Preliminaries

In this section, we define at first Lebesgue and Sobolev spaces with variable exponent and recall some of their properties. Let Ω be an open bounded set in $\mathbb{R}^N$, $N \geq 2$. The function $p(\cdot)$ satisfies the log-Hölder continuity on Ω if

$$|p(y) - p(x)| \leq \frac{C}{|log|y - x||}, \text{ for all } x,\ y \in \overline{\Omega} \text{such that } |y - x| \leq \frac{1}{2}, \tag{12}$$

with C being a positive constant.

We denote

$$C^+(\overline{\Omega}) = \big\{\text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \le p^+ < N\big\},$$

where

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

The variable exponent Lebesgue space is defined as

$$L^{p(\cdot)}(\Omega) = \{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

We can introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \quad \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\}.$$

The variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p_- < p_+ < \infty$ and continuous functions are dense in $L^{p(\cdot)}(\Omega)$ if $p_+ < \infty$, (see Kovăčik and Răkosnik [16]).

We denote by $L^{p'(\cdot)}(\Omega)$ the dual space of $L^{p(\cdot)}(\Omega)$ where $1/p(\cdot) + 1/p'(.) = 1$ (see [10],[12]). For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the generalized Hölder inequality

$$\Big|\int_\Omega uv\Big| \le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(x)} \parallel v \parallel_{p'(x)},$$

holds true.

Proposition 2.1 ([9],[19]). *if we denote*

$$\rho(u) = \int_\Omega |\; u \;|^{p(x)}\, dx, \forall u \in L^{p(\cdot)}(\Omega),$$

then the following assertions holds:

(i) $\parallel u \parallel_{p(x)} < 1$ *(respectively,* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(respectively,* $= 1, > 1$*);*

(ii) $\begin{cases} \parallel u \parallel_{p(x)} > 1 \Rightarrow \parallel u \parallel_{p(x)}^{p-} \le \rho(u) \le \parallel u \parallel_{p(x)}^{p+}, \\ \parallel u \parallel_{p(x)} < 1 \Rightarrow \parallel u \parallel_{p(x)}^{p+} \le \rho(u) \le \parallel u \parallel_{p(x)}^{p-} \end{cases}$;

(iii) $\parallel u \parallel_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$; $\parallel u \parallel_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

We define the variable Sobolev space by

$$W^{1.p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) and \mid \nabla u \mid \in L^{p(\cdot)}(\Omega)\}.$$

normed by,

$$\parallel u \parallel_{1.p(\cdot)} = \parallel u \parallel_{p(\cdot)} + \parallel \nabla u \parallel_{p(\cdot)} \quad \forall u \in W^{1.p(\cdot)}(\Omega). \tag{13}$$

We denote by $W_0^{1.p(\cdot)}(\Omega)$ *the closure of* $\mathcal{C}_0^\infty(\Omega)$ *in* $W^{1.p(\cdot)}(\Omega)$ *and*

$$p^*(.) = \begin{cases} \frac{Np(\cdot)}{N-p(\cdot)}, & \text{for } p(\cdot) < N, \\ \infty, & \text{for } p(\cdot) \ge N. \end{cases}$$

Proposition 2.2 ([9]). *(i) Assuming $p_- > 1$, the spaces $W^{1.p(\cdot)}(\Omega)$ and $W_0^{1.p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) if $q \in \mathcal{C}_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then

$$W_0^{1.p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega) \tag{14}$$

is compact and continuous (see, Theorem 8.4.2 [8]).

(iii) Let $p \in \mathcal{C}_+(\overline{\Omega})$. Then, for $u \in W_0^{1.p(\cdot)}(\Omega)$, the $p(\cdot)-$Poincaré inequality

$$\| u \|_{p(\cdot)} \leq C \| \nabla u \|_{p(\cdot)}$$

holds, where the positive constant C depends on $p(\cdot)$ and Ω.

Remark 2.1. By (iii) of Proposition 2.2, we know that $\| \nabla u \|_{p(\cdot)}$ and $\| u \|_{1.p(\cdot)}$ are equivalent norms on $W_0^{1.p(\cdot)}(\Omega)$.

Some technical Lemmas

Lemma 2.1 ([19]). *Let $q \in \mathcal{C}_+(\overline{\Omega})$, $g \in L^{q(\cdot)}(\Omega)$ and $(g_n)_n \in L^{q(\cdot)}(\Omega)$ with $\| g_n \|_{q(\cdot)} \leq C$, if $g_n(x) \to g(x)$ a.e in Ω , then $g_n(x) \to g(x)$ in $L^{q(x)}(\Omega)$, where C is a positive constant.*

Lemma 2.2 ([19]). *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipshitz function, with $F(0) = 0$ and $p \in \mathcal{C}_+(\overline{\Omega})$.*

If $u \in W_0^{1.p(\cdot)}(\Omega)$, then $F(u) \in W_0^{1.p(\cdot)}(\Omega)$. Moreover, if the set D of discontinuity point of F' in finite, then

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i}, & \text{for a.e. in } \{x \in \Omega \backslash u(x) \notin D\}, \\ 0, & \text{for a.e. in, } \{x \in \Omega \backslash u(x) \in D\}, \end{cases}$$

Lemma 2.3 ([9]). *Let $u \in W_0^{1.p(\cdot)}(\Omega)$, then, we have $T_k(u) \in W_0^{1.p(\cdot)}(\Omega)$, with $k > 0$. Moreover, we have also $T_k(u) \to u \in W_0^{1.p(\cdot)}(\Omega)$ when $k \to \infty$.*

Lemma 2.4 ([2]). *Let $(u_n)_n$ be a sequence in $W_0^{1.p(\cdot)}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1.p(\cdot)}(\Omega)$. then, $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1.p(\cdot)}(\Omega)$.*

Lemma 2.5 ([19]). *Assume that (2)-(4) hold, with $g \geq \lambda$ in (3), $\lambda > 0$ and let $(u_n)_n$ be a sequence in $W_0^{1.p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1.p(\cdot)}(\Omega)$ and*

$$\int_\Omega [a_n(x, u_n, \nabla u_n) - a_n(x, u_n, \nabla u)]\nabla(u_n - u) \to 0.$$

Then, $u_n \to u$ in $W_0^{1.p(\cdot)}(\Omega)$.

Notations and definitions

Definition 2.1. *For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ can be defined by*

$$T_k(s) = \begin{cases} s, & if \quad |s| \leq k \\ k \cdot sign(s), & if \quad |s| > k, \end{cases}$$

and we define $G_k(s) = s - T_k(s)$.

We define

$$\eta(s) = \int_0^s \frac{h(\tau)}{g(\tau)} d\tau, \tag{15}$$

we moreover define the space of functions

$$X_0^{p(\cdot)}(\Omega) = \{\varphi \in W_0^{1,p(\cdot)}(\Omega) \text{ such that } \int_\Omega \alpha(x)|\nabla\varphi|^{p(x)} < \infty\}, \tag{16}$$

and

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable such that } T_k(u) \in W_0^{1,p(\cdot)}(\Omega)\},$$

i.e. the set of measurable functions whose truncates belong to $W_0^{1,p(\cdot)}(\Omega)$ (we refer to [4], for more details). We note

$$A(s) = \int_0^s g(\tau)^{1/(p^+-1)} d\tau \text{ if } g \text{ is unbounded}, \tag{17}$$

$$A(s) = \int_0^s g(\tau)^{1/(p^--1)} d\tau \text{ if } g \text{ is bounded}. \tag{18}$$

With C_i, $i \in \mathbb{N}$, we indicate generic positive constants that may depend on the dimension N, on the domain Ω, and on the other data of the problem, while with ε_n we indicate a generic sequence that goes to zero as n diverges.

