On binary block codes associated to UP-algebras

Moin A. Ansari*
Department of Mathematics
College of Science
Post Box 2097, New Campus
Jazan University, Jazan
KSA
maansari@jazanu.edu.sa

Ali N.A. Koam
Department of Mathematics
College of Science
Post Box 2097, New Campus
Jazan University, Jazan
KSA
akoum@jazanu.edu.sa

Azeem Haider
Department of Mathematics
College of Science
Post Box 2097, New Campus
Jazan University, Jazan
KSA
aahaider@jazanu.edu.sa

Abstract. In this article, we define the notion of a UP-algebra valued function on a set and investigate related properties. We establish the binary block codes generated by UP-algebras valued function. We have also shown that for every binary block-code C, there exists a UP-algebra and UP-valued function which determines C whereas UP-algebras associated to a binary block code are not unique up to isomorphism.

Keywords: UP-algebras, binary block codes, partially ordered set.

1. Introduction

Logical algebras like BCI/BCK, BE and KU-algebras with their fuzzy and Intuitionistic concepts have become the keen interest for researchers in recent years and widely considered as a strong tool for information system and many other branches of Computer Sciences including fuzzy informatics with rough and soft concepts. Imai and Iseki [12] introduced BCK/BCI algebras as a generalization of the concept of set-theoretic difference and proportional calculi. In computer science, a block code is a type of channel coding. It adds redundancy to a message so that, at the receiver, one can decode the message with a minimum number of errors, where it is already provided that the information rate would not exceed the channel capacity. Block code simply encodes strings formed by an alphabet set $\mathcal{B}$ into code words by encoding each letter of $\mathcal{B}$ separately. Block codes can be source codes used in data compression, or channel codes used for detection and correction of channel errors [14]. Codes based on family of algorithms was constructed by Lempel and Ziv [20], which is applicable for real-world problems and sequences. A detailed terminology based on codes and decoding through graphs are discussed in [19].

Block codes in different logical algebras are being studied by numerous researchers in past few years. Jun and Song [13] introduced codes based on BCK-algebras whereas Surdive et al. studied coding theory in hyper BCK-algebras [18]. Further Fu and Xin [6] defined the concept of block codes in lattices and gave some properties based on them. Next Mostafa et al. [16] applied coding theory and gave some relation and connection between coding theory and KU-algebras.

UP-algebras was introduced by Iampan [7]. Further Iampan et. al contributed on different aspects related to UP-algebras in ([10], [8], [9], [11]). Recently Mosrijai and Iampan have introduced a new branch of bialgebraic structures: UP-bialgebras [15]. Senapati et. al [17] represented UP-algebras in inter-valued intuitionistic fuzzy environment. Moin et al. [5] introduced roughness in UP-algebras. Graphs of UP-algebras are studied by Moin et al. [4]. Further Moin et al. defined and studied concepts based on KU-algebras containing (α, β)-US soft sets in [1] whereas Koam et al. introduced the concept of n-ary block codes related to KU-algebras in [2]. Furthermore JU-algebras and p-closure ideals is introduced and studied by Moin et al. in [3].

In this paper, we introduce the UP-valued functions and investigate its several properties. Also, we establish block-codes by using the notion of UP-valued functions. We show that every finite UP-algebra determines a block-code.

Section 1, we have given introduction. Section 2 contains preliminaries and related definitions with some examples. Section 3 is based on main results. Section 4 is the conclusion of our work.

2. Preliminaries

In this section, we shall consider concepts based on UP-algebras, UP-subalgebras, UP-ideals, UP-valued function (cut function) and other important terminologies with examples and some related results. An UP-algebra is a type of logical algebras which is under investigation by several researchers.

Definition 2.1 ([7]). *By a UP-algebra we mean an algebra* $(U, *, \emptyset)$ *of type* $(2, 0)$ *with a single binary operation* $*$ *that satisfies the following identities: for any* $x, y, z \in U$,
(UP-1): $(y * z) * [(x * y) * (x * z)] = \emptyset$,
(UP-2): $\emptyset * x = x$,
(UP-3): $x * \emptyset = \emptyset$,
(UP-4): $x * y = y * x = \emptyset$ implies $x = y$.

An UP-algebras U is said to be commutative if $x * (x * y) = y * (y * x)$.

We define partial order relation in a UP-algebra U as $y \leq x$ if and only if $x * y = \emptyset$. If $(U, *, \emptyset)$ and $(V, \circ, \emptyset)$ are two UP-algebras then a map $f : U \longrightarrow C$ with the property $f(x * y) = f(x) \circ f(y)$, for all $x, y \in U$, is called a UP-algebra morphism. If f is bijective map, then f is an isomorphism of UP-algebras.

Example 2.1. Let $U = \{\emptyset, a, b, c\}$ be a set in which $*$ is defined by the following Cayley table

$*$	$\emptyset$	a	b	c
$\emptyset$	$\emptyset$	a	b	c
a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
b	$\emptyset$	a	$\emptyset$	c
c	$\emptyset$	a	b	$\emptyset$

It is easy to see that $U = \{\emptyset, a, b, c\}$ is a UP-algebra.

Example 2.2. Let $U = \{\emptyset, a, b, c, d\}$ be a set in which $*$ is defined by the following Cayley table

$*$	$\emptyset$	a	b	c	d
$\emptyset$	$\emptyset$	a	b	c	d
a	$\emptyset$	$\emptyset$	b	c	d
b	$\emptyset$	$\emptyset$	$\emptyset$	c	d
c	$\emptyset$	$\emptyset$	b	$\emptyset$	d
d	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Here $U = \{\emptyset, a, b, c, d\}$ is UP-algebra.

