Generalized relative order (α, β) and generalized relative type (α, β) oriented some growth analysis of composite analytic functions in the unit disc

Chinmay Biswas*
Department of Mathematics
Nabadwip Vidyasagar College
Nabadwip, Dist.- Nadia, PIN-741302, West Bengal
India
chinmay.shib@gmail.com

Tanmay Biswas
Rajbari, Rabindrapally, R.N. Tagore Road
P.O. Krishnagar, Dist-Nadia, PIN- 741101, West Bengal
India
tanmaybiswas_math@rediffmail.com

Abstract. In this paper we introduce the idea of generalized relative order (α, β) and generalized relative type (α, β) of an analytic function with respect to another analytic function in the unit disc where α and β are continuous non-negative on $(-\infty, +\infty)$ functions. Hence we study some growth properties relating to the composition of two analytic functions in the unit disc on the basis of generalized relative order (α, β) and generalized relative type (α, β) as compared to the growth of their corresponding left and right factors.

Keywords: growth, analytic function, composition, unit disc, generalized relative order (α, β), generalized relative hyper order (α, β), generalized relative logarithmic order (α, β),generalized relative type (α, β), generalized relative weak type (α, β).

1. Introduction, definitions and notations

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U = \{z : |z| < 1\}$ and $M_f(r)$ be the maximum of $|f(z)|$ on $|z| = r$. In [4], Sons defined the order $\varrho(f)$ and the lower order $\lambda(f)$ as

$$\begin{matrix} \varrho(f) \\ \lambda(f) \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} M_f(r)}{-\log(1-r)}.$$

Now, let L be a class of continuous non-negative on $(-\infty, \infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow \infty$ as $x \to \infty$. Further we assume that throughout the present paper α, $\beta \in L$. Now, considering this, we have introduced the definitions of the generalized order (α, β) and generalized

*. Corresponding author

lower order (α,β) of an analytic function f in the unit disc U which are as follows:

Definition 1.1. *The generalized order (α,β) denoted by $\varrho_{(\alpha,\beta)}[f]$ and generalized lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an analytic function f in the unit disc U are defined as:*

$$\begin{matrix}\varrho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(M_f(r))}{\beta\left(\frac{1}{1-r}\right)}.$$

Clearly $\varrho_{(\log\log r,\log r)}[f] = \varrho(f)$ and $\lambda_{(\log\log r,\log r)}[f] = \lambda(f)$.

Now, we can introduce the definitions of the generalized relative order (α,β) and generalized relative lower order (α,β) of an analytic function f with respect to another analytic function h in the unit disc U which are as follows:

Definition 1.2. *The generalized relative order (α,β) denoted by $\varrho_{(\alpha,\beta)}[f]_h$ and generalized relative lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[f]_h$ of an entire function f with respect to another entire function h in the unit disc U are defined as:*

$$\begin{matrix}\varrho_{(\alpha,\beta)}[f]_h \\ \lambda_{(\alpha,\beta)}[f]_h\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha\left(M_h^{-1}(M_f(r))\right)}{\beta\left(\frac{1}{1-r}\right)}.$$

Clearly, if $\alpha(r) = \beta(r) = \log r$ and $h(z) = \exp z$, then $\varrho_{(\alpha,\beta)}[f]_h = \varrho(f)$ and $\lambda_{(\alpha,\beta)}[f]_h = \lambda(f)$.

Now, one may give the definitions of generalized relative hyper order (α,β) and generalized relative logarithmic order (α,β) of an analytic function f with respect to another analytic function h in the unit disc U in the following way:

Definition 1.3. *The generalized relative hyper order (α,β) denoted by $\overline{\varrho}_{(\alpha,\beta)}[f]_h$ and generalized relative hyper lower order (α,β) denoted by $\overline{\lambda}_{(\alpha,\beta)}[f]_h$ of an entire function f with respect to entire function h in the unit disc U are defined as:*

$$\begin{matrix}\overline{\varrho}_{(\alpha,\beta)}[f]_h \\ \overline{\lambda}_{(\alpha,\beta)}[f]_h\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log(M_h^{-1}(M_f(r))))}{\beta\left(\frac{1}{1-r}\right)}.$$

Definition 1.4. *The generalized relative logarithmic order (α,β) denoted by $\underline{\varrho}_{(\alpha,\beta)}[f]_h$ and generalized relative logarithmic lower order (α,β) denoted by $\underline{\lambda}_{(\alpha,\beta)}[f]_h$ of an entire function f with respect to entire function h in the unit disc U are defined as:*

$$\begin{matrix}\underline{\varrho}_{(\alpha,\beta)}[f]_h \\ \underline{\lambda}_{(\alpha,\beta)}[f]_h\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(M_h^{-1}(M_f(r)))}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)}.$$

