The total game chromatic number of paths, cycles and stars

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Abstract. The total game coloring is one of the extensions of the game coloring problem in which two players alternatively color both vertices and edges of a given graph. In this paper, the total game chromatic number is determined for the classes of paths, cycles, and stars. The incidence game chromatic number for large paths is also discussed by relating the incidence and the total graphs of paths.

Keywords: game coloring, total coloring.

1. Introduction

Steven Brams defined the map-coloring game [1] in 1981, which was the introduction of graph coloring game problem. Further in 1991, Bodlaeder introduced the graph coloring game parameters [2]. The incidence game coloring and the total game coloring are some extensions of game coloring problem introduced by Andres [4], and Bartnicki and Miechowicz [3], respectively.

Let G be a graph and consider a set of colors $C = \{1, 2, 3, ..., k\}$. Further, consider two-players, Alice and Bob who take alternative turns to color vertices of an initially uncolored graph G using the colors in the set C. They both follow proper coloring rule, i.e., each pair of adjacent vertices must receive distinct colors. Skipping a turn is not allowed and the game ends when it is not possible for a turn to carry on. Alice wins if all the vertices of the graph are properly colored using the set C; otherwise, Bob wins. The minimum number of colors, say k, for which Alice has a winning strategy while playing on the graph G with the color set $\{1, 2, 3, ..., k\}$ is called the *game chromatic number* of G, i.e., $\chi_g(G) = k$.

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Total graph coloring is the assignment of colors or labels to both vertices and edges of the graph G such that each pair of adjacent or incident elements, such as vertex adjacent to vertex, edge adjacent to edge, and vertex incident upon edge, must receive different colors. The *total chromatic number* of a graph G, $\chi''(G)$ is the minimum number of colors required to color the graph.

Total game coloring of a graph G is considered as follows: Let C be a set of colors and let Alice and Bob be two players who alternatively make moves on the vertices and edges of the graph G by assigning colors from C. Here, it is assumed that Alice always starts the game and skipping a move is not permitted. The game ends when it is not possible to make a move. Alice wins if the graph is completely colored with the colors in C; otherwise, Bob wins. Moreover, it is always aimed to make Alice win so that the graph is completely colored. The minimum number of colors, say k, required for Alice to have a winning strategy on G is the *total game chromatic number* of G, i.e., $\chi''_g(G) = k$.

The *incidence graph* of G, denoted as G^I, is said to be the graph whose vertices are the incidences of G and whose edges denote adjacencies between the incidences of G. The *total graph* of a graph G, denoted as $T(G)$, has a vertex for each edge and vertex in G and an edge for every edge-edge, vertex-edge, and vertex-vertex adjacency in G.

2. The bounds of $\chi''_g(G)$

In this section, the upper and lower bounds of the function χ''_g and also some terminologies used in the up-coming theorems are discussed.

Proposition 2.1. $\chi''_g(G) \geq \Delta(G) + 1$.

It is obvious that, $\chi''_g(G) \geq \chi''(G)$ and $\chi''(G) \geq \Delta(G) + 1$. Hence, $\chi''_g(G) \geq \Delta(G) + 1$. Moreover, it is shown in [3] that $\chi''_g(G) \leq 2\Delta(G) + 1$. Thus, the bound for $\chi''_g(G)$ can be given as: $\Delta(G) + 1 \leq \chi''_g(G) \leq 2\Delta(G) + 1$.

The following terminologies are being used in the construction of proofs:

A *configuration* or a *set-up* is an arrangement of elements in a particular formation. A *strategy* is a plan of action designed to defeat the opponent. A *trap* is a strategy followed by a player to force a win against the opponent.

3. Total game chromatic number of some graphs

In this section, the exact value of the function χ''_g of paths, cycles, and stars are discussed. Let us consider each vertex and each edge as *positions*. Therefore, a path on n vertices will have $2n - 1$ positions.

Theorem 3.1. $\chi''_g(P_n) = 5$ *for* $n \geq 9$.

Proof. Let P_n be a path with $n \geq 9$. Since $\chi''_g(G) \leq 2\Delta(G) + 1$ is the upper bound, $\chi''_g(P_n) \leq 5$. Therefore, it remains to prove that Alice has a winning strategy on P_n with at least five colors, $\chi''_g(P_n) \geq 5$; in other words, Bob has a winning strategy on P_n with four or lesser colors. It is obvious for Alice to win with five colors and for Bob to win with three, hence it suffices to show that Bob wins on P_n with four colors. Assume that the players have four colors in the set, say $C = \{1, 2, 3, 4\}$, to use. In order to win, Bob needs the following set-up:

Figure 1: Required set-up for Bob to win.