3. Existence results

Theorem 3.1. *Assume that* (2)-(7) *hold true, then for every $f \in L^1(\Omega)$ there exists $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ entropy solution of* (1) *in the following sense $\forall k > 0$, $\int_\Omega \alpha(x)g(u)^{p'(x)}|\nabla T_k(u)|^{p(x)} < \infty$, and*

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi) = \int_\Omega f T_k(u-\phi) \quad \forall\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega). \tag{19}$$

If moreover $f \in L^{(p^*(.))'}(\Omega)$, then $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\int_\Omega \alpha(x)g(u)^{p'(x)}|\nabla u|^{p(x)} < \infty,$$

$$\int_\Omega a(x,u,\nabla u)\nabla\phi = \int_\Omega f\phi \quad \forall\phi \in W_0^{p(\cdot)}(\Omega).$$

Theorem 3.2. *Assume that* (2)-(7) *and* (9)-(11) *hold true, then for every* $f \in L^1(\Omega)$ *there exists* $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ *entropy solution of* (8) *in the following sense*

$$\forall k > 0, \ \int_\Omega \alpha(x) g(u)^{p'(x)} |\nabla T_k(u)|^{p(x)} < \infty, \ H(x,u,\nabla u) \in L^1(\Omega), \text{ and}$$

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi) + \int_\Omega H(x,u,\nabla u) T_k(u-\phi) = \int_\Omega f T_k(u-\phi),$$

$$\forall \phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

If, moreover $f \in L^{(p^*(\cdot))'}(\Omega)$*, then* $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\int_\Omega \alpha(x) g(u)^{p'(x)} |\nabla u|^{p(x)} < \infty, \ H(x,u,\nabla u) \in L^1(\Omega) \text{ and}$$

$$\int_\Omega a(x,u,\nabla u)\nabla \phi + \int_\Omega H(x,u,\nabla u)\phi = \int_\Omega f\phi, \ \forall \phi \in X_0^{p(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

We moreover state and prove a preliminary Lemma (see, [6] Lemma 3.3) that will be often used in the sequel.

Lemma 3.1. *Let* σ_n *be a sequence of nonnegative bounded functions, almost everywhere convergent to some function* σ, *and let* $\rho_n : \mathbb{R}^N \to \mathbb{R}^N$ *be a sequence of functions which is weakly convergent in* $(L^{q(x)}(\Omega))^N$ $(q(x) > 1)$ *to some function* ρ. *If the sequence* $\sigma_n |\rho_n|^{q(x)}$ *is bounded in* $L^1(\Omega)$, *then* $\sigma|\rho|^{q(x)}$ *belongs to* $L^1(\Omega)$ *and*

$$\int_\Omega \sigma |\rho|^{q(x)} \leq \liminf_{n\to+\infty} \sigma_n |\rho_n|^{q(x)}. \tag{20}$$

If moreover $\sigma_n \to \sigma$ *strongly in* $L^1(\Omega)$ *then it holds true also that*

$$\sigma_n^{1/q(x)} \rho_n \rightharpoonup \sigma^{1/q(x)} \rho \ \text{ weakly in } \ L^{q(x)}(\Omega)^N. \tag{21}$$

Proof. Since ρ_n weakly convergent to ρ in $(L^{q(x)}(\Omega))^N$, then there exists a functional $h \in (L^{q(x)}(\Omega))^N$ such that $|\rho_n| \leq h$.

Let $k > 0$ and $\psi \in (L^{q'(x)}(\Omega))^N$ we have $|T_k(\sigma_n)^{1/q(x)} \rho_n.\psi| \leq k^{1/\bar{q(x)}} h(x).\psi(x)$ and $T_k(\sigma_n)^{1/q(x)}\rho_n.\psi \to T_k(\sigma)^{1/q(x)}\rho.\psi$ a.e. in Ω, witch give

$$T_k(\sigma_n)^{1/q(x)} \rho_n \rightharpoonup T_k(\sigma)^{1/q(x)} \rho \ \text{ weakly in } \ (L^{q(x)}(\Omega))^N. \tag{22}$$

Hence by the lower semi-continuity of the $L^{q(x)}$-norm it results

$$\int_\Omega T_k(\sigma)|\rho|^{q(x)} \leq \liminf_{n\to+\infty} \int_\Omega T_k(\sigma_n)|\rho_n|^{q(x)} \leq \liminf_{n\to+\infty} \sigma_n |\rho_n|^{q(x)} \leq C. \tag{23}$$

Then letting k tend to infinity and using the monotone convergence theorem we obtain (20). Let now ψ be an arbitrary element of $(L^{q'(x)}(\Omega))^N$, then

$$\begin{aligned}\int_\Omega \sigma_n^{1/q(x)} \rho_n \psi &= \int_{\{\sigma_n \leq k\}} \sigma_n^{1/q(x)} \rho_n \psi + \int_{\{\sigma_n > k\}} \sigma_n^{1/q(x)} \rho_n \psi \\ &= \varepsilon_n + \int_{\{\sigma \leq k\}} \sigma_n^{1/q(x)} \rho_n \psi + \int_{\{\sigma_n > k\}} \sigma_n^{1/q(x)} \rho_n \psi.\end{aligned}$$

Moreover

$$\int_\Omega \sigma^{1/q(x)}\rho_n\psi = \int_{\{\sigma_n\le k\}} \sigma^{1/q(x)}\rho\psi + \int_{\{\sigma_n>k\}} \sigma^{1/q(x)}\rho\psi$$
$$= \varepsilon_n + \int_{\{\sigma\le k\}} \sigma^{1/q(x)}\rho\psi + \int_{\{\sigma_n>k\}} \sigma^{1/q(x)}\rho\psi.$$

It follows that

$$\left|\int_\Omega [\sigma_n^{1/q(x)}\rho_n - \sigma^{1/q(x)}\rho].\psi\right| \le \varepsilon_n + \int_{\{\sigma_n>k\}} \sigma_n^{1/q(x)}|\rho_n||\psi| + \int_{\{\sigma_n>k\}} \sigma^{1/q(x)}|\rho||\psi|.$$

Using Hölder inequality we have that

$$\int_{\{\sigma_n>k\}} \sigma_n^{1/q(x)}\rho_n\psi \le \|\sigma_n^{1/q(x)}\rho_n\|_{q(x)}\|\psi\|_{L^{q'(x)}(\sigma_n>k)},$$
$$\int_{\{\sigma_n>k\}} \sigma^{1/q(x)}\rho\psi \le \|\sigma^{1/q(x)}\rho\|_{q(x)}\|\psi\|_{L^{q'(x)}(\sigma_n>k)}. \qquad \square$$

Proof of Theorem 3.1

We divide the rest of the proof into two steps; in the first step, we assume only $f \in L^1(\Omega)$, while in the second one we consider $L^{(p^*(x))'}(\Omega)$ data.