Example 2.3. Let $U = \{\emptyset, a, b, c, d\}$ be a set in which $*$ is defined by the following Cayley table

$*$	$\emptyset$	a	b	c	d
$\emptyset$	$\emptyset$	a	b	c	d
a	$\emptyset$	$\emptyset$	b	a	d
b	$\emptyset$	a	$\emptyset$	c	d
c	$\emptyset$	$\emptyset$	b	$\emptyset$	d
d	$\emptyset$	a	b	c	$\emptyset$

Here $U = \{\emptyset, a, b, c, d\}$ is UP-algebra.

Example 2.4. Let $X = \{t_n | n = 1, 2, 3, \ldots 9\}$ and define a binary operation $*$ on X as $t_i * t_j = t_k \ \forall\ t_i, t_j \in X$ where $k = \frac{lcm(i,j)}{j}$. Then $(X, *, t_1)$ is a UP-algebra. The following table represents this operation:

$*$	t_1	t_2	t_3	t_4	t_5	t_6
t_1	t_1	t_2	t_3	t_4	t_5	t_6
t_2	t_1	t_1	t_3	t_2	t_5	t_3
t_3	t_1	t_2	t_1	t_4	t_5	t_2
t_4	t_1	t_1	t_3	t_1	t_5	t_3
t_5	t_1	t_2	t_3	t_4	t_1	t_6
t_6	t_1	t_1	t_1	t_2	t_5	t_1

Lemma 2.1 ([8])**.** *In a UP-algebra U the following properties hold for any $x, y, z \in U$:*
(UP-5) $x * x = \emptyset$,
(UP-6) $x * y = \emptyset$ *and* $y * z = \emptyset \Rightarrow x * z = \emptyset$,
(UP-7) $x * y = \emptyset \Rightarrow (z * x) * (z * y) = \emptyset$,
(UP-8) $x * y = \emptyset \Rightarrow (y * z) * (x * z) = \emptyset$,
(UP-9) $x * (y * x) = \emptyset$,
(UP-10) $(y * x) * x = \emptyset \iff x = y * x$, *and*
(UP-11) $x * (y * y) = \emptyset$

Lemma 2.2. *Let $U = (A, *, \emptyset)$ be UP-algebras, then define a binary relation $\leq$ on U as follows: for all $x, y, z \in A$*
(UP-12) $x \leq x$,
(UP-13) $\emptyset \leq x$.
(UP-14) $y * x \leq x$.
(UP-15) $x \leq y$ *and* $y \leq x \Rightarrow x = y$,
(UP-16) $y \leq x$ *and* $z \leq y \Rightarrow z \leq x$,
(UP-17) $y \leq x \Rightarrow z * y \leq z * x$,
(UP-18) $y \leq x \Rightarrow x * z \leq y * z$,
(UP-19) $(x * y) * (x * z) \leq y * z$.

3. UP-valued functions

Throughout we consider U to be a non-empty set and X to be a UP-algebra, respectively.

Definition 3.1. *A mapping $\tilde{U} : U \to X$ is called a UP-valued function or simply UP-function on U.*

Definition 3.2. *A cut function of $\tilde{U}$, for $t \in X$, is defined to be a mapping $\tilde{U_t} : U \to \{0, 1\}$, such that $(\tilde{U_t}(x) = 1 \iff \tilde{U}(x) * t = \emptyset)\ \forall\ x \in U$. It is obvious here that $\tilde{U_t}$ is the characteristics function of the following subset of U, called a cut subset or a t-cut of $\tilde{U}$: Here $U_t := \{x \in U | \tilde{U}(x) * t = \emptyset\}$. It is clear that $U_\emptyset = U$.*

Example 3.1. Let $U = \{l, m, n\}$ and $X = \{\emptyset, a, b, c, d\}$ is the UP-algebras given in example 2.

The function $\tilde{U} : U \to X$ given by $\tilde{U}(l) = a$, $\tilde{U}(m) = b$, $\tilde{U}(n) = c$, is a UP-function on U, and its cut subsets are $U_\emptyset = U$, $U_a = U$, $U_b = \{m\}$, $U_c = \{n\}$ and $U_d = \{\}$,

Proposition 3.1. *Every UP-function $\tilde{U} : U \to X$ on U is represented by the supremum of the set $\{t \in X | \tilde{U}_t(x) = 1\}$, that is $(\tilde{U}(x) = sup\{t \in X | \tilde{U}_t(x) = 1\})$ for all $x \in U$.*

Proof. For any $x \in U$, let $\tilde{U}(x) = r \in X$. Then $\tilde{U}(x) * r = \emptyset$, and so $\tilde{U}_r(x) = 1$. Assume that $\tilde{U}_t(x) = 1$ for $t \in X$. Then $\emptyset = \tilde{U}(x) * t = r * t$, i.e., $t \leq r$. Since $r \in \{t \in X | \tilde{U}_t(x) = 1\}$, it follows that $\tilde{U}(x) = r = sup\{t \in X | \tilde{U}_t(x) = 1\}$. □

Proposition 3.2. *If $\tilde{U} : U \to X$ is a UP-function on U, then $(\tilde{U}(x) = sup\{\tilde{U}_t(x) \diamond t | t \in X\})$ where*

$$\tilde{U}_t(x) \diamond t = \begin{cases} t, & if\ \tilde{U}_t(x) = 1, \\ \emptyset, & otherwise. \end{cases}$$

Proof. Follows from Proposition 3.1. □

Proposition 3.3. *Let $\tilde{U} : U \to X$ be a UP-function on U. Then $t * s = \emptyset \Rightarrow U_t \subseteq U_s$ for all $s, t \in X$.*