The fundamental strategy of Bob is to color five positions away from the position already colored by Alice with the same color.

Let $p : c$ denote that the position p is colored with c and $*$ denotes the position which is outside of the configuration. Consider the configuration in Figure 2, Table 1 explains Bob's strategy for winning when Alice colors anywhere in this configuration.

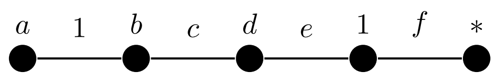

Figure 2: Configuration for trap 1.

Alice's move	Bob's move
$a : 2$	$e : 2$. This leads to Bob's win, since b, c, and d must receive three different colors.
$b : 2$	$e : 3$. This leads to Bob's win, since c and d must receive two different colors.
$c : 2$	$d : 3$. This brings a two-way winning chance for Bob in which he could either color $a : 4$ or $f : 4$ in his next move, but Alice can block only one of these.

Table 1: The above responses are considered as trap 1.

Other plays on d, e, and f follow by symmetry.

With respect to Alice's initial move, let the proof be seen in two cases.

Case 1. Suppose Alice does not start the game by coloring any of the first three positions from one end of P_n.

Without loss of generality, assume that Alice plays in the fourth position on one side of the path. Bob can always form the following set-up with the help of his fundamental strategic move:

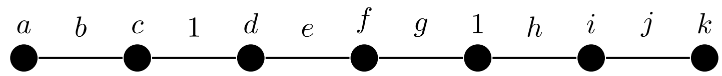

Figure 3: This formation is always possible in case 1.

The above set-up is always possible in this case. In order to win, Bob must build trap 2 given in Figure 4.

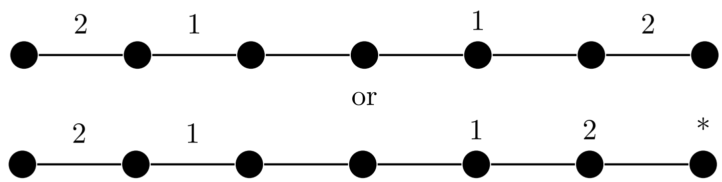

Figure 4: The above are the set-up for trap 2.

Consider Figure 3,

- If Alice colors either $i : 2$ or $j : 2$, then Bob can easily form trap 2 by coloring $b : 2$.
- Similarly, by symmetry, if Alice colors $a : 2$ or $b : 2$, then Bob can color $i : 2$.
- If Alice colors $k : 2$, then Bob has to use a different strategy by coloring $h : 3$. This forces a two-way winning for Bob, i.e., in his next move, he can color either $j : 4$ or $d : 3$. But Alice can block only one of these.
- If Alice colors anywhere away from this set-up, then Bob should color $i : 2$ (or $b : 2$). In this situation, Bob must be very careful about a (or j) because if it is already colored with 1, then this becomes a problem for Bob when Alice, in her next move, colors $b : 3$ (or $i : 3$). Therefore, the choice of i or b must be made with certainty.

The advantage of constructing trap 2 is that the usage of color 2 is limited within the segment. So, if Alice colors within it, then Bob wins with the help of trap 1. If she does not color inside the configuration, then Bob can color either $e : 2$ or $f : 2$. Now, the usage of 2 is completely ruled-out. This obviously leads to Bob's win.

Case 2. Suppose Alice starts the game by coloring any of the first three positions from one end of P_n.

As usual, Bob follows with his fundamental strategic move. In order to win, Bob should build trap 3 or trap 4.

If trap 3 (see Figure 5) is formed and if Alice colors inside the set-up, then Bob can win with the help of trap 1 and trap 2. Otherwise, Bob should color

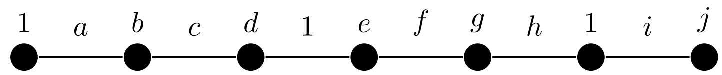

Figure 5: The above set-up is considered to be trap 3.

either d or e, which builds a two-way winning chance for him. Even if Alice, in her second move, colors inside this configuration, before the trap is formed, she cannot stop Bob from winning.

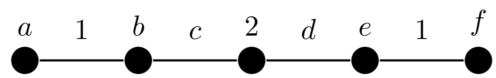

Figure 6: The above set-up is considered to be trap 4.

When trap 4 (see Figure 6) is formed, it must be made clear that either a or f or both can never acquire color 2. Assume if a cannot be colored with 2, then the usage of 1 and 2 are blocked in a, b, c, d, and e. Therefore, if Alice colors anywhere in the configuration, then Figure 1 can be easily built; otherwise, Bob can color $c : 3$ to build a two-way chance for him to win.