Step 1: Approximation and a priori estimates. Let $(f_n)_{u\in\mathbb{N}}$ be a sequence of smooth function such $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$, we consider the following problem

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\phi dx = \int_\Omega f_n\phi dx, \qquad \forall\phi \in W_0^{1.p(\cdot)}(\Omega), \tag{24}$$

where

$$a_n(x,s,\xi) = \frac{a(x,T_n(s),\xi)}{\alpha(x)}\alpha_n(x),$$
$$\alpha_n(x) = T_n(\alpha(x)), \quad g_n(s) = g(T_n(s)).$$

Note that g is continuous, so there exists $\lambda_n \ge 0$ such that $g_n(s) \ge \lambda_n$ and since $g_n^{\frac{1}{p^\pm-1}} \notin L^1([0.+\infty[) \cup L^1(]-\infty.0])$. Then by definition $a_n(x,s,\xi)$ satisfies

$$|a_n(x,s,\xi)| \le \gamma\alpha_n(x)(1+g_n(s))|\xi|^{p(x)-1},$$
$$a_n(x,s,\xi)\xi \ge \alpha_n(x)g_n(s)|\xi|^{p(x)} \ge \lambda_n\alpha_n(x)|\xi|^{p(x)},$$
$$[a_n(x,s,\xi) - a_n(x,s,\xi')][\xi-\xi'] > 0,; \text{ , for } \ \xi \ne \xi',$$

we can use the theorem 4.3 of [1] to infer the existence of $u_n \in W_0^{1.p(x)}(\Omega) \cap L^\infty(\Omega)$ solution to problem (24).

Case 1. The function g is unbounded, i.e. $\lim_{s\to+\infty} g(s) = +\infty$. Then, there exists $g_0 > 0$ such that $\alpha(s) > 1$, for all $s \geq g_0$ since $\lim_{s\to+\infty} g(s) = +\infty$. For every $k > A(g_0)$, with

$$A(s) = \int_0^s g(\tau)^{1/(p^+-1)} d\tau,$$

taking $T_k(A(u_n))$ as test function in (24), there exists $C_0 > 0$, and by assumption (3), we have

(25)

$$\begin{aligned} C_0\nu \int_{|A(u)|\leq k} |\nabla T_k(A(u_n))|^{p(x)} &\leq C_0 \int_{|A(u)|\leq k} \alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)} \\ &\leq C_0 \int_{|A(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^+-1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p^+}{p^+-1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)|\leq k} a_n(x, u_n, \nabla u_n)\nabla u_n g^{\frac{1}{p^+-1}}(u_n) dx \\ &\leq \int_\Omega |f_n||T_k(A(u_n))| dx \\ &\leq k\|f\|_{L^1}. \end{aligned}$$

Case 2. The function g is bounded, i.e. there exists a constant $M > 0$ such that $g(s) \leq M$, for every $s \in [0, +\infty[$. Taking $T_k(A(u_n))$ as an admissible test function in (24), with

$$A(s) = \int_0^s g(\tau)^{1/(p^--1)} d\tau.$$

There exists $C_1 > 0$, and by assumption (3), we have

(26)

$$\begin{aligned} C_1\nu \int_{|A(u)|\leq k} |\nabla T_k(A(u_n))|^{p(x)} &\leq C_1 \int_{|A(u)|\leq k} \alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)} \\ &\leq C_1 \int_{|A(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^--1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)|\leq k} a_n(x, u_n, \nabla u_n)\nabla u_n g^{\frac{1}{p^--1}}(u_n) dx \\ &\leq \int_\Omega |f_n||T_k(A(u_n))| dx \\ &\leq k\|f\|_{L^1}. \end{aligned}$$

And consequently there exist $C_2,\ C_3 > 0$ such that

$$\int_\Omega |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_2 \int_\Omega \alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)} \leq C_3 k. \tag{27}$$

Therefore,

$$\|\nabla T_k(A(u_n))\|^r_{p(\cdot)} dx \leq C_3 k, \tag{28}$$

with

$$r = \begin{cases} p^+, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)} < 1, \\ p^-, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)} \geq 1. \end{cases}$$

Let $k \geq 1$, we have

$$k \text{ meas } \{|A(u_n)| > k\} = \int_{|A(u_n)|>k} |T_k(A(u_n))| dx \leq C_3 k^{\frac{1}{r}},$$

which implies that

$$\text{meas}\{|A(u_n)| > k\} \leq C_3 \frac{1}{k^{1-\frac{1}{r}}} \longrightarrow 0 \text{ as } \ k \to \infty.$$

By the usual method we get that $T_k(A(u_n))$ is bounded in $W_0^{1.p(\cdot)}(\Omega)$, then there exists a subsequence still denoted $(T_k(A(u_n)))_{n\in\mathbb{N}}$ such that

$$T_k(A(u_n)) \rightharpoonup \eta_k \ \text{ weakly in } \ W_0^{1.p(\cdot)}(\Omega),$$

and by the Sobolev compact embedding, we have

$$T_k(A(u_n)) \to \eta_k \ \text{ strongly in } \ L^{p(\cdot)}(\Omega), \ \text{ and a.e. in } \ \Omega.$$

Consequently, we can assume that $(T_k(A(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus,

$$\text{meas}\{|T_k(A(u_n)) - T_k(A(u_m))| > \delta\} \leq \frac{\varepsilon}{2}, \ \text{ for all } m, n \geq n_0(\delta, \varepsilon).$$

We conclude that, for all $\delta, \varepsilon > 0$ there exists $n_0 = n_0(\delta, \varepsilon)$ such that $meas\{|u_n - u_m| > \delta\} \leq \varepsilon$ for all $\delta, \varepsilon > 0$. It follows that $(T_k(A(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, then it converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{aligned} &T_k(A(u_n)) \rightharpoonup T_k(A(u)) \ \text{ weakly in } \ W_0^{1.p(\cdot)}(\Omega), \\ &T_k(A(u_n)) \to T_k(A(u)) \ \text{ strongly in } \ L^{p(\cdot)}(\Omega), \ \text{ and a.e. in } \ \Omega. \end{aligned} \tag{29}$$

Moreover by (27), (29) and Lemma 3.1 we get

$$\begin{aligned} &\alpha_n(x)^{\frac{1}{p(x)}} \nabla T_k(A(u_n)) \rightharpoonup \alpha(x)^{\frac{1}{p(x)}} \nabla T_k(A(u)) \ \text{ weakly in } \ L^{p(\cdot)}(\Omega)^N, \text{and} \\ &\alpha(x)|\nabla T_k(A(u))|^{p(x)} \in L^1(\Omega). \end{aligned} \tag{30}$$

Step 2: Strong convergence of truncations. Let $w_n = T_{2k}(Z_n)$ be a test function in (24), where $Z_n = (A(u_n) - T_h(A(u_n)) + T_k(A(u_n)) - T_k(A(u)))$, for $h > k > 0$. Taking $M = 4k + h$ we get that,

$$\int_\Omega a_n(x, u_n, \nabla u_n) \nabla w_n dx \leq \int_\Omega |f_n||w_n| dx.$$

We have

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla w_n dx = \int_{\{|A(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n dx + \int_{\{|A(u_n)|\le k\}} a_n(x,u_n,\nabla u_n)\nabla w_n dx. \tag{31}$$

Concerning the first term on the right hand side of (31), since $\nabla w_n = 0$ on $\{|A(u_n)| > M\}$, we have

$$\begin{aligned}\int_{\{|A(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n dx &= \int_{\{|A(u_n)|>k\}} a(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla z_n \\ &\ge -\int_{\{|A(u_n)|>k\}} a_n(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla T_k(A(u))dx \\ &\ge -\varepsilon_0(n).\end{aligned}$$

Where $\widehat{M}$ is a positive constant depending on M such that $\{|A(u_n)| > M\}$, can be equivalently replaced by $\{|u_n| > \widehat{M}\}$.

