# Generalized relative order $(\alpha, \beta)$ and generalized relative type $(\alpha, \beta)$ oriented some growth properties of composite entire and meromorphic functions

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**Abstract.** In this paper we wish to prove some results relating to the growth rates of composite entire and meromorphic functions with their corresponding left and right factors on the basis of generalized relative order $(\alpha, \beta)$ and generalized relative type $(\alpha, \beta)$ where $\alpha$ and $\beta$ are continuous non-negative functions defined on $(-\infty, +\infty)$.
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## 1. Introduction, definitions and notations

Let $g(z)$ be an entire function defined in the open complex plane $\mathbb{C}$ and let $M_g(r) = \max\{|g(z)| : |z| = r\}$ .When $f$ is meromorphic, the Nevanlinna's characteristic function $T_f(r)$(see [6, p.4]) plays the same role as $M_f(r)$. All the standard notations of the Nevanlinna theory of meromorphic functions which are available in [6, 7, 11]. We also use the standard notations and definitions of the theory of entire functions which are available in [10] and therefore we do not explain those in details. The Nevanlinna's characteristic function of a meromorphic function $f$ is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r, a) \left(\overline{N}_f(r, a)\right)$ known as counting function of $a$-points (distinct $a$-points) of meromorphic $f$ is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

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$$\left(\overline{N}_f(r,a)=\int_0^r\frac{\overline{n}_f(t,a)-\overline{n}_f(0,a)}{t}dt+\overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)\,(\overline{n}_f(r,a))$ the number of $a$-points (distinct $a$-points) of $f$ in $|z|\leq r$ and an $\infty$ -point is a pole of $f$. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of $f$ is defined as:

$$m_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,\quad\text{where}$$
$$\log^+x=\max(\log x,0)\ \text{ for all } x\geqslant 0.$$

Also, we may employ $m(r,\frac{1}{f-a})$ by $m_f(r,a)$.

If $f$ is entire, then the Nevanlinna's characteristic function $T_f(r)$ of $f$ is defined as

$$T_f(r)=m_f(r).$$

Moreover, if $f$ is non-constant entire then $T_f(r)$ is also strictly increasing and continuous functions of $r$. Therefore its inverse $T_f^{-1}:(T_f(0),\infty)\to(0,\infty)$ exists and is such that $\lim_{s\to\infty}T_f^{-1}(s)=\infty$.

Now, let $L$ be a class of continuous non-negative functions $\alpha$ defined on $(-\infty,+\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$. For any $\alpha\in L$, we say that $\alpha\in L_1^0$, if $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$ and $\alpha\in L_2^0$, if $\alpha(\exp((1+o(1))x))=(1+o(1))\alpha(\exp(x))$ as $x\to+\infty$. Finally for any $\alpha\in L$, we also say that $\alpha\in L_1$, if $\alpha(cx)=(1+o(1))\alpha(x)$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$ and $\alpha\in L_2$, if $\alpha(\exp(cx))=(1+o(1))\alpha(\exp(x))$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$. Clearly, $L_1\subset L_1^0$, $L_2\subset L_2^0$ and $L_2\subset L_1$.

Considering this, the value

$$\varrho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log M_f(r))}{\beta(\log r)}\ (\alpha\in L,\beta\in L)$$

is called [9] generalized order $(\alpha,\beta)$ of an entire function $f$.

Now, Biswas et al. [3, 4] have introduced the definitions of the generalized order $(\alpha,\beta)$ of entire and meromorphic functions in the following way after giving a minor modification to the original definition of generalized order $(\alpha,\beta)$ of an entire function (see, e.g. [9]) which considerably extend the definition of $\varphi$-order introduced by Chyzhykov et al. [5].

**Definition 1.1** ([3, 4]). *The generalized order $(\alpha,\beta)$ denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order $(\alpha,\beta)$ denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function $f$ are defined as:*

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to\infty}\frac{\alpha(M_f(r))}{\beta(r)}\ \text{ and}$$
$$\lambda_{(\alpha,\beta)}[f]=\liminf_{r\to\infty}\frac{\alpha(M_f(r))}{\beta(r)}\ \text{ where } \alpha,\beta\in L_1$$

*If $f$ is a meromorphic function, then*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and}$$
$$\lambda_{(\alpha,\beta)}[f] = \liminf_{r\to\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ where } \alpha\in L_2 \text{ and } \beta\in L_1$$