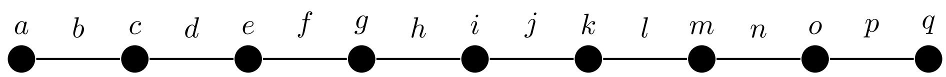

Figure 7: A set-up with 17 positions.

Consider Figure 7, assume that, Alice in her first move colors $a : 1$ and Bob colors $f : 1$. Alice is now forced to color only at $i : 1$, $j : 1$, $k : 2$, or $m : 1$; otherwise, Bob can form trap 3 or trap 4.

- If Alice in her second move colors $i : 1$ (or $j : 1$), then Bob must color at five positions away from i (or j) with 1. This set-up also serves as a trap because wherever Alice colors inside the set-up, except d, Bob can win. If Alice colors away from the set-up or colors $d : 2$, then Bob should color $g : 3$ which brings a two-way chance to win, i.e., Bob, in his next move, can color either $b : 3$ or $p : 3$ (or $q : 3$) to form trap 1 or trap 2, respectively.

- If Alice in her second move colors $k : 2$, then Bob should color $g : 3$ which again forms a two-way winning chance for him.

- If Alice colors $m : 1$, then Bob should color $i : 2$. This forces Alice to play either at $j : 1$ or at $i : 2$; otherwise, Bob colors $j : 3$. Now, Bob, at his third move, can easily form trap 4 by coloring $c : 2$. Thus Bob wins.

Similarly, when Alice colors at b or c, Bob can easily win with the above-mentioned strategies. Thus, Alice requires a minimum of five colors to win. Hence, $\chi''_g(P_n) = 5$. □

The above-mentioned traps and strategies can be used to improve the incidence game chromatic number of paths. The trap 1 discussed in this paper was also discussed by Kim in [5]. He proved that the incidence game chromatic number of paths, $i_g(P_n) = 5$ for $n \geq 13$. In order to prove the following corollary, it is necessary to establish a relationship between the incidence and the total graphs of paths. The total game coloring of a graph G can also be defined as the vertex coloring game on the total graph of G, $T(G)$, i.e., $\chi''_g(G) = \chi_g(T(G))$. Since, $T(P_n)$ has $2n-1$ vertices, it can be concluded that $\chi_g(T(P_n)) = 5$ for $|V(T(P_n))| \geq 17$. Similarly, the incidence game coloring of a graph G can be defined as the vertex coloring game on the incidence graph of the graph, G^I, i.e., $i_g(G) = \chi_g(G^I)$. Moreover, the incidence graph of a path can be obtained by deleting exactly one vertex of $T(P_n)$ with degree $\delta(=2)$; therefore, ${P_n}^I$ has $2n-2$ vertices.

Corollary 3.1. $i_g(P_n) = 5$, *for* $|V(P_n)| \geq 10$.

Let C_n be a cycle on n vertices, and it has $2n$ positions. The total game chromatic number of cycles can also be found using the same way as that of paths. Moreover, the traps discussed in paths will again be recalled in the proof of cycles.

Theorem 3.2. $\chi''_g(C_n) = 5$, *for* $n \geq 3$.

Proof. Let C_n be a cycle with $n \geq 3$, where n denotes the number of vertices. It is trivial from the upper bound that $\chi''_g(C_n) \leq 5$. Following the proof of paths, it is enough to show that Bob has a winning strategy on C_n with four colors. Assume that the players have four colors in the color set, say $C = \{1, 2, 3, 4\}$, to use. In order to win, Bob needs to build the set-up in Figure 1.

Figure 8: The game on C_3.

Case 1. Consider C_3, there are six positions in C_3, namely, p_1, p_2, p_3, p_4, p_5, and p_6. If Alice, in her first move, colors some position p_i with 1, then Bob should color the position not adjacent to it with a different color (see Figure 8 - left). Thus, the usage of these two colors are not allowed anywhere in C_3, this forces Alice to use another color, and Bob should repeat the same (see Figure 8 - right). This finally helps Bob win by forming the set-up discussed in Figure 1.

Case 2. Consider C_4 and C_5, let Alice start the game by coloring any position, say p_0, with 1. Bob has to color five positions away from p_0 with 1. Now, the usage of 1 is not allowed anywhere in the graph. Thus, in Alice's second move, wherever she colors, Bob can win with the help of trap 1.

Case 3. Consider C_6, here, the players start the game as described in case 2 (see Figure 9).