For the second term on the right hand side of (31); we have for $\widehat{k}$ defined as $\widehat{M}$

$$\int_{\{|A(u_n)|\le k\}} a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n)) - \nabla T_k(A(u)))dx \le \int_\Omega |f_n||w_n|dx + \varepsilon_0(n).$$

On the other hand, we have

$$\begin{aligned}&\int_{\{|A(u_n)|\le k\}} a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n)) - \nabla T_k(A(u)))dx \\ &= \int_{\{|A(u_n)|\le k\}} (a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n)) - a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))) \\ &\times (\nabla T_k(A(u_n)) - \nabla T_k(A(u)))dx \\ &+ \int_{\{|A(u_n)|\le k\}} a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))(\nabla T_k(A(u_n)) - \nabla T_k(A(u)))dx.\end{aligned}$$

By (30) and the definition of $a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))$ the third terms of the right hand tend to 0, as n tend to infinity. So, we have

$$\begin{aligned}&\int_{\{|A(u_n)|\le k\}} (a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n)) - a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))) \\ &\times (\nabla T_k(A(u_n)) - \nabla T_k(A(u)))dx \\ &= \int_{\{|A(u_n)|\le k\}} a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n)) - \nabla T_k(A(u)))dx + \varepsilon_1(n) \\ &\le \int_\Omega |f_n(x)||w_n|dx + \varepsilon_1(n) + \varepsilon_0(n) \le \varepsilon_2(n),\end{aligned} \tag{32}$$

then

$$\int_\Omega [a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) - a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n))] \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u)))dx \le \varepsilon_2(n),$$

as $f_n \to f$ strongly in $L^1(\Omega)$, and $w_n \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$. So, by the Lemma 2.5, we conclude that

$$\begin{aligned} &T_k(A(u_n)) \to T_k(A(u)) \text{ in } W_0^{1,p(\cdot)}(\Omega), \\ &\nabla T_k(A(u_n)) \to \nabla T_k(A(u)) \text{ almost everywhere in } \Omega. \end{aligned}$$

If g is unbounded, we have

$$\begin{aligned} |\nabla T_k(A(u_n))|^{p(x)} =& |\nabla A(u_n)|^{p(x)} \chi_{|A(u_n)| \le k} \\ =& g^{\frac{p(x)}{p^+-1}}(u_n) \chi_{|A(u_n)| \le k} |\nabla u_n|^{p(x)}, \\ g^{p(x)/(p^+-1)}(u_n) \chi_{|u_n| \le \widehat{k}} |\nabla u_n|^{p(x)} =& g^{p(x)/(p^+-1)}(T_{\widehat{k}}(u_n)) |\nabla T_{\widehat{k}}(u_n)|^{p(x)}, \\ |\nabla T_{\widehat{k}}(u_n)|^{p(x)} =& \frac{|\nabla T_k(A(u_n))|^{p(x)}}{g^{\frac{p(x)}{p^+-1}}(T_{\widehat{k}}(u_n))}, \end{aligned}$$

and if g is bounded we have

$$\begin{aligned} &|gT_k(A(u_n))|^{p(x)} = |\nabla A(u_n)|^{p(x)} \chi_{|A(u_n)| \le k} = g^{\frac{p(x)}{p^--1}}(u_n) \chi_{|A(u_n)| \le k} |\nabla u_n|^{p(x)}, \\ &g^{\frac{p(x)}{p^--1}}(u_n) \chi_{|u_n| \le \widehat{k}} |\nabla u_n|^{p(x)} = g^{\frac{p(x)}{p^--1}}(T_{\widehat{k}}(u_n)) |\nabla T_m(u_n)|^{p(x)}, \\ &|\nabla T_{\widehat{k}}(u_n)|^{p(x)} = \frac{|\nabla T_k(A(u_n))|^{p(x)}}{g^{\frac{p(x)}{p^--1}}(T_{\widehat{k}}(u_n))}, \end{aligned}$$

since g continuous we have $g(T_{\widehat{k}}(u_n)) \ge \min_{[0,\widehat{k}]}(g(s)) = g_{\widehat{k}}$. Finally, we have

$$|\nabla T_m(u_n)|^{p(x)} \le C |\nabla T_k(A(u_n))|^{p(x)}. \tag{33}$$

and on the other, by Lemma 3.1, that for every $k > 0$

$$\alpha(x) |\nabla T_k(u)|^{p(x)} \in L^1(\Omega),$$

and

$$\alpha_n(x)^{\frac{1}{p(x)}} \nabla T_k(u_n) \rightharpoonup \alpha(x)^{\frac{1}{p(x)}} \nabla T_k(u) \ \ \textit{weakly in} \ \ (L^{p(x)}(\Omega))^N. \tag{34}$$

Step 3: Entropy solutions.

Case 1. The function α is unbounded, taking $T_k(A(u_n))$ as test function in (24), with $A(s) = \int_0^s g(\tau)^{1/(p^- -1)} d\tau$, we obtaining

$$\begin{aligned} C_4 \int_{|A(u)|\le k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} \\ \le \int_{|A(u)|\le k} a_n(x,u_n,\nabla u_n)\nabla T_k(u_n) g^{\frac{1}{p^- -1}}(u_n)dx \\ \le \int_\Omega |f_n||T_k(A(u_n))|dx \\ \le k\|f\|_{L^1}. \end{aligned} \tag{35}$$

Case 2. The function g is bounded, taking $T_k(A(u_n))$ as test function in (24), with $A(s) = \int_0^s g(\tau)^{1/(p^+ -1)} d\tau$, we have

$$\begin{aligned} C_5 \int_{|A(u)|\le k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} \\ \le \int_{|A(u)|\le k} a_n(x,u_n,\nabla u_n)\nabla T_k(u_n) g^{\frac{1}{p^+ -1}}(u_n)dx \\ \le \int_\Omega |f_n||T_k(A(u_n))|dx \\ \le k\|f\|_{L^1}. \end{aligned} \tag{36}$$

By means of (35) and (36), we obtain

$$\int_\Omega \alpha_n(x) g(u_n)^{p'(x)} |\nabla T_{\widehat{k}}(u_n)|^{p(x)} \le C_6 k\|f\|_{L^1}. \tag{37}$$

Let $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, taking $T_k(u_n - \phi)$ as a test function in the approximate problem (24), we get

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)dx = \int_\Omega f_n T_k(u_n-\phi)dx.$$

Adding and subtracting in the equation above the term

$$\int_\Omega a_n(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)dx,$$

and taking advantage of the monotonicity condition (4) we get

$$\int_\Omega a_n(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)dx \le \int_\Omega f_n T_k(u_n-\phi)dx. \tag{38}$$

Noticing that $\{|u_n-\phi|\le k\} \subset \{|u_n| \le k + \|\phi\|_{L^\infty}\}$ and that

$$\int_\Omega a_n(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)dx = \int_\Omega \frac{a(x,T_n(u_n),\nabla\phi)}{\alpha(x)} \alpha_n(x)\nabla T_k(u_n-\phi)dx,$$

we can pass to the limit in (38) using assumption (2), (33), (34) and (37). Hence we obtain, for every $k > 0$,

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)dx \le \int_\Omega fT_k(u-\phi)dx. \tag{39}$$

To recover (19), let h and k be positive real numbers, let $t = \pm 1$, and let ψ be a function in $W_0^{1.p(x)}(\Omega) \cap L^\infty(\Omega)$. Choose $\phi = T_h(u) + tT_k(u-\psi)$ in the inequality in (39) we obtain

$$\begin{aligned}\int_\Omega a(x,u,\nabla T_h(u){+}t\nabla T_k(u-\psi))\nabla T_k(G_h(u)-tT_k(u-\psi))dx\\ \le \int_\Omega fT_k(G_h(u)-tT_k(u-\psi))dx,\end{aligned} \tag{40}$$

and we follow the same steps as the Lemma 7 in [5] we get

$$-t\int_\Omega a(x,u,\nabla u+t\nabla T_k(u-\psi))\nabla T_k(u-\psi)dx \le -t\int_\Omega fT_k(u-\psi)dx. \tag{41}$$

for every $\psi \in W_0^{1.p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, and for every $k > 0$. Choosing $t > 0$, dividing by t, and then letting t tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\psi)dx \ge \int_\Omega fT_k(u-\psi)dx.$$

While the reverse inequality is obtained choosing $t < 0$, dividing by $-t$, and then letting t tend to zero.