Now, in order to refine the growth scale namely the generalized relative order (α, β), we introduce the definitions of another growth indicators, called generalized relative type (α, β) and generalized relative lower type (α, β) respectively of an analytic function f with respect to another analytic function h in the unit disc U which are as follows:

Definition 1.5. *The generalized relative type* (α, β) *and generalized relative lower type* (α, β) *of an entire function* f *with respect to another entire function* h *in the unit disc* U *having finite positive generalized relative order* (α, β) $(0 < \varrho_{(\alpha,\beta)}[f]_h < \infty)$ *are defined as :*

$$\begin{matrix} \sigma_{(\alpha,\beta)}[f]_h \\ \overline{\sigma}_{(\alpha,\beta)}[f]_h \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(M_h^{-1}(M_f(r))))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\varrho_{(\alpha,\beta)}[f]_h}}.$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta)}[f]_h \leq \sigma_{(\alpha,\beta)}[f]_h \leq \infty$.

Analogously, to determine the relative growth of two analytic functions in the unit disc U having same non zero finite generalized relative lower order (α, β), one can introduced the definitions of generalized relative weak type (α, β) and generalized relative upper weak type (α, β) of an analytic function f with respect to analytic function h in the unit disc U of finite positive generalized relative lower order (α, β) , $\lambda_{(\alpha,\beta)}[f]_h$ in the following way:

Definition 1.6. *The generalized relative upper weak type* (α, β) *and generalized relative weak type* (α, β) *of an entire function* f *with respect to another entire function* h *in the unit disc* U *having finite positive generalized relative lower order* (α, β) $(0 < \lambda_{(\alpha,\beta)}[f]_h < \infty)$ *are defined as :*

$$\begin{matrix} \overline{\tau}_{(\alpha,\beta)}[f]_h \\ \tau_{(\alpha,\beta)}[f]_h \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(M_h^{-1}(M_f(r))))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\lambda_{(\alpha,\beta)}[f]_h}}.$$

It is obvious that $0 \leq \tau_{(\alpha,\beta)}[f]_h \leq \overline{\tau}_{(\alpha,\beta)}[f]_h \leq \infty$.

In this paper, we study some growth properties relating to the composition of two analytic function of in the unit disc on the basis of generalized relative order (α, β), generalized relative hyper order (α, β), generalized relative logarithmic order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) as compared to the growth of their corresponding left and right factors. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [1], [2] and [3].

2. Theorems

In this section we present the main results of the paper.

Theorem 2.1. *Let f , g, h and k be non-constant entire functions in the unit disc U such that $0 < \lambda_{(\alpha,\beta)}[f(g)]_h \leq \varrho_{(\alpha,\beta)}[f(g)]_h < \infty$ and $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$. Then*

$$\frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k} \leq \liminf_{r\to 1} \frac{\alpha\left(M_h^{-1}(M_{f(g)}(r))\right)}{\alpha(M_k^{-1}(M_f(r)))}$$
$$\leq \min\left\{\frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k}\right\} \leq \max\left\{\frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k}\right\}$$
$$\leq \limsup_{r\to 1} \frac{\alpha\left(M_h^{-1}(M_{f(g)}(r))\right)}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}.$$

Proof of Theorem 2.1. From the definitions of $\lambda_{(\alpha,\beta)}[f(g)]_h$, $\varrho_{(\alpha,\beta)}[f(g)]_h$, $\lambda_{(\alpha,\beta)}[f]_k, \varrho_{(\alpha,\beta)}[f]_k$ and we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha(M_h^{-1}(M_{f(g)}(r))) \geqslant (\lambda_{(\alpha,\beta)}[f(g)]_h - \varepsilon)\beta((1-r)^{-1}), \tag{1}$$
$$\alpha(M_h^{-1}(M_{f(g)}(r))) \leq (\varrho_{(\alpha,\beta)}[f(g)]_h + \varepsilon)\beta((1-r)^{-1}), \tag{2}$$
$$\alpha(M_k^{-1}(M_f(r))) \geqslant (\lambda_{(\alpha,\beta)}[f]_k - \varepsilon)\beta((1-r)^{-1}), \tag{3}$$
$$\alpha(M_k^{-1}(M_f(r))) \leq (\varrho_{(\alpha,\beta)}[f]_k + \varepsilon)\beta((1-r)^{-1}). \tag{4}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\alpha(M_h^{-1}(M_{f(g)}(r))) \leq (\lambda_{(\alpha,\beta)}[f(g)]_h + \varepsilon)\beta((1-r)^{-1}), \tag{5}$$
$$\alpha(M_h^{-1}(M_{f(g)}(r))) \geqslant (\varrho_{(\alpha,\beta)}[f(g)]_h - \varepsilon)\beta((1-r)^{-1}), \tag{6}$$
$$\alpha(M_k^{-1}(M_f(r))) \leq (\lambda_{(\alpha,\beta)}[f]_k + \varepsilon)\beta((1-r)^{-1}), \tag{7}$$
$$\text{and } \alpha(M_k^{-1}(M_f(r))) \geqslant (\varrho_{(\alpha,\beta)}[f]_k - \varepsilon)\beta((1-r)^{-1}). \tag{8}$$