Proof. Let $s, t \in X$ be such that $t * s = \emptyset$. For any $x \in U_t$, we have $\tilde{U}(x) * t = \emptyset$. Now by Lemma 1(4) we have that $(t * s) * (\tilde{U}(x) * s) = \emptyset \Rightarrow \emptyset * (\tilde{U}(x) * s) = \emptyset \Rightarrow \tilde{U}(x) * s = \emptyset$ (by UP-2) $\Rightarrow x \in U_s$. Hence $U_t \subseteq U_s$. □

Proposition 3.4. *Let $\tilde{U} : U \to X$ be a UP-algebra on U. Then:*

1. *$\tilde{U}(x) \neq \tilde{U}(y) \Longleftrightarrow U_{\tilde{U}(x)} \neq U_{\tilde{U}(y)}$ for all $x, y \in U$.*
2. *$\tilde{U}(x) * t = \emptyset \Longleftrightarrow U_{\tilde{U}(x)} \subseteq U_{(t)}$ for all $t \in X, x \in U$.*

Proof. 1. If $U_{\tilde{U}(x)} \neq U_{\tilde{U}(y)}$, then it is obvious that $\tilde{U}(x) \neq \tilde{U}(y)$ for all $x, y \in U$. Then $\tilde{U}(x) * \tilde{U}(y) \neq \emptyset$ or $\tilde{U}(y) * \tilde{U}(x) \neq \emptyset$. Thus $U_{\tilde{U}(x)} = \{z \in U | \tilde{U}(z) * \tilde{U}(x) = \emptyset\} \neq \{z \in U | \tilde{U}(y) * \tilde{U}(z) = \emptyset\} = U_{\tilde{U}(y)}$.

2. If $\tilde{U}(x) * t = \emptyset$, then by Proposition 3.3 $U_{\tilde{U}(x)} \subseteq U_t$. Let $t \in X$ and $x \in U$ be such that $U_{\tilde{U}(x)} \subseteq U_t$. If $\tilde{U}(x) * t \neq \emptyset$, then $x \notin U_t$. Since $\tilde{U}(x) * \tilde{U}(x) = \emptyset$, it follows that $x \in U_{\tilde{U}(x)}$ so that $U_{\tilde{U}(x)} \nsubseteq U_t$ which is a contradiction. □

Corollary 3.1. *Let $\tilde{U} : U \to X$ be a UP-function on U. Then $\tilde{U}(x) * \tilde{U}(y) = \emptyset \Longleftrightarrow \tilde{U}_{U(x)} \subseteq \tilde{U}_{U(y)}$ for all $x, y \in U$.*

Proof. It follows directly from Proposition 3.3 and Proposition 3.4. □

For a UP-function $\tilde{U} : U \to X$ on U, consider the following sets:
$U_X := \{U_t | t \in X\}, \tilde{U}_X := \{\tilde{U}_t | t \in X\}$.

Proposition 3.5. *Let $\tilde{U} : U \to X$ be a UP-function on U then there exists sup Y in X such that $U_{\sup\{t|t\in Y\}} = \bigcap\{U_t|t \in Y\}$, for all $Y \subseteq X$.*

Proof. We suppose that for every subset Y of X, there exists a sup Y in X. We have $x \in U_{\sup\{t|t\in Y\}} \iff \tilde{U}(x) * \sup\{t|t \in Y\} = \emptyset \iff \tilde{U}(x) * r = \emptyset \ \forall r \in Y \iff x \in U_r \ \forall r \in Y \iff x \in \bigcap\{U_t|t \in Y\}$. □

Corollary 3.2. *Let $\tilde{U} : U \to X$ be a UP-function on U, where X is a bounded UP-algebra. Then, $U_{\sup\{t|t\in S\}} = \bigcap\{U_t|t \in S\} \ \forall S \subseteq X$.*

Corollary 3.3. *Let $\tilde{U} : U \to X$ be a UP-function on U. Suppose that for any subset Y of X, there exists a sup of Y. Then, $U_s \cap U_t \in U_X \ \forall s, t \in Y$.*

Converse of the above corollary may not be true as shown in the following example:

Example 3.2. Let $U = \{x, y\}$ be a set and $X = \{\emptyset, a, b, c, d\}$ be a UP-algebra given in Example 2.3.Then, for a function $\tilde{U} : U \to X$ such that $\tilde{U}(x) = a$ and $\tilde{U}(y) = b$. Here, we obtained cut sets of $\tilde{U}$ are as follows: $U_\emptyset = U, U_a = \{x\}, U_b = \{y\}, U_c = U_d = \emptyset$.

It is clear here that $U_a \cap U_b \in U_X$ but $\sup\{a, b\}$ does not exists in X.

Proposition 3.6. *Let $\tilde{U} : U \to X$ be a UP-function on U. Then, $\bigcup_{t\in X} U_t = U$.*

Proof. For every $x \in U$, let $\tilde{U}(x) = t \in X$. Then $x \in U_t$, hence $x \in \bigcup_{t\in X} U_t$. Thus $U \subseteq \bigcup_{t\in X} U_X$. □

Proposition 3.7. *Let $\tilde{U} : U \to X$ be a UP-function on U. Then $\bigcap_{x\in U_t} U_t \in U_x \ \forall x \in U$.*

Proof. For any $x \in U$, $x \in U_t \iff \tilde{U}_t(x) = 1$. It follows from Proposition 5 that $\bigcap\{U_t|x \in U_t\} = \bigcap\{U_t|\tilde{U}(x) = 1\} = U_{\sup\{t|\tilde{U}_t(x)=1\}} \in U_X$. □

Next we define equivalence relation $\sim$ on X. Let $\tilde{U} : U \to X$ be a UP-function on U and let $\sim$ be a binary relation on X defined by $s \sim t \iff U_s = U_t$ $\forall s, t \in X$. This relation is an equivalence relation on X. The corresponding equivalence class of an element $t \in X$ is denoted by $[t]^\sim$.