Step 4: weak solutions. In this last part of the proof we consider the stronger assumption $f \in L^{(p^*(\cdot))'}(\Omega)$. Recalling that $u_n \in W_0^{1.p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we can choose $A(u_n)$ as test function in (24), where $A(s) = \int_0^s g(\tau)^{1/(p^+-1)}d\tau$. We obtain

$$\begin{aligned}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n g^{\frac{1}{p^+-1}}(u_n)dx &\le \int_\Omega |f||T_k(A(u_n))|dx\\ &\le \|A(u_n)\|_{L^{p*(x)}}\|f\|_{L^{(p*(x))'}}.\end{aligned} \tag{42}$$

Using Sobolev inequality, we get

$$\int_\Omega \alpha_n(x)g(u_n)^{p'(x)}|\nabla u_n|^{p(x)} \le \frac{(C_7\|f\|_{L^{(p^*(x))'}})^{p^{+'}}}{\nu^{\frac{1}{1+p^+}}}. \tag{43}$$

Thanks to the a.e. convergence of ∇u_n to ∇u and Fatou Lemma,

$$\int_\Omega \alpha(x)g(u)^{p'(x)}|\nabla u|^{p(x)} \le \frac{(C_7\|f\|_{L^{(p^*(x))'}})^{p^{+'}}}{\nu^{\frac{1}{1+p^+}}}. \tag{44}$$

Thus, we have that the sequence

$$\frac{a(x,T_n(u_n),\nabla u_n)}{\alpha(x)}\alpha_n(x)^{\frac{1}{p'(x)}} \text{ is bounded in } (L^{p'(\cdot)}(\Omega))^N,$$

and converges almost everywhere to $a(x,u,\nabla u)\alpha(x)^{\frac{1}{p'(x)}-1}$. Hence, it also converges weakly in $(L^{p'(\cdot)}(\Omega))^N$ to its a.e.-limit. Noticing that, for every $\phi \in X_0^{p(\cdot)}(\Omega)$

$$\alpha_n(x)^{\frac{1}{p(x)}}\nabla\phi \to \alpha(x)^{\frac{1}{p(x)}}\nabla\phi \text{ strongly in } L^{p(\cdot)}(\Omega)^N,$$

we can pass to the limit in (24) and conclude the proof.

3.1 Proof of Theorem 3.2

Let us divide the proof in five steps:

Step 1: Approximation and a priori estimates.
Step 2: Strong convergence of truncations.
Step 3: The equi-integrability of $H_n(x,u_n,\nabla u_n)$.
Step 4: Entropy solutions.
Step 5: Weak solutions case $f \in L^{{p^*}'(\cdot)}(\Omega)$.

Step 1: Approximation and a priori estimates. Let $(f_n)_{u\in\mathbb{N}}$ be a sequence of smooth function such $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$, we consider the following problem

$$\begin{cases} -\operatorname{div} a_n(x,u_n,\nabla u_n) + H_n(x,u_n,\nabla u_n) = f_n, \quad \text{in} \ \ \Omega, \\ u_n \in W_0^{1.p(\cdot)}(\Omega), \end{cases} \tag{45}$$

where

$$H_n(x,s,\xi) = \min[T_n(\beta_n(x)h_n(s)|\xi|^{p(x)}), \max(-T_n(\beta_n(x)h_n(s)|\xi|^{p(x)}), H_n(x,s,\xi))],$$
$$h_n(s) = g_n(s)\frac{h(s)}{g(s)}, \quad \beta_n(x) = T_n(\beta(x)), \text{and } \beta_n(x) \le \theta\alpha_n(x).$$

Note that, the function H_n is bounded and

$$|H_n(x,s,\xi)| \le T_n(\beta_n(x)h_n(s)|\xi|^{p(x)-1}) \le \beta_n(x)h_n(s)|\xi|^{p(x)-1}. \tag{46}$$

We also, observe that $\frac{h_n}{g_n} = \frac{h}{g} \in L^1(\mathbb{R})$ and

$$h_n \le \max\{g(s) :| s |\le n\}\frac{h}{g}, \text{ so that } \ h_n \in L^1(\mathbb{R}).$$

Let $(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}$ be a test function in (45), we have

$$\begin{aligned}
&\int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(A(u_n))e^{\theta\eta(u_n)}dx \\
&\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \\
&\qquad + \int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n \frac{\theta h(u_n)}{g(u_n)}(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \\
&\qquad = \int_\Omega f_n (T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx.
\end{aligned} \tag{47}$$

By (3) and (9), we have

$$\int_\Omega H_n(x,u_n,\nabla u_n)(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx + \int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n \frac{\theta h(u_n)}{g(u_n)}(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \geq 0.$$

It follows that

$$\int_{A(u_n)>0} a_n(x,u_n,\nabla u_n)\nabla (T_k(A(u_n)))e^{\theta\eta(u_n)}dx \leq \int_\Omega |f_n|(T_k(A(u_n)))^+ e^{\theta\eta(u_n)},$$

so we get, if g unbounded,

$$\begin{aligned}
&\int_{0<A(u_n)\leq k} a_n(x,u_n,\nabla u_n)\nabla u_n g^{\frac{1}{p^+-1}}(u_n)e^{\theta\eta(u_n)}dx \\
&\leq \int_\Omega |f_n||T_k(A(u_n))|e^{\theta\eta(u_n)}dx.
\end{aligned} \tag{48}$$

By the assumption (3), there exist $C_8,\ C_9 > 0$ such that

$$\begin{aligned}
&\nu \int_{0<A(u_n)\leq k} |\nabla T_k(A(u_n))|^{p(x)}dx \leq C_8 \int_{0<A(u_n)\leq k} \alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)} \\
&\leq \int_{0<A(u_n)\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^+-1}}|\nabla u_n|^{p(x)} \leq C_9 k\|f\|_{L^1},
\end{aligned} \tag{49}$$

and if g bounded we obtain

$$\begin{aligned}
&\int_{0<A(u_n)\leq k} a_n(x,u_n,\nabla u_n)\nabla u_n g^{\frac{1}{p^--1}}(u_n)e^{\theta\eta(u_n)}dx \\
&\leq \int_\Omega |f_n||T_k(A(u_n))|e^{\theta\eta(u_n)}dx.
\end{aligned} \tag{50}$$

Using again the assumption (3), there exist $C_{10}, C_{11} > 0$ such that

$$\begin{aligned}
&\nu \int_{0<A(u_n)\leq k} |\nabla T_k(A(u_n))|^{p(x)}dx \leq C_{10} \int_{0<A(u_n)\leq k} \alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)} \\
&\leq \int_{0<A(u_n)\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^--1}}|\nabla u_n|^{p(x)} \leq C_{11} k\|f\|_{L^1},
\end{aligned} \tag{51}$$

By using the test function $-(T_k(A(u_n)))^- e^{-\eta(u_n)}$, and reasoning as before we get

$$\begin{aligned}&\nu\int_{-k<A(u_n)\le 0}|\nabla T_k(A(u_n))|^{p(x)}dx\\ &\le C_{12}\int_{-k<A(u_n)\le 0}\alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)}\le C_{13}k\|f\|_{L^1}.\end{aligned}\tag{52}$$

Combining (51) and (52), we get

$$\nu\int_\Omega|\nabla T_k(A(u_n))|^{p(x)}\le C_{14}\int_\Omega\alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)}\le C_{15}k\|f\|_{L^1}.\tag{53}$$