Now, from (1) and (4) it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geqslant \frac{\lambda_{(\alpha,\beta)}[f(g)]_h - \varepsilon}{\varrho_{(\alpha,\beta)}[f]_k + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geqslant \frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k}, \tag{9}$$

which is the first part of the theorem.

Combining (5) and (3) we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\lambda_{(\alpha,\beta)}[f(g)]_h + \varepsilon}{\lambda_{(\alpha,\beta)}[f]_k - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to 1}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\leq\frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}. \tag{10}$$

Now, from (1) and (7) we obtain for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\geq\frac{\lambda_{(\alpha,\beta)}[f(g)]_h-\varepsilon}{\lambda_{(\alpha,\beta)}[f]_k+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\geq\frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}. \tag{11}$$

Now, it follows from (3) and (2) , for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\leq\frac{\varrho_{(\alpha,\beta)}[f(g)]_h+\varepsilon}{\lambda_{(\alpha,\beta)}[f]_k-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\leq\frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}. \tag{12}$$

Which is the last part of the theorem.

Now, from (2) and (8) , it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\leq\frac{\varrho_{(\alpha,\beta)}[f(g)]_h+\varepsilon}{\varrho_{(\alpha,\beta)}[f]_k-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\leq\frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k}. \tag{13}$$

So, combining (4) and (6) , we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\geqslant\frac{\varrho_{(\alpha,\beta)}[f(g)]_h-\varepsilon}{\varrho_{(\alpha,\beta)}[f]_k+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\geqslant\frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k}. \tag{14}$$

Thus, the second part of the theorem follows from (10) and (13) ,the third part is trivial and fourth part follows from (11) and (14) .

Thus, the theorem follows from (9) , (10) , (11), (12) , (13) and (14) .

Remark 2.1. If we take "$0 < \lambda_{(\alpha,\beta)}[g]_k \leq \varrho_{(\alpha,\beta)}[g]_k < \infty$" instead of "$0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$" and other conditions remain same, the conclusion of Theorem 2.1 remains true with "$\lambda_{(\alpha,\beta)}[f]_k$", "$\varrho_{(\alpha,\beta)}[f]_k$" and "$\alpha(M_k^{-1}(M_f(r)))$" replaced by "$\lambda_{(\alpha,\beta)}[g]_k$", "$\varrho_{(\alpha,\beta)}[g]_k$" and "$\alpha(M_k^{-1}(M_g(r)))$" respectively in the denominator.

Theorem 2.2. *Let f , g, h and k be non-constant entire functions in the unit disc U such that $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$ and $\lambda_{(\alpha,\beta)}[f(g)]_h = \infty$. Then*

$$\lim_{r\to 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} = \infty.$$

Proof of Theorem 2.2. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of $\frac{1}{1-r}$ tending to infinity

$$\alpha(M_h^{-1}(M_{f(g)}(r))) \leq \Delta \cdot \alpha(M_k^{-1}(M_f(r))). \tag{15}$$

Again from the definition of $\varrho_{(\alpha,\beta)}[f]_k$, it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha(M_k^{-1}(M_f(r))) \leq (\varrho_{(\alpha,\beta)}[f]_k + \epsilon)\beta((1-r)^{-1}). \tag{16}$$

Thus, from (15) and (16), we have for a sequence of values of r tending to 1 that

$$\begin{aligned}
&\alpha(M_h^{-1}(M_{f(g)}(r))) \leq \Delta(\varrho_{(\alpha,\beta)}[f]_k + \epsilon)\beta((1-r)^{-1}),\\
&i.e.,\ \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta((1-r)^{-1})} \leq \Delta(\varrho_{(\alpha,\beta)}[f]_k + \epsilon),\\
&i.e.,\ \liminf_{r\to 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta((1-r)^{-1})} = \lambda_{(\alpha,\beta)}[f(g)]_h < \infty.
\end{aligned}$$

This is a contradiction.