Let $\tilde{U}(U) := \{t \in X|\tilde{U}(x) = t \text{ for some } x \in U\}$ and for $t \in X$, let $[t) := \{x \in X|x * t = \emptyset\}$ and $(t] := \{x \in X|t * x = \emptyset\}$.

Proposition 3.8. *For a UP-function $\tilde{U} : U \to X$ on U, we have $s \sim t \iff [s) \cap \tilde{U}(U) = [t) \cap \tilde{U}(U) \ \forall s, t \in X$.*

Proof. If $s \sim t \iff U_s = U_t \iff \tilde{U}(x) * s = \tilde{U}(x) * t = \emptyset \ \tilde{U}(x) \in \tilde{U}(U)$

$\iff s < \tilde{U}(x) \iff t < \tilde{U}(x) \ \tilde{U}(x) \in \tilde{U}(U) \iff [s) \bigcap \tilde{U}(U) = [t) \bigcap \tilde{U}(U)$

Given $s \sim s \iff U_s = U_t \iff s * \tilde{U}(x) = \emptyset \ \forall x \in U \iff \tilde{U}(x) \in [s) \ \forall x \in U = \tilde{U}(x) \in [t) \ \forall x \in U \iff [s) \cap \tilde{U}(U) = [t) \cap \tilde{U}(U)$. □

Example 3.3. Consider $U = \{a, b, c, d\}$ and X be a UP-algebra given in Example 2.4. Define a UP-function $\tilde{U} : U \to X$ by $\tilde{U}(a) = t_2, \tilde{U}(b) = t_1, \tilde{U}(c) = t_5, \tilde{U}(d) = t_3$. Then corresponding cut sets of $\tilde{U}$ are given as:

$U_{t_1} = U$, $U_{t_2} = \{a\}$, $U_{t_3} = \{d\}$, $U_{t_5} = \{c\}$, $U_{t_4} = U_{t_6} = \{\}$.

Lemma 3.1. *Let $\tilde{U} : U \to X$ be a UP-function on U. For every $x \in U$, we have $\tilde{U}(x) = \sup([\tilde{U}(x)]^{\sim})$, that is, $\tilde{U}(x)$ is the maximal element of the $[[\tilde{U}(x)]^{\sim}$ where ever it belongs.*

4. Codes based on UP-algebras

Consider $U = \{1, 2, 3, \ldots, n\}$ and let X be a UP-algebra. Every UP-function $\tilde{U} : U \to X$ on U determines a binary block-code C of length n in such a way that to every $[x]^{\sim}$, where $x \in X$, there would be codewords $c_x = x_1x_2x_3\ldots x_n$ with $x_i = j \iff \tilde{U}_x(i) = j$, for $i \in U$ and $j \in \{0, 1\}$. Let $c_x = x_1x_2x_3\ldots x_n$ and $c_y = y_1y_2y_3\ldots y_n$ be two codewords belonging to a binary block-code C. We define an order relation $\preceq$ on the set of codewords belonging to a binary block-code C as $c_x \preceq c_y \iff y_i \leq x_i$ for $i = 1, 2, 3, \ldots, n$.

Example 4.1. Let $U = X = \{\emptyset, a, b, c\}$ be a UP-algebra given in Example 2.1. Let $\tilde{U} : U \to X$ be a UP-function on X given by $\tilde{U}(\emptyset) = \emptyset, \tilde{U}(a) = a, \tilde{U}(b) = b, \tilde{U}(c) = c$ with its cut sets as: $U_\emptyset = U$, $U_a = \{a\}$, $U_b = \{a, b\}$ and $U_c = \{a, c\}$. We have the following table

$\tilde{U}_x$	$\emptyset$	a	b	c
$\tilde{U}_\emptyset$	1	1	1	1
$\tilde{U}_a$	0	1	0	0
$\tilde{U}_b$	0	1	1	0
$\tilde{U}_c$	0	1	0	1

Therefore, $C = \{1111, 0100, 0110, 0101\}$ and graphically it is represented as:

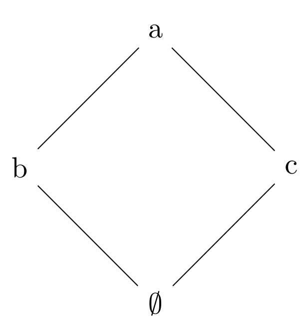

Fig. (i), $(X, \leq)$

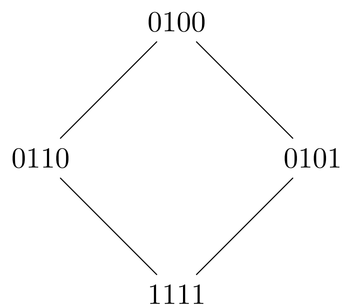

Fig. (ii), $(C, \preceq)$.