Therefore,

$$\|\nabla T_k(A(u_n))\|^{r_1}_{p(\cdot)}dx\le C_{15}k,\tag{54}$$

$$r_1=\begin{cases}p^+, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)}<1,\\ p^-, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)}\ge 1.\end{cases}$$

That implies $(T_k(A(u_n)))_{n\in\mathbb{N}}$ converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{aligned}&T_k(A(u_n))\rightharpoonup T_k(A(u)) \ \text{ weakly in } \ W_0^{1.p(\cdot)}(\Omega),\\ &T_k(A(u_n))\to T_k(A(u)) \ \text{ strongly in } \ L^{p(\cdot)}(\Omega), \ \text{ and a.e. in } \ \Omega,\end{aligned}$$

and on the other, by Lemma 3.1, that for every $k>0$

$$\begin{aligned}&\alpha_n(x)^{\frac{1}{p(x)}}\nabla T_k(A(u_n))\rightharpoonup\alpha(x)^{\frac{1}{p(x)}}\nabla T_k(A(u)) \ \text{ weakly in } \ L^{p(\cdot)}(\Omega)^N, \text{and}\\ &\alpha(x)|\nabla T_k(A(u))|^{p(x)}\in L^1(\Omega).\end{aligned}\tag{55}$$

Step 2: Strong convergence of truncations. Let $w_n^+e^{\theta\eta(u_n)}$ be a test function in (45), where $w_n=T_{2k}(Z_n)$ be a test function in (24), where $Z_n=(A(u_n)-T_h(A(u_n))+T_k(A(u_n))-T_k(A(u)))$, for $h>k>0$. Taking $M=4k+h$, we have

$$\begin{aligned}&\int_{w_n>0}a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx+\int_{w_n>0}H_n(x,u_n,\nabla u_n)w_n^+e^{\theta\eta(u_n)}dx\\ &+\theta\int_{w_n>0}a_n(x,u_n,\nabla u_n)\nabla u_n\frac{h(u_n)}{g(u_n)}w_n^+e^{\theta\eta(u_n)}dx=\int_{w_n>0}f_nw_n^+e^{\theta\eta(u_n)}dx,\end{aligned}\tag{56}$$

by (3) and (9), we have

$$\begin{aligned}&\int_{w_n>0}H_n(x,u_n,\nabla u_n)w_n^+e^{\theta\eta(u_n)}dx\\ &\qquad\qquad+\theta\int_{w_n>0}a_n(x,u_n,\nabla u_n)\nabla u_n\frac{h(u_n)}{g(u_n)}w_n^+e^{\theta\eta(u_n)}dx\ge 0.\end{aligned}$$

So, we get that,

$$\int_{w_n>0} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx \leq \int_{w_n>0} f_n w_n^+ e^{\gamma(u_n)}dx.$$

We have

$$\begin{aligned}
&\int_{\{w_n>0\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx\\
&= \int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx \qquad (57)\\
&+ \int_{\{w_n>0\}\cap\{|A(u_n)|\leq k\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx.
\end{aligned}$$

Concerning the first term on the right hand side of (57); since $\nabla w_n = 0$ on $\{|A(u_n)|>M\}$, we have

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx\\
&= \int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a_n(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla Z_n e^{\theta\eta(u_n)}\\
&\geq -\int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a_n(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla T_k(A(u))e^{\theta\eta(u_n)}dx\\
&\geq -e^{\theta\eta(\infty)}\int_{\{|A(u_n)|>k\}} a_n(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla T_k(A(u))dx\\
&\geq -\varepsilon_3(n).
\end{aligned}$$

For the second term on the right hand side of (57); we have for $\widehat{k}>0$

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|A(u_n)|\leq k\}} a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n))-\nabla T_k(A(u)))e^{\theta\eta(u_n)}dx\\
&\qquad\qquad \leq e^{\theta\eta(\infty)}\int_{\{w_n>0\}} |f_n||w_n|dx + \varepsilon_3(n).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|A(u_n)|\leq k\}} a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))[\nabla T_k(A(u_n))-\nabla T_k(A(u))]e^{\theta\eta(u_n)}dx\\
&= \int_{\{w_n>0\}\cap\{|A(u_n)|\leq k\}} [a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n)) - a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))]\\
&\times [\nabla T_k(A(u_n))-\nabla T_k(A(u))]e^{\theta\eta(u_n)}dx\\
&+ \int_{\{w_n>0\}\cap\{|A(u_n)|\leq k\}} a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))[\nabla T_k(A(u_n))-\nabla T_k(A(u))]e^{\theta\eta(u_n)}dx.
\end{aligned}$$

The third terms of the right hand tend to 0, as n tend to infinity. Thanks to step 2 of theorem 3.1. So, we have

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|A(u_n)|\leq k\}}[a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))\\
&-a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))][\nabla T_k(A(u_n))-\nabla T_k(A(u))]e^{\theta\eta(u_n)}dx\\
&=\int_{\{|A(u_n)|\leq k\}}a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))[\nabla T_k(A(u_n))-\nabla T_k(A(u))]e^{\theta\eta(u_n)}dx\\
&+\varepsilon_4(n)\leq e^{\theta\eta(\infty)}\int_{\{w_n>0\}}|f_n(x)||w_n|dx+\varepsilon_4(n)+\varepsilon_3(n)\leq\varepsilon_5(n),
\end{aligned} \tag{58}$$

as $f_n \to f$ strongly in $L^1(\Omega)$, and $w_n \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$.

Let $-(w_n)^- \exp(-\theta\eta(u_n))$ be a test function in the problem (45), we obtain

$$\begin{aligned}
&\int_{w_n<0}a_n(x,u_n,\nabla u_n)\nabla w_n e^{-\theta\eta(u_n)}dx\\
&+\int_{w_n<0}-H_n(x,u_n,\nabla u_n)(w_n)^- e^{-\theta\eta(u_n)}dx\\
&+\theta\int_{w_n<0}a_n(x,u_n,\nabla u_n)-\nabla u_n\frac{h(u_n)}{g(u_n)}-(w_n)^- e^{-\theta\eta(u_n)}dx\\
&=\int_{w_n<0}-f_n(w_n)^- e^{-\theta\eta(u_n)}dx,
\end{aligned} \tag{59}$$

Reasoning as before, we get that

$$\int_{\{w\leq 0\}\cap\{|A(u_n)|>k\}}a(x,u_n,\nabla u_n)\nabla w_n e^{-\theta\eta(u_n)}dx\geq-\varepsilon_6(n),$$

where $\varepsilon_6(n)$ tend to 0 as n tend to infinity.

$$\begin{aligned}
\int_{w\leq 0}a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_{\widehat{k}}(u_n)-\nabla T_{\widehat{k}}(u))e^{-\theta\eta(u_n)}dx&\\
\leq\int_{w\leq 0}|f_n||w_n|e^{-\theta\eta(u_n)}dx+\varepsilon_6(n).&
\end{aligned}$$

Since w_n tend to 0 weakly * in $L^\infty(\Omega)$, and f_n converge strongly to f in $L^1(\Omega)$ we conclude that

$$\int_{w\leq 0}a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n))-\nabla T_k(A(u)))e^{-\theta\eta(u_n)}dx\leq\varepsilon_7(n).$$

By adding the term to the last expression, we get

$$\begin{aligned}
\int_{w\leq 0}&[a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))-a_n(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))]\\
&\times(\nabla T_k(A(u_n))-\nabla T_k(A(u)))e^{-\theta\eta(u_n)}dx\leq\varepsilon_7(n).
\end{aligned} \tag{60}$$

Combining (58) and (60), we get

$$\int_{\Omega}[a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) - a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n))]\times \\ (\nabla T_k(A(u_n)) - \nabla T_k(A(u)))e^{-\theta\eta(u_n)}dx \leq \varepsilon_8(n),$$

so by the Lemma 2.5, we conclude that

$$T_k(A(u_n)) \to T_k(A(u)) \text{ in } W_0^{1,p(\cdot)}(\Omega), \nabla A(u_n) \to \nabla A(u) \text{a.e. in } \Omega,$$

moreover we have

$$|\nabla T_m(u_n)|^{p(x)} \leq c|\nabla T_k(A(u_n))|^{p(x)}, \tag{61}$$

thanks to the continuity of g.