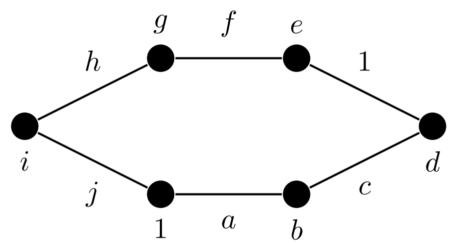

Figure 9: The situation on C_6 after the first move of the players.

In the above figure, Alice cannot color anywhere at a, b, c, d, e, f, i, and j; otherwise, Bob wins with the help of trap 1 or trap 2. Therefore, the only places left for Alice to color are h and g. Moreover, Alice cannot use color 2 at these places, because Bob can easily win by forming trap 4. If Alice colors any of them with 1, then Bob should form trap 2 by coloring i or f with 2. Thus Bob wins.

Case 4. Consider C_7, C_8, and C_9, the first move follows as already discussed. If Alice in her second move uses any color other than 1, then Bob can form trap 3 by coloring five positions away from any of the previously colored 1's. Therefore, Alice, in her second move, is forced to use 1.

In case of C_7,

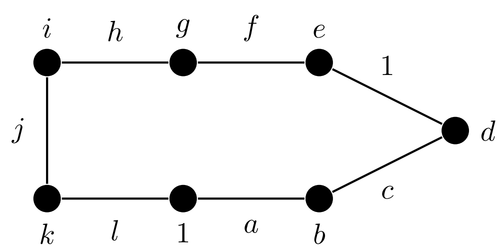

Figure 10: The situation on C_7 after the first move of the players.

Alice can only color either g or j with 1; otherwise, as mentioned, Alice will herself form trap 3. With this, Bob can easily form trap 4.

In case of C_8,

Alice can only color h or k with 1; otherwise, Bob can form trap 3 or Alice herself forms trap 3. Without loss of generality (see Figure 11), let Alice colors $h : 1$, then Bob should color $l : 2$. This forces Alice to color either $i : 2$ or $k : 1$, thus Bob can color $c : 2$ and win by the formation of trap 4.

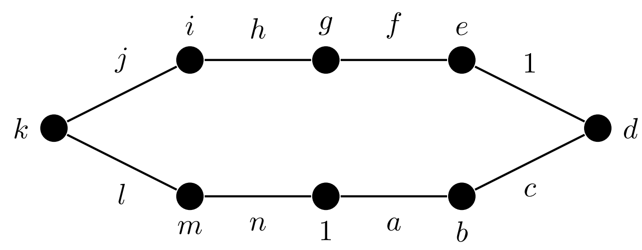

Figure 11: The situation on C_8 after the first move of the players.

In case of C_9, with the similar construction, it can be seen that Alice can possibly color either j or k with 1. Moreover, even if Alice colors at j or k, Bob can easily form trap 4.

Case 5. For cycles with $n \geq 10$, irrespective of Alice's move, trap 3 can easily be formed by Bob in his second move.

Therefore, Alice requires at least five colors to win on cycles, $\chi''_g(C_n) \geq 5$. Hence, it can be concluded that $\chi''_g(C_n) = 5$ for cycles with $n \geq 3$. □

A star, $K_{1,n}$ is a tree with a central vertex adjacent to all other outer vertices, where the degree of all the outer vertices are one. Also, $\Delta(K_{1,n}) = n$.

Theorem 3.3. $\chi''_g(K_{1,n}) = \Delta + 1$, *for* $n \geq 2$.

Proof. Let $K_{1,n}$ be a star with $n \geq 2$. Let Alice, always, in her first move color the central vertex, say v_0, with 1. Since v_0 is adjacent to all other vertices and edges, 1 cannot be repeated anywhere on the graph. Thus, Bob has to opt for another color. Alice has to follow a simple strategy that for every color Bob places on a vertex (or edge), Alice should always have to use the same color on an edge (or a vertex) which is not incident to it. Therefore, with n vertices, other than v_0, and n edges, Alice can always win with $n + 1$ colors. □

The following theorem is a direct consequence of the results related to total coloring of a graph:

Theorem 3.4. *Let G be a graph*

1. $\chi''_g(G) = 1$ *if and only if G is an empty graph.*
2. *If G is a connected graph, then* $\chi''_g(G) \geq 3$.
3. *There exists no graph G such that* $\chi''_g(G) = 2$.

4. Conclusion

Certainly, game coloring is a topic with enormous ideas directly or indirectly providing several real world applications, especially while dealing with two or more uncooperative partners. It can also be used in the fields of optimization, network security, traffic maintenance, etc. In this paper, three general classes of graphs are considered and the total game chromatic number of those graphs are obtained. These results can also be extended to other graphs.

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