Using the inequality $T_f(r) \le \log M_f(r) \le 3T_f(2r)$ $\{cf.\ [6]\}$, for an entire function $f$, one may easily verify that

$$\begin{matrix}\rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to\infty}\begin{matrix}\sup \\ \inf\end{matrix}\ \frac{\alpha(M_f(r))}{\beta(r)} = \lim_{r\to\infty}\begin{matrix}\sup \\ \inf\end{matrix}\ \frac{\alpha(\exp(T_f(r)))}{\beta(r)},$$

when $\alpha\in L_2$ and $\beta\in L_1$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see, e.g., [1, 2, 8]) will come. Now, in order to make some progress in the study of relative order, Biswas et al. [3] have introduced the definitions of generalized relative order $(\alpha,\beta)$ and generalized relative lower order $(\alpha,\beta)$ of a meromorphic function with respect to an entire function in the following way:

**Definition 1.2** ([3]). *Let $\alpha,\beta\in L_1$. The generalized relative order $(\alpha,\beta)$ and generalized relative lower order $(\alpha,\beta)$ denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively of a meromorphic function $f$ with respect to an entire function $g$ are defined as:*

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

Similarly one may give the definitions of generalized relative hyper order $(\alpha,\beta)$ and generalized relative logarithmic order $(\alpha,\beta)$ of a meromorphic function $f$ with respect to an entire function $g$ in the following way:

**Definition 1.3.** *Let $\alpha,\beta\in L_1$. The generalized relative hyper order $(\alpha,\beta)$ denoted by $\overline{\rho}_{(\alpha,\beta)}[f]_g$ and generalized relative hyper lower order $(\alpha,\beta)$ denoted by $\overline{\lambda}_{(\alpha,\beta)}[f]_g$ of an meromorphic function $f$ with respect to an entire function $g$ are defined as:*

$$\overline{\rho}_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty}\frac{\alpha(\log(T_g^{-1}(T_f(r))))}{\beta(r)},$$
$$\text{and } \overline{\lambda}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty}\frac{\alpha(\log(T_g^{-1}(T_f(r))))}{\beta(r)}.$$

**Definition 1.4.** *Let $\alpha,\beta \in L_1$. The generalized relative logarithmic order $(\alpha,\beta)$ denoted by $\underline{\rho}_{(\alpha,\beta)}[f]_g$ and generalized relative logarithmic lower order $(\alpha,\beta)$ denoted by $\underline{\lambda}_{(\alpha,\beta)}[f]_g$ of a meromorphic function $f$ with respect to an entire function $g$ are defined as:*

$$\begin{aligned}\underline{\rho}_{(\alpha,\beta)}[f]_g &= \limsup_{r\to\infty}\frac{\alpha\left(T_g^{-1}\left(T_f(r)\right)\right)}{\beta(\log(r))},\\ \text{and } \underline{\lambda}_{(\alpha,\beta)}[f]_g &= \liminf_{r\to\infty}\frac{\alpha\left(T_g^{-1}\left(T_f(r)\right)\right)}{\beta(\log(r))}.\end{aligned}$$

To compare the relative growth of two meromorphic functions of having same non-zero finite generalized relative order $(\alpha,\beta)$ with respect to an entire function, Biswas et al. [3] have introduced the definitions of generalized relative type $(\alpha,\beta)$ and generalized relative lower type $(\alpha,\beta)$ in the following way:

**Definition 1.5** ([3]). *Let $\alpha,\beta \in L_1$. The generalized relative type $(\alpha,\beta)$ denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type $(\alpha,\beta)$ denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of a meromorphic function $f$ with respect to an entire function $g$ having non-zero finite generalized relative order $(\alpha,\beta)$ are defined as:*

$$\begin{aligned}\sigma_{(\alpha,\beta)}[f]_g &= \limsup_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}},\\ \text{and } \overline{\sigma}_{(\alpha,\beta)}[f]_g &= \liminf_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}.\end{aligned}$$

Analogously, to determine the relative growth of a meromorphic function $f$ having same non-zero finite generalized relative lower order $(\alpha,\beta)$ with respect to an entire function $g$, Biswas et al.[3] have introduced the definitions of generalized relative upper weak type $(\alpha,\beta)$ denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type $(\alpha,\beta)$ denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of $f$ with respect to $g$ of finite positive generalized relative lower order $(\alpha,\beta)$ in the following way:

**Definition 1.6** ([3]). *Let $\alpha,\beta \in L_1$. The generalized relative upper weak type $(\alpha,\beta)$ denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and generalized relative weak type $(\alpha,\beta)$ denoted by $\tau_{(\alpha,\beta)}[f]_g$ of an meromorphic function $f$ with respect to entire function $g$ having finite positive generalized relative lower order $(\alpha,\beta)$ $(0<\lambda_{(\alpha,\beta)}[f]_g<\infty)$ are defined as :*

$$\begin{aligned}\overline{\tau}_{(\alpha,\beta)}[f]_g &= \limsup_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}},\\ \text{and } \tau_{(\alpha,\beta)}[f]_g &= \liminf_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.\end{aligned}$$

It is obvious that $0\le\tau_{(\alpha,\beta)}[f]_g\le\overline{\tau}_{(\alpha,\beta)}[f]_g\le\infty$.

In this paper we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of generalized relative order $(\alpha, \beta)$, generalized relative type $(\alpha, \beta)$ and generalized relative weak type $(\alpha, \beta)$.

## 2. Main results

In this section we present the main results of the paper. Below we suppose that functions $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ belong to the class $L_1$.

**Theorem 2.1.** *Let $f$ be a meromorphic function and $g$, $h$, $k$ be an entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f \circ g]_h \leq \rho_{(\alpha_1,\beta_1)}[f \circ g]_h < \infty$ and $0 < \lambda_{(\alpha_2,\beta_2)}[f]_k \leq \rho_{(\alpha_2,\beta_2)}[f]_k < \infty$. Then*

$$\begin{aligned} \frac{\lambda_{(\alpha_1,\beta_1)}[f \circ g]_h}{\rho_{(\alpha_2,\beta_2)}[f]_k} &\leq \liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \\ &\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2,\beta_2)}[f]_k}, \frac{\rho_{(\alpha_1,\beta_1)}[f \circ g]_h}{\rho_{(\alpha_2,\beta_2)}[f]_k}\right\} \\ &\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2,\beta_2)}[f]_k}, \frac{\rho_{(\alpha_1,\beta_1)}[f \circ g]_h}{\rho_{(\alpha_2,\beta_2)}[f]_k}\right\} \\ &\leq \limsup_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2,\beta_2)}[f]_k}. \end{aligned}$$

**Proof.** From the definition of $\lambda_{(\alpha_2,\beta_2)}[f]_k$, $\rho_{(\alpha_2,\beta_2)}[f]_k$, $\lambda_{(\alpha_1,\beta_1)}[f \circ g]_h$ and $\rho_{(\alpha_1,\beta_1)}[f \circ g]_h$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large positive numbers of $r$ that

$$\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))) \geqslant (\lambda_{(\alpha_2,\beta_2)}[f]_k - \varepsilon)\beta_1(r), \tag{1}$$

$$\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))) \leq (\rho_{(\alpha_2,\beta_2)}[f]_k + \varepsilon)\beta_1(r), \tag{2}$$

$$\alpha_1(T_h^{-1}(T_{f\circ g}(r))) \geqslant (\lambda_{(\alpha_1,\beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r), \tag{3}$$

$$\alpha_1(T_h^{-1}(T_{f\circ g}(r))) \leq (\rho_{(\alpha_1,\beta_1)}[f \circ g]_h + \varepsilon)\beta_1(r). \tag{4}$$

Again, we get for a sequence of positive numbers of $r$ tending to infinity that

$$\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))) \leq (\lambda_{(\alpha_2,\beta_2)}[f]_k + \varepsilon)\beta_1(r), \tag{5}$$

$$\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))) \geqslant (\rho_{(\alpha_2,\beta_2)}[f]_k - \varepsilon)\beta_1(r), \tag{6}$$

$$\alpha_1(T_h^{-1}(T_{f\circ g}(r))) \leq (\lambda_{(\alpha_1,\beta_1)}[f \circ g]_h + \varepsilon)\beta_1(r), \tag{7}$$

$$\alpha_1(T_h^{-1}(T_{f\circ g}(r))) \geqslant (\rho_{(\alpha_1,\beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r). \tag{8}$$

Now, from (2) and (3) , it follows for all sufficiently large positive numbers of $r$ that