By Lemma 3.1, for every $k > 0$ we have

$$\alpha(x)|\nabla T_k(u)| \in L^1(\Omega), \tag{62}$$

and

$$\alpha_n(x)^{\frac{1}{p(x)}}\nabla T_k(u_n) \rightharpoonup \alpha(x)^{\frac{1}{p(x)}}\nabla T_k(u) \quad \text{weakly in } (L^{p(x)}(\Omega))^N. \tag{63}$$

Step 3: The equi-integrability of $H_n(x, u_n, \nabla u_n)$. In order to pass to the limit in the approximate problem, we shall show that

$$H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \text{ in } L^1(\Omega). \tag{64}$$

Let E a set of Ω such that meas$(E) = 0$, and $l > 0$. We have

$$\begin{aligned}
&\int_E |H_n(x, u_n, \nabla u_n)|dx \leq \int_E \beta(x)h(u_n)|\nabla u_n|^{p(x)}dx \\
&= \int_{E\cap\{|u_n|>l\}} \beta(x)h(u_n)|\nabla u_n|^{p(x)}dx + \int_{E\cap\{|u_n|\leq l\}} \beta(x)h(u_n)|\nabla u_n|^{p(x)}dx \\
&= \int_{E\cap\{|u_n|>l\}} \beta(x)h(u_n)|\nabla u_n|^{p(x)}dx + \int_{E\cap\{|u_n|\leq l\}} \beta(x)h(T_l(u_n))|\nabla T_l(u_n)|^{p(x)}dx \\
&\leq \int_{E\cap\{|u_n|>l\}} \beta(x)h(u_n)|\nabla u_n|^{p(x)}dx + \max_{[0,l]}(h(s)) \int_{E\cap\{|u_n|\leq l\}} \beta(x)|\nabla T_l(u_n)|^{p(x)}dx.
\end{aligned}$$

From (62), we deduce that the second term of the right hand side of the last inequality equal to 0 as meas$(E) = 0$. Prove that

$$\int_{E\cap\{|u_n|>l\}} \beta(x)h(u_n)|\nabla T_l(u_n)|^{p(x)}dx \to 0.$$

Let $(T_1(u_n - T_l(u_n))^+ \exp(2\theta\eta(u_n))$ be a test function in the problem (45). We have

$$\begin{aligned}&\int_{l<u_n\le l+1} a_n(x,u_n,\nabla u_n)\nabla T_1(u_n - T_l(u_n))\exp(2\theta\eta(u_n))dx\\&+\int_{l<u_n} 2\theta a_n(x,u_n,\nabla u_n)\nabla u_n\frac{h(u_n)}{g(u_n)}(T_1(u_n - T_l(u_n)))^+\exp(2\theta\eta(u_n))dx\\&+\int_{l<u_n} H_n(x,u_n,\nabla u_n)T_1(u_n - T_l(u_n))^+\exp(2\theta\eta(u_n))dx\\&=\int_{l<u_n} f_n(T_1(u_n - T_l(u_n)))^+\exp(2\eta(u_n))dx.\end{aligned}$$

By assumption (3) and (9) we have

$$\begin{aligned}&\int_{l<u_n}|\nabla u_n|^{p(x)}\beta(x)h(u_n)(T_1(u_n - T_l(u_n))^+\exp(2\theta\eta(u_n))dx\\&\le C_{16}\int_{l<u_n}|f|dx.\end{aligned}\tag{65}$$

Let $-(T_1(u_n - T_l(u_n))^- \exp(-2\theta\eta(u_n))$ be a test function in the problem (45). Reason as above, we get

$$\begin{aligned}&\int_{u_n<-l}|\nabla u_n|^{p(x)}\beta(x)h(u_n)(T_1(u_n - T_l(u_n))^+\exp(2\eta(u_n))dx\\&\le C_{17}\int_{u_n<-l}|f|dx.\end{aligned}\tag{66}$$

From (65) and (66) we conclude that

$$\int_{l<|u_n|}|\nabla u_n|^{p(x)}\beta(x)h(u_n)dx \le C_{18}\int_{l<|u_n|}|f|dx.\tag{67}$$

Let l tend to infinity we get

$$\int_{l<|u_n|}|\nabla u_n|^{p(x)}\beta(x)h(u_n)dx \to 0.$$

Finally, we get the equi-integrability of H.

Step 4: Entropy solutions. Let $(T_k(A(u_n)))^+e^{\theta\eta(u_n)}$ be a test function in (45), we have

$$\begin{aligned}&\int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(A(u_n))e^{\theta\eta(u_n)}dx\\&\qquad+\int_\Omega H_n(x,u_n,\nabla u_n)(T_k(A(u_n)))^+e^{\theta\eta(u_n)}dx\\&\qquad+\int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n\frac{\theta h(u_n)}{g(u_n)}(T_k(A(u_n)))^+e^{\theta\eta(u_n)}dx\\&\qquad=\int_\Omega f_n(T_k(A(u_n)))^+e^{\theta\eta(u_n)}dx.\end{aligned}\tag{68}$$

By (3) and (9), we have

$$\int_\Omega H_n(x,u_n,\nabla u_n)(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx + \int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n \frac{\theta h(u_n)}{g(u_n)}(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \geq 0.$$

It follows that

$$\int_{A(u_n)>0} a_n(x,u_n,\nabla u_n)\nabla(T_k(A(u_n)))e^{\theta\eta(u_n)}dx \leq \int_\Omega |f_n|(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}.$$

If the function g is unbounded, taking $A(s)=\int_0^s g(\tau)^{1/(p^- -1)}d\tau$, and by (3) we have

$$\begin{aligned} &C_{19}\int_{0<A(u)\leq k} \alpha_n(x) g_n(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta\eta(u_n)} \\ &\leq \int_{0<A(u)\leq k} a_n(x,u_n,\nabla u_n)\nabla T_k(u_n) g^{\frac{1}{p^- -1}}(u_n) e^{\theta\eta(u_n)}dx \\ &\leq \int_\Omega |f_n||T_k(A(u_n))|e^{\theta\eta(u_n)}dx \\ &\leq C_{20}k\|f\|_{L^1}. \end{aligned} \tag{69}$$

Else, taking $A(s)=\int_0^s g(\tau)^{1/(p^+ -1)}d\tau$, and by (3) we obtain

$$\begin{aligned} &C_{21}\int_{0<A(u)\leq k} \alpha_n(x) g_n(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta\eta(u_n)} \\ &\leq \int_{0<A(u)\leq k} a_n(x,u_n,\nabla u_n)\nabla T_k(u_n) g^{\frac{1}{p^+ -1}}(u_n) e^{\theta\eta(u_n)}dx \\ &\leq \int_\Omega |f_n||T_k(A(u_n))|e^{\theta\eta(u_n)}dx \\ &\leq C_{22}k\|f\|_{L^1}. \end{aligned} \tag{70}$$