Thus the theorem follows.

Remark 2.2. If we take "$0 < \lambda_{(\alpha,\beta)}[g]_k \leq \varrho_{(\alpha,\beta)}[g]_k < \infty$" instead of "$0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$" and other conditions remain same, the conclusion of Theorem 2.2 remains true with "$\alpha(M_k^{-1}(M_f(r)))$" replaced by "$\alpha(M_k^{-1}(M_g(r)))$" in the denominator.

Remark 2.3. Theorem 2.2 and Remark 2.2 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta)}[f(g)]_h = \infty$" is replaced by "$\varrho_{(\alpha,\beta)}[f(g)]_h = \infty$" and the other conditions remain the same.

We may now state the following theorem without proof based on Definition 1.3.

Theorem 2.3. *Let f , g, h and k be non-constant entire functions in U such that $0<\overline{\lambda}_{(\alpha,\beta)}[f(g)]_h \le \overline{\varrho}_{(\alpha,\beta)}[f(g)]_h<\infty$ and $0<\overline{\lambda}_{(\alpha,\beta)}[f]_k \le \overline{\varrho}_{(\alpha,\beta)}[f]_k<\infty$. Then*

$$\begin{aligned}
&\frac{\overline{\lambda}_{(\alpha,\beta)}[f(g)]_h}{\overline{\varrho}_{(\alpha,\beta)}[f]_k} \le \liminf_{r\to 1}\frac{\alpha(\log(M_h^{-1}(M_{f(g)}(r))))}{\alpha(\log\left(M_k^{-1}(M_f(r))\right))}\\
&\le \min\left\{\frac{\overline{\lambda}_{(\alpha,\beta)}[f(g)]_h}{\overline{\lambda}_{(\alpha,\beta)}[f]_k}, \frac{\overline{\varrho}_{(\alpha,\beta)}[f(g)]_h}{\overline{\varrho}_{(\alpha,\beta)}[f]_k}\right\} \le \max\left\{\frac{\overline{\lambda}_{(\alpha,\beta)}[f(g)]_h}{\overline{\lambda}_{(\alpha,\beta)}[f]_k}, \frac{\overline{\varrho}_{(\alpha,\beta)}[f(g)]_h}{\overline{\varrho}_{(\alpha,\beta)}[f]_k}\right\}\\
&\le \limsup_{r\to 1}\frac{\alpha(\log(M_h^{-1}(M_{f(g)}(r))))}{\alpha(\log\left(M_k^{-1}(M_f(r))\right))} \le \frac{\overline{\varrho}_{(\alpha,\beta)}[f(g)]_h}{\overline{\lambda}_{(\alpha,\beta)}[f]_k}.
\end{aligned}$$

Remark 2.4. If we take "$0<\overline{\lambda}_{(\alpha,\beta)}[g]_k \le \overline{\varrho}_{(\alpha,\beta)}[g]_k<\infty$" instead of "$0<\overline{\lambda}_{(\alpha,\beta)}[f]_k \le \overline{\varrho}_{(\alpha,\beta)}[f]_k<\infty$" and other conditions remain same, the conclusion of Theorem 2.3 remains true with "$\overline{\lambda}_{(\alpha,\beta)}[f]_k$", "$\overline{\varrho}_{(\alpha,\beta)}[f]_k$" and "$\alpha(\log(M_k^{-1}(M_f(r))))$" replaced by "$\overline{\lambda}_{(\alpha,\beta)}[g]_k$", "$\overline{\varrho}_{(\alpha,\beta)}[g]_k$" and "$\alpha(\log(M_k^{-1}(M_g(r))))$" respectively in the denominator.

We may now state the following theorem without proof based on Definition 1.4.