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geqslant \frac{(\lambda_{(\alpha_1,\beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r)}{(\rho_{(\alpha_2,\beta_2)}[f]_k + \varepsilon)\beta_1(r)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\geqslant\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h}{\rho_{(\alpha_2,\beta_2)}[f]_k}. \tag{9}$$

Combining (1) and (7) , we get for a sequence of positive numbers of $r$ tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{(\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h+\varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2,\beta_2)}[f]_k-\varepsilon)\beta_1(r)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h}{\lambda_{(\alpha_2,\beta_2)}[f]_k}. \tag{10}$$

Now, from (3) and (5) , we obtain for a sequence of positive numbers of $r$ tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\geq\frac{(\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h-\varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2,\beta_2)}[f]_k+\varepsilon)\beta_1(r)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\geq\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h}{\lambda_{(\alpha_2,\beta_2)}[f]_k}. \tag{11}$$

Now, it follows from (1) and (4) for all sufficiently large positive numbers of $r$ that

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{(\rho_{(\alpha_1,\beta_1)}[f\circ g]_h+\varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2,\beta_2)}[f]_k-\varepsilon)\beta_1(r)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\rho_{(\alpha_1,\beta_1)}[f\circ g]_h}{\lambda_{(\alpha_2,\beta_2)}[f]_k}. \tag{12}$$

Now, from (4) and (6) , it follows for a sequence of positive numbers of $r$ tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{(\rho_{(\alpha_1,\beta_1)}[f\circ g]_h+\varepsilon)\beta_1(r)}{(\rho_{(\alpha_2,\beta_2)}[f]_k-\varepsilon)\beta_1(r)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\rho_{(\alpha_1,\beta_1)}[f\circ g]_h}{\rho_{(\alpha_2,\beta_2)}[f]_k}. \tag{13}$$

Again combining (2) and (8) , we get for a sequence of positive numbers of $r$ tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geqslant \frac{(\rho_{(\alpha_1,\beta_1)}[f\circ g]_h - \varepsilon)\beta_1(r)}{(\rho_{(\alpha_2,\beta_2)}[f]_k + \varepsilon)\beta_1(r)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geqslant \frac{\rho_{(\alpha_1,\beta_1)}[f\circ g]_h}{\rho_{(\alpha_2,\beta_2)}[f]_k}. \tag{14}$$

Thus the theorem follows from (9) , (10) , (11), (12), (13) and (14) . □

**Remark 2.1.** If we take "$0 < \lambda_{(\alpha_2,\beta_2)}[g]_k \leq \rho_{(\alpha_2,\beta_2)}[g]_k < \infty$" instead of "$0 < \lambda_{(\alpha_2,\beta_2)}[f]_k \leq \rho_{(\alpha_2,\beta_2)}[f]_k < \infty$" and other conditions remain same, the conclusion of Theorem 2.1 remains true with "$\lambda_{(\alpha_2,\beta_2)}[g]_k$", "$\rho_{(\alpha_2,\beta_2)}[g]_k$" and "$\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r)))))$" in place of "$\lambda_{(\alpha_2,\beta_2)}[f]_k$", "$\rho_{(\alpha_2,\beta_2)}[f]_k$" and "$\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))$" respectively in the denominator.

We may now state the following theorem without proof based on Definition 1.3.

**Theorem 2.2.** *Let $f$ be a meromorphic function and $g$, $h$, $k$ be an entire functions such that $0 < \overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]_h \leq \overline{\rho}_{(\alpha_1,\beta_1)}[f\circ g]_h < \infty$ and $0 < \overline{\lambda}_{(\alpha_2,\beta_2)}[f]_k \leq \overline{\rho}_{(\alpha_2,\beta_2)}[f]_k < \infty$. Then*

$$\begin{aligned}
&\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\rho}_{(\alpha_2,\beta_2)}[f]_k} \leq \liminf_{r\to+\infty}\frac{\alpha_1(\log(T_h^{-1}(T_{f\circ g}(r))))}{\alpha_2(\log(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \\
&\leq \min\left\{\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\lambda}_{(\alpha_2,\beta_2)}[f]_k}, \frac{\overline{\rho}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\rho}_{(\alpha_2,\beta_2)}[f]_k}\right\} \\
&\leq \max\left\{\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\lambda}_{(\alpha_2,\beta_2)}[f]_k}, \frac{\overline{\rho}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\rho}_{(\alpha_2,\beta_2)}[f]_k}\right\} \\
&\leq \limsup_{r\to+\infty}\frac{\alpha_1(\log(T_h^{-1}(T_{f\circ g}(r))))}{\alpha_2(\log(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\overline{\rho}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\lambda}_{(\alpha_2,\beta_2)}[f]_k}.
\end{aligned}$$