By means of (69) and (70), we obtain

$$\int_{0<A(u)\leq k} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \leq C_{23}k\|f\|_{L^1}. \tag{71}$$

Let $(T_k(A(u_n)))^- \exp(-\theta\eta(u_n))$ be a test function in the problem (45). Reason as above, we get

$$\int_{-k<A(u)\leq 0} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \leq C_{24}k\|f\|_{L^1}. \tag{72}$$

Combining (71) and (72), we have

$$\int_\Omega \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \leq C_{25}k\|f\|_{L^1}. \tag{73}$$

Let $\varphi \in X_0^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, where $X_0^{p(\cdot)}(\Omega)$ is the space defined in (16), taking $\phi = T_k(u_n - \varphi)$ as a test function in the approximate problem, we get

$$\int_\Omega a_n(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)dx + \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - \varphi)dx$$
$$= \int_\Omega f_n T_k(u_n - \varphi)dx.$$

Adding and subtracting in the equation above the term

$$\int_\Omega a_n(x, u_n, \nabla\varphi)\nabla T_k(u_n - \varphi)dx$$

we obtain

$$\int_\Omega \Big[a_n(x, u_n, \nabla u_n) - a_n(x, u_n, \nabla\varphi)\Big]\nabla T_k(u_n - \varphi)dx$$
$$+ \int_\Omega a_n(x, u_n, \nabla\varphi)\nabla T_k(u_n - \varphi)dx$$
$$+ \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - \varphi)dx = \int_\Omega f_n T_k(u_n - \varphi)dx.$$

We then have, once again by (4)

$$\int_\Omega a_n(x, u_n, \nabla\varphi)\nabla T_k(u_n - \varphi)dx$$
$$+ \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - \varphi)dx \le \int_\Omega f_n T_k(u_n - \varphi)dx. \tag{74}$$

Noticing that $\{|u_n - \varphi| \le k\} \subset \{|u_n| \le k + \|\varphi\|_{L^\infty}\}$ and that

$$\int_\Omega a_n(x, u_n, \nabla\varphi)\nabla T_k(u_n - \varphi)dx = \int_\Omega \frac{a(x, T_n(u_n), \nabla\varphi)}{\alpha(x)}\alpha_n(x)\nabla T_k(u_n - \varphi)dx$$

we can pass to the limit in (74) using assumption (2), the a.e. convergence of u_n and the weak convergence of $\alpha(x)^{\frac{1}{p(x)}}\nabla T_k(u_n)$ proved in Step 2. Hence, we obtain, for every $k > 0$,

$$\int_\Omega a(x, u, \nabla\varphi)\nabla T_k(u - \varphi)dx$$
$$+ \int_\Omega H(x, u, \nabla u)T_k(u - \varphi)dx \le \int_\Omega f T_k(u - \varphi)dx. \tag{75}$$

As in the last part of Theorem 3.1 we take advantage of Lemma 7 of [5] to infer that, for every $k > 0$,

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi) + \int_\Omega H(x, u, \nabla u)T_k(u - \varphi) = \int_\Omega f T_k(u - \varphi),$$

$\forall \varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Step 5: weak solutions. Let $f \in L^{(p^*(\cdot))'}(\Omega)$, for each $n \in \mathbb{N}$, there exists a weak solution $u_n \in W_0^{1.p(\cdot)}(\Omega)$, which is an admissible test function in the weak sense (45). By Theorem 2.10 in [1], we have $u_n \in L^\infty(\Omega)$, and so $A(u_n) \in W_0^{1.p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ for all $n \in \mathbb{N}$, where $A(s) = \int_0^s g(\tau)^{1/(p^- -1)} d\tau$.

Let $v = e^{\theta\eta(u_n)}(A(u_n))^+$ be a test function in (45), we obtain

$$\begin{aligned} &\int_{A(u_n)\geq 0} a_n(x,u_n,\nabla u_n)\nabla A(u_n) e^{\theta\eta(u_n)} dx \\ &+ \int_{A(u_n)\geq 0} H_n(x,u_n,\nabla u_n) A(u_n) e^{\theta\eta(u_n)} dx \\ &+ \int_{A(u_n)\geq 0} a_n(x,u_n,\nabla u_n) A(u_n) \frac{\theta h(u_n)}{g(u_n)} e^{\theta\eta(u_n)} \nabla u_n dx \\ &= \int_{A(u_n)\geq 0} f A(u_n) e^{\theta\eta(u_n)} dx. \end{aligned} \tag{76}$$

On the other hand, by (3) and (9) we have

$$\begin{aligned} \int_{A(u_n)\geq 0} a_n(x,u_n,\nabla u_n) A(u_n) \frac{\theta h(u_n)}{g(u_n)} e^{\theta\eta(u_n)} \nabla u_n dx \\ + \int_{A(u_n)\geq 0} H_n(x,u_n,\nabla u_n) A(u_n) e^{\theta\eta(u_n)} dx \geq 0 \end{aligned}$$

So, (76) becomes

$$\int_{A(u_n)\geq 0} a_n(x,u_n,\nabla u_n)\nabla A(u_n) e^{\theta\eta(u_n)} dx \leq \int_{A(u_n)\geq 0} f A(u_n) e^{\theta\eta(u_n)} dx. \tag{77}$$

By the assumption (3), there exists $C_{26} > 0$ such that

$$\begin{aligned} &C_{26} \int_{A(u_n)\geq 0} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta\eta(u_n)} \\ &\leq \int_{A(u_n)\geq 0} f A(u_n) e^{\theta\eta(u_n)} dx. \end{aligned} \tag{78}$$

Let $v = -e^{-\theta\eta(u_n)}(A(u_n))^-$ be a test function in (45), and reasing as above we get

$$\begin{aligned} &C_{27} \int_{A(u_n)\leq 0} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta\eta(u_n)} \\ &\leq \int_{A(u_n)\leq 0} f A(u_n) e^{\theta\eta(u_n)} dx \end{aligned} \tag{79}$$

Adding up (78) and (79), we conclude that there exists $C_{28} > 0$ such that

$$\begin{aligned}
&\int_\Omega \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} \\
&\le C_{28} \int_\Omega f A(u_n) dx \le C_{28} \|A(u_n)\|_{L^{p*(x)}} \|f\|_{L^{(p*(x))'}}. \qquad (80)
\end{aligned}$$

Using Sobolev inequality and, thanks to the a.e. convergence of ∇u_n to ∇u and Fatou Lemma,

$$\int_\Omega \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} \le C_{29} (\|f\|_{(p^*(\cdot))'})^{p^{+'}}.$$

Thus, we have that the sequence

$$\frac{a(x, T_n(u_n), \nabla u_n)}{\alpha(x)} \alpha_n(x)^{\frac{1}{p'(x)}} \text{ is bounded in } (L^{p'(\cdot)}(\Omega))^N, \qquad (81)$$

and converges almost everywhere to $a(x, u, \nabla u) g(|u|)^{\frac{1}{p'(x)}-1}$ and weakly in $L^{p'(x)}(\Omega)^N$ to its a.e.-limit. Noticing that, for every $\phi \in X_0^{p(\cdot)}(\Omega)$

$$\alpha_n(x)^{\frac{1}{p(x)}} \nabla \phi \to \alpha(x)^{\frac{1}{p(x)}} \nabla \phi \text{ strongly in } L^{p(\cdot)}(\Omega)^N.$$

Let $\varphi \in X_0^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, then

$$\int_\Omega a_n(x, u_n, \nabla u_n) \varphi dx + \int_\Omega H_n(x, u_n, \nabla u_n) \varphi dx = \int_\Omega f \varphi dx, \qquad (82)$$

it follows from (64) and (81) that we may pass to the limit in (82) and conclude the proof.

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