Hadamard product of meromorphic multivalent functions with positive coefficient

Osama Ogilat*
Department of Basic Sciences
Faculty of Arts and Science
Al-Ahliyya Amman University
Amman, P.O.Box 19328
Jordan
o.oqilat@ammanu.edu.jo
oqily2008@gmail.com

Radwan Al-Smadi
Department of Basic Sciences
Faculty of Arts and Sciences
Ahliyya Amman University
Amman, P.O.Box 19328
Jordan
ralsmadi@ammanu.edu.jo

Abstract. In the present paper, we introduced a class $S(\gamma, \tau, \delta, p)$ of a meromorphic multivalent function and used this class of function Hadamard product (or convolution) to prove some nice and attractive properties such that the coefficient inequalities, distortion theorem, radii of starlikeness and convexity. Moreover, we verified that the class $S(\gamma, \tau, \delta, p)$ is closed under convex linear combination.

Keywords: multivalent function, analytic functions, negative coefficients, convexity, distortion theorem.

1. Introduction

Let L_p denoted the class of functions given by the formula

$$f_1(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{p+k-1} z^{p+k-1}, p \in N \tag{1}$$

and

$$f_2(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_{p+k-1} z^{p+k-1}, \tag{2}$$

*. Corresponding author

which are normalized meromorphic p-valent in the unit disk $U = \{z : 0 < |z| < 1\}$([9],[20]). Let $f_1, f_2 \in L_p$, then the Hadamard product of $f_1 * f_2$ is defined by

$$W(z) = (f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} z^{p+k-1}. \tag{3}$$

The class L_p of analytic p-valent functions, have been studied by many authors, for Significant properties and characteristics of various subclasses of the class L_p, such as: Owa [22], Saibah [21], Aouf ([12],[13]), Uralgeddi [3], Moa'ath ([10], [16]), Srivastava ([7], [8]), Morga ([14], [15]), Kulkarni [23], Osama ([17], [18], [19]), Aabed [1], Gahmin [5], Kamali [11] and Markinde [4]. The class of functions $S(\gamma, \tau, \delta, p)$ define as follows:

A function $W(z)$ given by (3) is said to be a member of the class $S(\gamma, \tau, \delta, p)$ if it satisfies

$$\left| \frac{(1-2\delta) z W'(z) - \delta z^2 W''(z)}{(\tau - 1) W(z) + \tau z W'(z)} \right| \le \gamma, \tag{4}$$

where $\gamma > 0$, $0 \le \tau < 1$, $\delta \ge 0$, $\forall\ z \in U$. This class of function is motivated from the work done by Alghalary and Joshi [2].

2. Coefficient inequalities

Alghalary and Joshi [2] presented a basic properties such as coefficient inequality, growth and distortion and several others. In this Section, we provide a sufficient condition for a function W, analytic in a punctured desk U in $S(\gamma, \tau, \delta, p)$.

Theorem 2.1. *If $W \in L_p$ satisfies*

$$\begin{aligned} &\sum_{k=1}^{\infty} \{(p+k-1)[1 - \tau\gamma - \delta(p+k)] + \gamma(1-\tau)\} a_{p+k-1} b_{p+k-1} \\ &\qquad \le \gamma(\tau + 1 - \tau p) - p(1-\delta) - \delta p^2, \end{aligned} \tag{5}$$

then $W \in S(\gamma, \tau, \delta, P)$, where $\gamma > 0, 0 \le \tau < 1, \delta \ge 0$ for all $z \in U$.

Proof. Let us suppose that

$$\begin{aligned} &\sum_{k=1}^{\infty} \{(p+k-1)[1 - \tau\gamma - \delta(p+k)] + \gamma(1-\tau)\} a_{p+k-1} b_{p+k-1} \\ &\qquad \le \gamma(\tau + 1 - \tau p) - p(1-\delta) - \delta p^2, \end{aligned} \tag{6}$$

for $W \in S(\gamma, \tau, \delta, p)$. It suffices to show that

$$\left| \frac{(1-2\delta) z W'(z) - \delta z^2 W''(z)}{(\tau - 1) W(z) + \tau z W'(z)} \right| \le \gamma, \tag{7}$$