Theorem 2.4. *Let f , g, h and k be non-constant entire functions in the unit disc U such that $0<\underline{\lambda}_{(\alpha,\beta)}[f(g)]_h \le \underline{\varrho}_{(\alpha,\beta)}[f(g)]_h<\infty$ and $0<\underline{\lambda}_{(\alpha,\beta)}[f]_k \le \underline{\varrho}_{(\alpha,\beta)}[f]_k<\infty$. Then*

$$\begin{aligned}
&\frac{\underline{\lambda}_{(\alpha,\beta)}[f(g)]_h}{\underline{\varrho}_{(\alpha,\beta)}[f]_k} \le \liminf_{r\to 1}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))}\\
&\le \min\left\{\frac{\underline{\lambda}_{(\alpha,\beta)}[f(g)]_h}{\underline{\lambda}_{(\alpha,\beta)}[f]_k}, \frac{\underline{\varrho}_{(\alpha,\beta)}[f(g)]_h}{\underline{\varrho}_{(\alpha,\beta)}[f]_k}\right\} \le \max\left\{\frac{\underline{\lambda}_{(\alpha,\beta)}[f(g)]_h}{\underline{\lambda}_{(\alpha,\beta)}[f]_k}, \frac{\underline{\varrho}_{(\alpha,\beta)}[f(g)]_h}{\underline{\varrho}_{(\alpha,\beta)}[f]_k}\right\}\\
&\le \limsup_{r\to 1}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \le \frac{\underline{\varrho}_{(\alpha,\beta)}[f(g)]_h}{\underline{\lambda}_{(\alpha,\beta)}[f]_k}.
\end{aligned}$$

Remark 2.5. If we take "$0<\underline{\lambda}_{(\alpha,\beta)}[g]_k \le \underline{\varrho}_{(\alpha,\beta)}[g]_k<\infty$" instead of "$0<\underline{\lambda}_{(\alpha,\beta)}[f]_k \le \underline{\varrho}_{(\alpha,\beta)}[f]_k<\infty$" and other conditions remain same, the conclusion of Theorem 2.4 remains true with "$\underline{\lambda}_{(\alpha,\beta)}[f]_k$", "$\underline{\varrho}_{(\alpha,\beta)}[f]_k$" and "$\alpha(M_k^{-1}(M_f(r)))$" replaced by "$\underline{\lambda}_{(\alpha,\beta)}[g]_k$", "$\underline{\varrho}_{(\alpha,\beta)}[g]_k$" and "$\alpha(M_k^{-1}(M_g(r)))$" respectively in the denominator.

In the line of Theorem 2.1 , one can easily deduce the Theorem 2.5 and Theorem 2.6, so proofs are omitted.

Theorem 2.5. *Let f , g, h and k be non-constant entire functions in the unit disc U such that $0 < \overline{\sigma}_{(\alpha,\beta)}[f(g)]_h \leq \sigma_{(\alpha,\beta)}[f(g)]_h < \infty$, $0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$ and $\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$. Then*

$$\begin{aligned}\frac{\overline{\sigma}_{(\alpha,\beta)}[f(g)]_h}{\sigma_{(\alpha,\beta)}[f]_k} &\leq \liminf_{r\to 1}\frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))}\\ &\leq \min\left\{\frac{\overline{\sigma}_{(\alpha,\beta)}[f(g)]_h}{\overline{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(g)]_h}{\sigma_{(\alpha,\beta)}[f]_k}\right\} \leq \max\left\{\frac{\overline{\sigma}_{(\alpha,\beta)}[f(g)]_h}{\overline{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(g)]_h}{\sigma_{(\alpha,\beta)}[f]_k}\right\}\\ &\leq \limsup_{r\to 1}\frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(g)]_h}{\overline{\sigma}_{(\alpha,\beta)}[f]_k}.\end{aligned}$$

Remark 2.6. If we take "$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$" and "$\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho^{(\alpha,\beta)}[g]_k$" instead of "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" and "$\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$" and other conditions remain same, the conclusion of Theorem 2.5 remains true with "$\sigma_{(\alpha,\beta)}[f]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[f]_k$" and "$\exp(\alpha(M_k^{-1}(M_f(r))))$" replaced by "$\sigma_{(\alpha,\beta)}[g]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(M_k^{-1}(M_g(r))))$" respectively in the denominator.

Remark 2.7. If we take "$0 < \tau_{(\alpha,\beta)}[f]_k \leq \overline{\tau}_{(\alpha,\beta)}[f]_k < \infty$" and "$\varrho_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[f]_k$" instead of "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" and "$\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$" and other conditions remain same, the conclusion of Theorem 2.5 remains true with "$\sigma_{(\alpha,\beta)}[f]_k$" and "$\overline{\sigma}_{(\alpha,\beta)}[f]_k$" replaced by "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" and "$\tau_{(\alpha,\beta)}[f]_k$" respectively in the denominator.