**Remark 2.2.** If we take "$0 < \overline{\lambda}_{(\alpha_2,\beta_2)}[g]_k \leq \overline{\rho}_{(\alpha_2,\beta_2)}[g]_k < \infty$" instead of "$0 < \overline{\lambda}_{(\alpha_2,\beta_2)}[f]_k \leq \overline{\rho}_{(\alpha_2,\beta_2)}[f]_k < \infty$" and other conditions remain same, the conclusion of Theorem 2.2 remains true with "$\overline{\lambda}_{(\alpha_2,\beta_2)}[g]_k$", "$\overline{\rho}_{(\alpha_2,\beta_2)}[g]_k$" and "$\alpha_2(\log(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$" in place of "$\overline{\lambda}_{(\alpha_2,\beta_2)}[f]_k$", "$\overline{\rho}_{(\alpha_2,\beta_2)}[f]_k$" and "$\alpha_2(\log(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$" respectively in the denominator.

Now, we may state the following theorem without proof based on Definition 1.4.

**Theorem 2.3.** *Let $f$ be a meromorphic function and $g$, $h$, $k$ be an entire functions such that* $0 < \underline{\lambda}_{(\alpha_1,\beta_1)}[f \circ g]_h \leq \underline{\rho}_{(\alpha_1,\beta_1)}[f \circ g]_h < \infty$ *and* $0 < \underline{\lambda}_{(\alpha_2,\beta_2)}[f]_k \leq \underline{\rho}_{(\alpha_2,\beta_2)}[f]_k < \infty$. *Then*

$$\begin{aligned}
&\frac{\underline{\lambda}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\underline{\rho}_{(\alpha_2,\beta_2)}[f]_k} \leq \liminf_{r \to +\infty} \frac{\alpha_1\left(T_h^{-1}\left(T_{f\circ g}(r)\right)\right)}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \\
&\leq \min\left\{\frac{\underline{\lambda}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\underline{\lambda}_{(\alpha_2,\beta_2)}[f]_k}, \frac{\underline{\rho}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\underline{\rho}_{(\alpha_2,\beta_2)}[f]_k}\right\} \\
&\leq \max\left\{\frac{\underline{\lambda}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\underline{\lambda}_{(\alpha_2,\beta_2)}[f]_k}, \frac{\underline{\rho}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\underline{\rho}_{(\alpha_2,\beta_2)}[f]_k}\right\} \\
&\leq \limsup_{r \to +\infty} \frac{\alpha_1\left(T_h^{-1}\left(T_{f\circ g}(r)\right)\right)}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{\underline{\rho}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\underline{\lambda}_{(\alpha_2,\beta_2)}[f]_k}.
\end{aligned}$$

**Remark 2.3.** If we take "$0 < \underline{\lambda}_{(\alpha_2,\beta_2)}[g]_k \leq \underline{\rho}_{(\alpha_2,\beta_2)}[g]_k < \infty$" instead of "$0 < \underline{\lambda}_{(\alpha_2,\beta_2)}[f]_k \leq \underline{\rho}_{(\alpha_2,\beta_2)}[f]_k < \infty$" and other conditions remain same, the conclusion of Theorem 2.3 remains true with "$\underline{\lambda}_{(\alpha_2,\beta_2)}[g]_k$", "$\underline{\rho}_{(\alpha_2,\beta_2)}[g]_k$" and "$\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r)))))$" in place of "$\underline{\lambda}_{(\alpha_2,\beta_2)}[f]_k$", "$\underline{\rho}_{(\alpha_2,\beta_2)}[f]_k$" and "$\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))$" respectively in the denominator.

The proof of the following theorem can be carried out as of the Theorem 2.1, therefore we omit the details.