Now, we have the following theorem:

Theorem 4.1. *Every finite UP-algebra X determines a block-code C such that $(X, \leq) \simeq (C, \preceq)$.*

Proof. Let $X = \{t_1, t_2, \ldots, t_n\}$ be a finite UP-algebra in which t_1 is the smallest element and $\tilde{U} : U \to X$ be the identity UP-function on X. The decomposition of $\tilde{U}$ gives a family $\{\tilde{U}_t | t \in X\}$ which is the required code under the order defined by the relation $c_x \preceq c_y \Longleftrightarrow y_i \leq x_i$ for $i = 1, 2, 3, \ldots, n$. Next suppose that $f : X \to \{\tilde{U}_t | t \in X\}$ be a function defined by $f(t) = \tilde{U}_t$ for all $t \in X$. Hence by Lemma 3.1, every equivalence class contains exactly one element. Thus f is one-to-one. Let $x, y \in X$ be such that $y * x = t_1$. Then $U_y \subseteq U_x$ by using Proposition 3.3. Therefore $\tilde{U}_x \leq \tilde{U}_y$. Hence f is an isomorphism. □

Example 4.2. Consider $U = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and X be a UP-algebra given in Example 2.4. Define a UP-function $\tilde{U} : U \to X$ by $\tilde{U}(a_i) = t_i,\ i = 1, 2, \ldots 6$. Then we have the table below:

	a_1	a_2	a_3	a_4	a_5	a_6
$\tilde{U}_{t_1}$	1	1	1	1	1	1
$\tilde{U}_{t_2}$	0	1	0	1	0	1
$\tilde{U}_{t_3}$	0	0	1	0	0	1
$\tilde{U}_{t_4}$	0	0	0	1	0	0
$\tilde{U}_{t_5}$	0	0	0	0	1	0
$\tilde{U}_{t_6}$	0	0	0	0	0	1

Cut sets of $\tilde{U}$ are: $U_{t_1} = U$, $U_{t_2} = \{a_2, a_4, a_6\}$, $U_{t_3} = \{a_3, a_6\}$, $U_{t_4} = \{a_4\}$, $U_{t_5} = \{a_5\}$, $U_{t_6} = \{a_6\}$. Therefore, $C = \{111111, 010101, 001001, 000100, 000010, 000001\}$ and its graph representation is as below

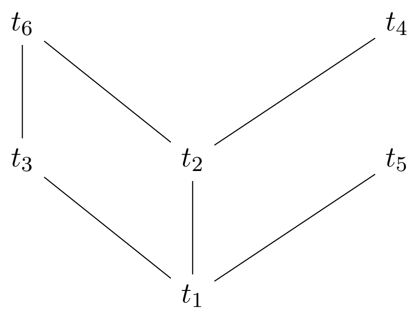

Fig. (iii), $(X, \leq)$

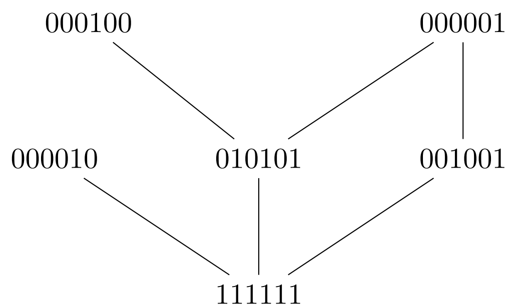

Fig. (iv), $(C, \preceq)$

Proposition 4.1. *Suppose that $(X, \leq)$ be a finite partial order set with the minimum element $\emptyset$. We define a binary relation $*$ on X as follows:*

1. $\emptyset * x = x, x * x = \emptyset,\ x \in X,$
2. $x * y = \emptyset$ *if* $y \leq x,\ x, y \in X,$
3. $x * y = y,$ *otherwise*

Then the structure $(X, *, \emptyset)$ is a UP-algebra.

Proof. We just need to show that $(y*z)*((x*y)*(x*z)) = \emptyset$, for all $z, y, z \in X$. We have three cases here:

Case i At least one element is $\emptyset$.

1. $x = \emptyset$; $(y * z) * ((\emptyset * y) * (\emptyset * z)) = (y * z) * (y * z) = \emptyset.$
2. $y = \emptyset$; $(\emptyset * z) * ((x * \emptyset) * (x * z)) = z * (x * z) = z * z = \emptyset.$
3. $z = \emptyset$; $(y * \emptyset) * ((x * y) * (x * \emptyset)) = \emptyset * ((x * y) * \emptyset) = \emptyset * \emptyset = \emptyset.$

Case ii One element is comparable with another.

1. $x \leq y$; $(y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset,$
2. $x \leq z$; $(y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset,$
3. $y \leq z$; $(y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset,$
4. $y \leq x$; $(y*z)*((x*y)*(x*z)) = (y*z)*(\emptyset*(x*z)) = (y*z)*(x*z) = z * z = \emptyset,$
5. $z \leq x$; $(y*z)*((x*y)*(x*z)) = (y*z)*((x*y)*\emptyset) = (y*z)*\emptyset = \emptyset,$
6. $z \leq y$; $(y * z) * ((x * y) * (x * z)) = \emptyset * (y * z) = y * z = \emptyset.$

Case iii Two elements are comparable with the third.

1. $x \le y$ and $z \le y$; $(y * z) * ((x * y) * (x * z)) = \emptyset * (y * z) = y * z = \emptyset$,
2. $x \le z$ and $y \le z$; $(y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset$,
3. $y \le x$ and $z \le x$; $(y * z) * ((x * y) * (x * z)) = (y * z) * (\emptyset * \emptyset) = (y * z) * \emptyset = \emptyset$.

Rest cases will follow from transitive property of partial order set $(X, \le)$. □

Let C be a binary block code with n codewords of length n. Consider the matrix $M_C = (m_{i,j}) \in \mathbb{M}_C(\{0, 1\})$ with the rows consisting of the codewords of C. We say $\mathbb{M}_C(\{0, 1\})$ associated matrix to code C.

A question was asked by Jun and Song in [13] for BCK-algebras that for every block code C do we have a BCK-algebra for which we can determine C. Here we have answered that question in positive sense but partially for a UP-algebra in the following results.