Remark 2.8. If we take "$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$" and "$\varrho_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[g]_k$" instead of "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" and "$\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$" and other conditions remain same, the conclusion of Theorem 2.5 remains true with "$\overline{\sigma}_{(\alpha,\beta)}[f]_k$", "$\sigma_{(\alpha,\beta)}[f]_k$" and "$\exp(\alpha(M_k^{-1}(M_f(r))))$" replaced by "$\tau_{(\alpha,\beta)}[g]_k$", "$\overline{\tau}_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(M_k^{-1}(M_g(r))))$" respectively in the denominator.

Theorem 2.6. *Let f , g, h and k be non-constant entire functions in the unit disc U such that $0 < \tau_{(\alpha,\beta)}[f(g)]_h \leq \overline{\tau}_{(\alpha,\beta)}[f(g)]_h < \infty$, $0 < \tau_{(\alpha,\beta)}[f]_k \leq \overline{\tau}_{(\alpha,\beta)}[f]_k < \infty$ and $\lambda_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[f]_k$. Then*

$$\begin{aligned}\frac{\tau_{(\alpha,\beta)}[f(g)]_h}{\overline{\tau}_{(\alpha,\beta)}[f]_k} &\leq \liminf_{r\to 1}\frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))}\\ &\leq \min\left\{\frac{\tau_{(\alpha,\beta)}[f(g)]_h}{\tau_{(\alpha,\beta)}[f]_k}, \frac{\overline{\tau}_{(\alpha,\beta)}[f(g)]_h}{\overline{\tau}_{(\alpha,\beta)}[f]_k}\right\} \leq \max\left\{\frac{\tau_{(\alpha,\beta)}[f(g)]_h}{\tau_{(\alpha,\beta)}[f]_k}, \frac{\overline{\tau}_{(\alpha,\beta)}[f(g)]_h}{\overline{\tau}_{(\alpha,\beta)}[f]_k}\right\}\\ &\leq \limsup_{r\to 1}\frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))} \leq \frac{\overline{\tau}_{(\alpha,\beta)}[f(g)]_h}{\tau_{(\alpha,\beta)}[f]_k}.\end{aligned}$$

Remark 2.9. If we take "$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[g]_k$" instead of "$0 < \tau_{(\alpha,\beta)}[f]_k \leq \overline{\tau}_{(\alpha,\beta)}[f]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[f(g)]_h =$

$\lambda_{(\alpha,\beta)}[f]_k$" and other conditions remain same, the conclusion of Theorem 2.6 remains true with "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" and "$\exp(\alpha(M_k^{-1}(M_f(r))))$" replaced by "$\tau_{(\alpha,\beta)}[g]_k$", "$\overline{\tau}_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(M_k^{-1}(M_g(r))))$" respectively in the denominator.

Remark 2.10. If we take "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$" instead of "$0 < \tau_{(\alpha,\beta)}[f]_k \leq \overline{\tau}_{(\alpha,\beta)}[f]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[f]_k$" and other conditions remain same, the conclusion of Theorem 2.6 remains true with "$\tau_{(\alpha,\beta)}[f]_k$" and "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" replaced by "$\overline{\sigma}_{(\alpha,\beta)}[f]_k$" and "$\sigma_{(\alpha,\beta)}[f]_k$" respectively in the denominator.

Remark 2.11. If we take "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[g]_k$" instead of "$0 < \tau_{(\alpha,\beta)}[f]_k \leq \overline{\tau}_{(\alpha,\beta)}[f]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[f]_k$" and other conditions remain same, the conclusion of Theorem 2.6 remains true with "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" and "$\exp(\alpha(M_k^{-1}(M_f(r))))$" replaced by "$\overline{\sigma}_{(\alpha,\beta)}[g]_k$", "$\sigma_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(M_k^{-1}(M_g(r))))$" respectively in the denominator.

References

[1] A. K. Agarwal, *On the properties of an entire function of two complex variables*, Canadian J. Math., 20 (1968), 51-57.

[2] B. A. Fuks, *Introduction to the theory of analytic functions of several complex variables*, Translations of Mathematical Monographs, Amer. Math. Soc., Providence, Rhode Island, 1963.

[3] O. P. Juneja, G. P. Kapoor, *Analytic functions-growth aspects*, Pitman avanced publishing program, Pitman Pub., Boston, 296p., 1985.

[4] L. R. Sons, *Regularity of growth and gaps*, J. Math. Anal. Appl., 24 (1968), 296-306.

Accepted: November 23, 2020