**Theorem 2.4.** *Let $f$ be a meromorphic function and $g$, $h$, $k$ be an entire functions such that* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f \circ g]_h \leq \sigma_{(\alpha_1,\beta_1)}[f \circ g]_h < \infty$, $0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k \leq \sigma_{(\alpha_2,\beta_2)}[f]_k < \infty$ *and* $\rho_{(\alpha_1,\beta_1)}[f \circ g]_h = \rho_{(\alpha_2,\beta_2)}[f]_k$. *Then*

$$\begin{aligned}
&\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\sigma_{(\alpha_2,\beta_2)}[f]_k} \leq \liminf_{r \to +\infty} \frac{\exp(\alpha_1(T_h^{-1}(T_{f\circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \\
&\leq \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k}, \frac{\sigma_{(\alpha_1,\beta_1)}[f \circ g]_h}{\sigma_{(\alpha_2,\beta_2)}[f]_k}\right\} \\
&\leq \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f \circ g]_h}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k}, \frac{\sigma_{(\alpha_1,\beta_1)}[f \circ g]_h}{\sigma_{(\alpha_2,\beta_2)}[f]_k}\right\} \\
&\leq \limsup_{r \to +\infty} \frac{\exp(\alpha_1(T_h^{-1}(T_{f\circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\sigma_{(\alpha_1,\beta_1)}[f \circ g]_h}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k}.
\end{aligned}$$

**Remark 2.4.** If we take "$0 < \tau_{(\alpha_2,\beta_2)}[f]_k \leq \overline{\tau}_{(\alpha_2,\beta_2)}[f]_k < \infty$" and "$\rho_{(\alpha_1,\beta_1)}[f \circ g]_h = \lambda_{(\alpha_2,\beta_2)}[f]_k$" instead of "$0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k \leq \sigma_{(\alpha_2,\beta_2)}[f]_k < \infty$" and "$\rho_{(\alpha_1,\beta_1)}[f \circ g]_h = \rho_{(\alpha_2,\beta_2)}[f]_k$" respectively and other conditions remain same, the conclusion of Theorem 2.4 remains true with "$\tau_{(\alpha_2,\beta_2)}[f]_k$" and "$\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k$" in place of "$\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k$" and "$\sigma_{(\alpha_2,\beta_2)}[f]_k$" respectively in the denominator.

**Remark 2.5.** If we take "$0<\overline{\sigma}_{(\alpha_2,\beta_2)}[g]_k \le \sigma_{(\alpha_2,\beta_2)}[g]_k<\infty$" and "$\rho_{(\alpha_1,\beta_1)}[f\circ g]_h=\rho_{(\alpha_2,\beta_2)}[g]_k$" instead of "$0<\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k\le\sigma_{(\alpha_2,\beta_2)}[f]_k<\infty$" and "$\rho_{(\alpha_1,\beta_1)}[f\circ g]_h=\rho_{(\alpha_2,\beta_2)}[f]_k$" respectively and other conditions remain same, the conclusion of Theorem 2.4 remains true with "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]_k$", "$\sigma_{(\alpha_2,\beta_2)}[g]_k$" and "$\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$" in place of "$\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k$", "$\sigma_{(\alpha_2,\beta_2)}[f]_k$" and "$\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$".

**Remark 2.6.** If we take "$0<\tau_{(\alpha_2,\beta_2)}[g]_k\le\overline{\tau}_{(\alpha_2,\beta_2)}[g]_k<\infty$" and "$\rho_{(\alpha_1,\beta_1)}[f\circ g]_h=\lambda_{(\alpha_2,\beta_2)}[g]_k$" instead of "$0<\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k\le\sigma_{(\alpha_2,\beta_2)}[f]_k<\infty$" and "$\rho_{(\alpha_1,\beta_1)}[f\circ g]_h=\rho_{(\alpha_2,\beta_2)}[f]_k$" and other conditions remain same, the conclusion of Theorem 2.4 remains true with "$\tau_{(\alpha_2,\beta_2)}[g]_k$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]_k$" and "$\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$" in place of "$\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k$", "$\sigma_{(\alpha_2,\beta_2)}[f]_k$" and "$\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$" respectively in the denominator.