Theorem 4.2. *Let $\mathbb{M}_C(\{0, 1\})$ be associated matrix to code C, if the codeword $1111\ldots1$ is in C and the matrix M_C is upper triangular with $m_{ii} = 1$, for all $i \in \{1, 2, 3, \ldots, n\}$, then there exists a set U with n elements, a UP-algebra X and a UP-function $f : U \to X$ such that f determines C.*

Proof. Considering the lexicographic order $\le_{lex}$ in C we find that $(C, \le_{lex})$ is a totally ordered set. Let $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ with the lexi order $c_n \le_{lex} c_{n-1} \le_{lex} \ldots \le_{lex} c_2 \le_{lex} c_1$. Hence $c_1 = 111\ldots1$ and $c_n = 00\ldots01$. Then by partial order in C, $(C, \preceq)$ is a POS with $c_1 \preceq c_i, i \in \{1, 2, 3\ldots, n\}$. Here zero element and maximal element in $(C, \preceq)$ is $\emptyset = c_1$ and c_n respectively. We define a binary operation $\circ$ on $X = C$ as defined in Proposition 4.1. Then we get that $X = (C, \circ, c_1)$ is a UP-algebra and C is isomorphic to $\mathcal{C}_n$ as a UP-algebra, where $\mathcal{C}_n$ is a UP-algebra with n elements defined in Proposition 4.1. We suppose that $U = C$ and the identity map $f : U \to C, f(t) = t$ as a UP-function. The decomposition of f provides a family of maps $C_{c_n} = \{f_r : A \to \{0, 1\} | f_r(a) = 1,$ if and only if $f(a) \circ r = \emptyset, \forall a \in A, r \in X\}$. This set forms a binary block code C relative to the partial order relation $\preceq$. Further let $c_k \in C, 1 < k < n, c_k = 00\ldots0a_{i_k}\ldots a_{i_n}$, $a_{i_j} \in \{0, 1\}$. If $a_{i_j} = 0$, then $c_k \preceq c_{i_j}$. Also $c_{i_j} \circ c_k = \emptyset$. Again $a_{i_j} = 1$, we get that $c_{i_j} \preceq c_k$ or a_{i_j} and c_k can't be compared, hence $c_{i_j} \circ c_k = c_k$. □

Two different UP-algebras having same binary block code are said to be code similar algebras and denoted by $U_1 \sim U_2$. Let $\mathfrak{U}_n$ be set of all finite UP-algebra with n elements then the relation code similar is an equivalence relation on $\mathfrak{U}_n$ and that $\mathfrak{Q}_n$ denotes the quotient set then for $C \in \mathfrak{C}_n$, where $\mathfrak{C}_n$ is the set of block codes having similar properties given in Theorem 4.2. An equivalence class in $\mathfrak{Q}_n$ is denoted by $\breve{C} = \{U \in \mathfrak{U}_n | U$ determines the binary block code C$\}$.

Proposition 4.2. *The quotient set* $\mathfrak{Q}_n$ *has* $2^{\frac{(n-1)(n-2)}{2}}$ *elements, which is the same cardinal as in the case of* $\mathfrak{C}_n$.

Proof. Let M_C be the associated matrix of $C \in \mathfrak{C}_n$ then we shall compute the cardinal of the set M_C which is an upper triangular matrix with $m_{ii} = 1$ for all $i \in \{1, 2, 3, \dots, n\}$. Second row of the matrix M_C has the form $(0, 1, a_3, \dots, a_n)$, where $a_3, \dots a_n \in \{0, 1\}$. Hence number of different rows of this type of matrix is 2^{n-2}. The third row of the matrix M_C has the form $(0, 0, 1, a_4, a_5, \dots a_n)$, where $a_4 \dots a_n \in \{0, 1\}$. Hence number of different rows of this type is 2^{n-3}. Therefore we get that the cardinal of the set $\mathfrak{C}_n$, is $2^{n-2}2^{n-3}\dots 2 = 2^{\frac{(n-1)(n-2)}{2}}$. □

For finite non-isomorphic UP-algebras with n elements cardinality is $\geq 2^{\frac{(n-1)(n-2)}{2}}$. We take the following definitions:

Definition 4.1. *A totally ordered relation* $\succeq_{lex}$ *on* $\mathfrak{C}_n$ *for the matrix* $M_{C_j}, j \in \{1, 2, \dots n\}$ *is defined as* $C_1 \succeq_{lex} C_2$ *if there is* $i \in \{2, 3, \dots n\}$ *such that* $r_1^{C_1} = r_2^{C_2} \dots r_{i-1}^{C_1} = r_{i-1}^{C_2}$ *and* $r_i^{C_1} \succeq_{lex} r_i^{C_2}$ *where* $\succeq_{lex}$ *represents lexicographic order.*

Definition 4.2. *For two associated matrix* M_{C_1} *and* M_{C_2}, *the partial order on* $\mathfrak{C}_n$ *for* $C_1, C_2 \in \mathfrak{C}_n$ *is defined as* $C_2 \gg C_1$ *such that* $r_1^{C_1} = r_1^{C_2} \dots r_{i-1}^{C_1} = r_{i-1}^{C_2}$ *and* $r_i^{C_1} \preceq r_i^{C_2}$.

Let $\Phi = (\phi_{ij}) \in \mathbb{M}\{0, 1\}$ be a matrix such that $\phi_{ij} = 1, i \leq j$, for all $i, j \in \{1, 2, 3 \dots, n\}$ and $\phi_{ij} = 0$ in the rest part. We get a minimum element $M_\omega = \Phi$ for the code ω in the partial order set $(\mathcal{C}_\mathfrak{n}, \gg)$ where elements in $\mathcal{C}_\mathfrak{n}$ are ascending order relative to $\succeq_{lex}$. Then we see that $(\mathcal{C}_\mathfrak{n}, \circ, \omega)$ is a non-commutative and non-implicative UP-algebra. This UP-algebra determines a binary block code $C_{\mathcal{C}_\mathfrak{n}}$ of length $2^{\frac{(n-1)(n-2)}{2}}$. It is clear that $\breve{C}_{\mathcal{C}_\mathfrak{n}} \in \mathfrak{C}_{2^{\frac{(n-1)(n-2)}{2}}}$.

Proposition 4.3. *Let* $A = (a_{i,j}) \in \mathbb{M}_{n\times m}(\{0, 1\})$ *be a matrix with rows lexicographic in ascending order. Consider a matrix* $B = (b_{i,j}) \in \mathbb{M}_q(\{0, 1\})$, $q = n + m$, *such that* B *is an upper triangular matrix with* $b_{ii} = 1$, $i \in \{1, 2, \dots q\}$. *In this way we see that* A *becomes a submatrix of* B.

Proof. We introduce new columns in the matrix A on the left side (from right to left) $\overbrace{00\dots01}^{n}, \overbrace{00\dots10}^{n}, \dots \overbrace{10\dots00}^{n}$ and obtained a new matrix E with n rows and $n+m$ columns. Now introducing the following m rows in the bottom of the matrix E:
$\overbrace{00\dots0}^{n}\overbrace{10\dots00}^{m}, \overbrace{00\dots0}^{n+1}\overbrace{01\dots00}^{m-1}, \dots \overbrace{00\dots01}^{n+m-1}$ to get matrix B. □

Theorem 4.3. *Assuming the above notations, let* C *be a binary block code with* n *codewords of length* $m, n \neq m$, *or a block code with* n *codewords of length* n *such that the codeword* $\overbrace{11\dots1}^{n\ times}$ *is not in* C, *or a block code with* n *codewords of* *length n such that the matrix M_C is not upper triangular. There are a natural number $q \geq max\{m, n\}$, a set A with m elements and a UP-function $f : A \to \mathcal{C}_q$ such that we obtain block code C_{C_n} contains the block code C as a subset.*

Proof. Assume C to be a binary block code, $C = \{c_1, c_2, \ldots c_n\}$, with codeword of length m and with its lexicographic order $c_1 \geq_{lex} c_2 \geq_{lex} \ldots \geq_{lex} c_n$. Let $M \in \mathbb{M}_{n,m}(\{0,1\})$ be the associated matrix with rows $c_1 \ldots c_n$ in this order. By using Theorem 4.2 we can extend the matrix M to a square matrix $B \in \mathbb{M}_q(\{0,1\}), q = m + n$, such that $B = (m'_{i,j})_{i,j \in \{1,2,3,\ldots q\}}$ is an upper triangular matrix with $m_{ii} = 1$, for all $i \in \{1, 2, \ldots q\}$. If the first row of B is not $\overbrace{11\ldots1}^{q}$ then we insert the first row $\overbrace{11\ldots1}^{q+1}$ and first column $1\overbrace{0\ldots0}^{q}$ to obtain a new matrix M'. Now for the matrix M', we obtain a UP-algebra $\mathcal{C}_q = \{c_1, c_2 \ldots c_q\}$, with $c_1 = \phi$ the zero of $\mathcal{C}_q$ and a binary code $C_{\mathcal{C}_q}$. Assuming that initial columns of M have in the new matrix M' positions $i_{j_1}, i_{j_2}, \ldots i_{j_m} \in \{1, 2, 3 \ldots q\}$, let $A = \{x_{j_1}, x_{j_2}, \ldots x_{j_m}\} \subseteq \mathcal{C}_q$. The UP-function $f : A \to \mathcal{C}_q, f(x_{j_i}) = x_j, i \in \{1, 2, 3 \ldots, m\}$, determines the binary block code $C_{\mathcal{C}_q}$ such that $C \subseteq C_{\mathcal{C}_q}$. □

Example 4.3. Let $C = \{0110, 0010, 1111, 0001\}$ be a binary block code. Code C can be written as $C = \{1111, 0110, 0010, 0001\} = \{c_1, c_2, c_3, c_4\}$ by lexicographic order. From Theorem 4.2, $c_1 \preceq c_i$, $i \in \{2, 3, 4\}$, $c_2 \preceq c_3$ where as c_2 can't be compared with c_4 and c_3 can't be compared with c_4. The $\circ$ operation on C is given in the table below

$\circ$	c_1	c_2	c_3	c_4
c_1	c_1	c_2	c_3	c_4
c_2	c_1	c_1	c_3	c_4
c_3	c_1	c_1	c_1	c_4
c_4	c_1	c_2	c_3	c_1

It is clear that C with $\circ$ is a UP-algebra.

For $A = \{\emptyset, 1, 2, 3\}$ the same binary block code C can be obtained from UP-algebra $(A, *, \emptyset)$ as given in the table below with UP-function $f : A \to A$, $f(x) = x$.

$*$	$\emptyset$	1	2	3
$\emptyset$	$\emptyset$	1	2	3
1	$\emptyset$	$\emptyset$	1	3
2	$\emptyset$	$\emptyset$	$\emptyset$	3
3	$\emptyset$	1	2	$\emptyset$

Since $(A, *, \emptyset)$ is a commutative UP-algebra and $(C, \circ, c_1)$ is a non-commutative UP-algebra. Therefore, $(A, *, \emptyset)$ is not isomorphic to $(C, \circ, c_1)$. Hence we can say that a UP-algebra associated to a binary block code as in Theorem 4.2 is not unique up to an isomorphism. If we start from the commutative UP-algebra $(A, *, c_1)$ to obtain the code C, and construct UP-algebra $(C, \circ, c_1)$ as in the Theorem 4.2, then we get that the algebra $(C, \circ, c_1)$ lost the commutative property even they are code similar.