**Theorem 2.5.** *Let $f$ be a meromorphic function and $g$, $h$, $k$ be an entire functions such that* $0<\tau_{(\alpha_1,\beta_1)}[f\circ g]_h\le\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]_h<\infty$, $0<\tau_{(\alpha_2,\beta_2)}[f]_k\le\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k<\infty$ *and* $\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h=\lambda_{(\alpha_2,\beta_2)}[f]_k$. *Then*

$$\begin{aligned}\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k} &\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(T_h^{-1}(T_{f\circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))}\\ &\le \min\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]_h}{\tau_{(\alpha_2,\beta_2)}[f]_k},\frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k}\right\}\\ &\le \max\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]_h}{\tau_{(\alpha_2,\beta_2)}[f]_k},\frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(T_h^{-1}(T_{f\circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))}\le\frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]_h}{\tau_{(\alpha_2,\beta_2)}[f]_k}.\end{aligned}$$

**Remark 2.7.** If we take "$0<\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k\le\sigma_{(\alpha_2,\beta_2)}[f]_k<\infty$" and "$\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h=\rho_{(\alpha_2,\beta_2)}[f]_k$" instead of "$0<\tau_{(\alpha_2,\beta_2)}[f]_k\le\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k<\infty$" and "$\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h=\lambda_{(\alpha_2,\beta_2)}[f]_k$" respectively and other conditions remain same, the conclusion of Theorem 2.5 remains true with "$\overline{\sigma}_{(\alpha_2,\beta_2)}[f]_k$" and "$\sigma_{(\alpha_2,\beta_2)}[f]_k$" in place of "$\tau_{(\alpha_2,\beta_2)}[f]_k$" and "$\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k$" respectively in the denominator.

**Remark 2.8.** If we take "$0<\tau_{(\alpha_2,\beta_2)}[g]_k\le\overline{\tau}_{(\alpha_2,\beta_2)}[g]_k<\infty$" and "$\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h=\lambda_{(\alpha_2,\beta_2)}[g]_k$" instead of "$0<\tau_{(\alpha_2,\beta_2)}[f]_k\le\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k<\infty$" and "$\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h=\lambda_{(\alpha_2,\beta_2)}[f]_k$" respectively and other conditions remain same, the conclusion of Theorem 2.5 remains true with "$\tau_{(\alpha_2,\beta_2)}[g]_k$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]_k$" and "$\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$" in place of "$\tau_{(\alpha_2,\beta_2)}[f]_k$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k$" and "$\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$" respectively in the denominator.

**Remark 2.9.** If we take "$0<\overline{\sigma}_{(\alpha_2,\beta_2)}[g]_k\le\sigma_{(\alpha_2,\beta_2)}[g]_k<\infty$" and "$\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h=\rho_{(\alpha_2,\beta_2)}[g]_k$" instead of "$0<\tau_{(\alpha_2,\beta_2)}[f]_k\le\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k<\infty$" and "$\lambda_{(\alpha_1,\beta_1)}[f\circ g]_h=\lambda_{(\alpha_2,\beta_2)}[f]_k$" respectively and other conditions remain same, the conclusion of Theorem 2.5 remains true with "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]_k$", "$\sigma_{(\alpha_2,\beta_2)}[g]_k$"

and "$\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$" in place of "$\tau_{(\alpha_2,\beta_2)}[f]_k$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[f]_k$" and "$\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$" respectively in the denominator.

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**References**

[1] L. Bernal-Gonzaléz, *Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito*, Doctoral Thesis, Universidad de Sevilla, Spain, 1984.

[2] L. Bernal, *Orden relative de crecimiento de funciones enteras*, Collect. Math., 39 (1988), 209-229.

[3] T. Biswas and C. Biswas, *Some results on generalized relative order $(\alpha, \beta)$ and generalized relative type $(\alpha, \beta)$ of meromorphic function with respect to an entire function*, Ganita, 70 (2020), 239-252.

[4] T. Biswas, C. Biswas, R. Biswas, *A note on generalized growth analysis of composite entire functions*, Poincare J. Anal. Appl., 7 (2020), 277-286.

[5] I. Chyzhykov, N. Semochko, *Fast growing entire solutions of linear differential equations*, Math. Bull. Shevchenko Sci. Soc., 13 (2016), 1-16.

[6] W.K. Hayman, *Meromorphic functions*, The Clarendon Press, Oxford, 1964.

[7] I. Laine, *Nevanlinna theory and complex differential equations*, De Gruyter, Berlin, 1993.

[8] B. K. Lahiri, D. Banerjee, *Relative order of entire and meromorphic functions*, Proc. Natl. Acad. Sci. India, Sect. A, 69 (1999), 339-354.

[9] M. N. Sheremeta, *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion* (in Russian), Izv. Vyssh. Uchebn. Zaved Mat., 2 (1967), 100-108.

[10] G. Valiron, *Lectures on the general theory of integral functions*, Chelsea Publishing Company, NY, 1949.

[11] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993.

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