Example 4.4. Let $C = \{11110, 10010, 10011, 00001\}$ be a binary block code. Using lexicographic order, we can write $C = \{11110, 10011, 10010, 00000\} = \{c_1, c_2, c_3, c_4\}$. Let $M_c \in \mathcal{M}_{4\times5}(\{0,1\})$ be the associated matrix,

$$\mathcal{M}_c = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using Proposition 4.3 we have the following upper triangular matrix which starts from the matrix M_C as given below

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since the first 1^st row is not $\overbrace{11\ldots1}^{9}$. Using Proposition 4.3 we insert a new row $\overbrace{11\ldots1}^{10}$ as a first row and a new column $\overbrace{10\ldots0}^{10}$ as a fixed column, then we obtain the following matrix:

$$M' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The binary block code $C = \{c_1, c_2, c_3, c_4\}$ whose codewords are the row of the matrix M', determines a UP-algebra $(C, \circ, c_1)$. Let $V = \{c_6, c_7, c_8, c_9, c_{10}\}$ and $f : V \to X, f(c_i) = c_i, i \in \{6, 7, 8, 9, 10\}$ be a UP-function which determines the binary block code $U = \{11111, 11110, 10011, 10010, 00000, 10000, 01000, 00100, 00010, 00001\}$ the code C is a subset of the code U.

5. Conclusion

In this work we have studied a branch of coding theory namely, *Binary Block Codes* associated to UP-algebras. At first we have defined and investigated UP-valued functions and their properties. By using these UP-functions we have associated every UP-algebra X to a block code C through cut-function. We have also partially answered the question raised by Jun and Song [13] in case of a UP-algebra that for each binary block code C we determined a UP-algebra X such that the binary block code generated by X, C_X contains code C. We have also shown that UP-algebras associated to a binary block code are not unique up to isomorphism.

References

[1] Moin A. Ansari, Ali N. A. Koam, Azeem Haider, *On KU-Algebras containing (α, β)-US soft sets*, Korean J. Math., 28 (2020), 89-104.

[2] Ali N. A. Koam, Moin A. Ansari, Azeem Haider, Journal of Taibah University For Science, 14 (2020), 172-176.

[3] Moin A. Ansari, Azeem Haider, Ali N. A. Koam, *On JU-algebras and p-closure ideals*, International Journal of Mathematics and Computer Science, 15 (2020), 135-154.

[4] Moin A. Ansari, Azeem Haider, Ali N. A. Koam, *On graph associated to UP-algebras*, Mathematical and Computational Applications, 23 (2018).

[5] Moin A. Ansari, Ali N. A. Koam, Azeem Haider, *Rough set theory applied in UP-algebras*, Italian Journal of Pure and Applied Mathematics, 42 (2019), 388-402.

[6] Yulong Fu, Xiao Long Xin, *Lattices and block codes*, U.P.B. Sci. Bull., Series A., 79 (2017).

[7] A. Iampan, *A new branch of the logical algebra: UP-algebras*, Journal of Algebra and Related Topics, 5 (2017), 35-54.

[8] A. Iampan, *The UP-isomorphism theorems for UP-algebras*, Discuss. Math. Gen. Algebra Appl., 3 (2019), 113-123.

[9] A. Iampan, *Derivations of UP-algebras by means of UP-endomorphisms*, Algebraic Structures and Their Applications, 3 (2016), 1-20.

[10] Aiyared Iampan, Phakawat Mosrijai, Akarachai Satirad, *Introducing partial transformation UP-algebras*, European Journal of Pure and Applied Mathematics, 11 (2018), 876-881.

[11] A. Iampan, P. Mosrijai, A Satirad, *Introducing partial transformation UP-algebras*, European Journal of Pure and Applied Mathematics, 11 (2018), 876-881.

[12] Y. Imai, K. Iséki, *On axiom systems of propositional calculi*, XIV. Proceedings of the Japan Academy, 42 (1966), 19-22.

[13] Young Bae Jun, Seok Zun Song, *Codes based on BCK-algebras*, Information Sciences, 181 (20110), 5102–5109.

[14] Shu Lin, Daniel J. Costello, *Error control coding: fundamentals and applications*, New Jersey, Prentice Hall, 1983.

[15] P. Mosrijai, A. Iampan, *A new branch of bialgebraic structures: UP-bialgebras*, Journal of Taibah University for Science, 13 (2019), 450-459.

[16] Samy Mohammed Mostafa, Bayumy Youssef, Hussein Ali Jad, *Coding theory applied to KU-algebras*, Journal of New Theory. 6 (2015), 43-53.

[17] Tapan Senapati, G. Muhiuddin, K. P. Shum, *Representation of UP-algebras in intervalued-valued intuitionistic fuzzy environment*, Italian Journal of Pure and Applied Mathematics, 38 (2017), 497-518.

[18] Atamewoue Tsafack Surdive, Ndjeya Slestin, Lele Clestin, *Coding theory and hyper BCK-algebras*, Journal of Hyperstructures, 7 (2018), 82-93.

[19] N. Wiberg, *Codes and decoding on general graphs*, Sweden: Department of Electrical Engineering, Linkping University, 1996.

[20] J. Ziv, A. Lempel, *Compression of individual sequences via variable-rate coding*, IEEE Trans. Inform. Theory, 24 (1978),530-536.

Accepted: July 01, 2020