where $\gamma > 0, 0 \leq \tau < 1, \delta \geq 0$ for all $z \in U$. We know

$$(1-2\delta)zW'(z) - \delta z^2 W''(z)$$

$$= (1-2\delta)[-pz^{-p-1} + \sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2}]$$
$$- \delta z^2[-p(-p-1)z^{-p-2}] + \sum_{k=1}^{\infty}(p+k-1(p+k-2)a_{p+k-1}b_{p+k-1}z^{p+k-3})$$
$$= -pz^{-p}(1-\delta) - \delta p^2 + \sum_{k=1}^{\infty}[(p+k-1)(1-\delta(p+k))]a_{p+k-1}b_{p+k-1},$$

and

$$(\tau-1)[z^{-p} + \sum_{k=1}^{\infty}a_{p+k-1}b_{p+k-1}z^{p+k-1}] + \tau z - pz^{-p} + \sum_{k=1}^{\infty}a_{p+k-1}b_{p+k-1}z^{p+k-1}$$
$$= \tau z^{-p} - z^{-p} - \tau p z^{-p+1} + \sum_{k=1}^{\infty}[(\tau-1) + \tau(p+k-1)]a_{p+k-1}b_{p+k-1}.$$

Then

$$\left|\frac{-pz^{-p}(1-\delta) - \delta p^2 + \sum_{k+1}^{\infty}[(p+k-1)(1-\delta(p+k))]a_{p+k-1}b_{p+k-1}z^{p+k-1}}{\tau z^{-p} - z^{-p} - \tau p z^{-p-1} + \sum_{k=1}^{\infty}[(\tau-1)+\tau(p+k-1)]a_{p+k-1}b_{p+k-1}z^{p+k-1}}\right|$$
$$\leq \frac{p(1-\delta) + \sum_{k=1}^{\infty}[(p+k-1)(1-\delta)(p+k)]a_{p+k-1}b_{p+k-1}}{(\tau+1-\tau p) - \sum_{k=1}^{\infty}[(1-\tau)-\tau(p+k-1)]a_{p+k-1}b_{p+k-1}},$$

that is

$$\sum_{k=1}^{\infty}[(p+k-1)(1-\tau\gamma-\delta(p+k)) + \gamma(1-\tau)]a_{p+k-1}b_{p+k-1}$$
$$\leq \gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2. \tag{8}$$

When $k = 1$, we have

$$a_p b_p \leq \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{p[(1-\tau\gamma-\delta(p+1))] + \gamma(1-\tau)}, \tag{9}$$

which is equivalent to our condition of the theorem, so that $W \in S(\gamma, \tau, \delta, p)$.

Theorem 2.2. *Let the function W defined by 3, then $W \in S(\gamma, \tau, \delta, p)$ if and only if*

$$\left|\frac{(1-2\delta)zW'(z) - \delta z^2 W''(z)}{(\tau-1)W(z) + \tau z W'(z)}\right| \leq \gamma \tag{10}$$

is satisfied.

Proof. With the aid of Theorem 2.1 it suffices to show the 'only if' $W \in S(\gamma, \tau, \delta, p,)$ then

$$\begin{aligned} &\left|\frac{(1-2\delta)zW'(z)-\delta z^2W''(z)}{(\tau-1)W(z)+\tau zW'(z)}\right| \\ &= \left|\frac{-pz^{-p}(1-\delta)-\delta p^2+\sum[(p+k-1)(1-\delta(p+k))]a_{p+k-1}b_{p+k-1}z^{p+k-2}}{\tau z^{-p}-z^{-p}+\sum_{k=1}^{\infty}[(\tau-1)+\tau(p+k-1)]a_{p+k-1}b_{p+k-1}z^{p+k-1}}\right| \\ &\leq \frac{p(1-\delta)+\delta p^2+\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}}{\tau+1-\tau p-\sum\gamma(p+k-1)a_{p+k-1}b_{p+k-1}} \leq \gamma, \end{aligned}$$

this implies that

$$\begin{aligned} &\sum_{k=1}^{\infty}(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)a_{p+k-1}b_{p+k-1} \\ &\leq \gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2. \end{aligned} \tag{11}$$

Similarly, the method of proving in Theorem 2.1 applied and obtained the required result. The result is sharp for function W of the form

$$W_{p+k-1}(z)=z^{-p}+\frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}z^{p+k-1}, k\geq 1. \tag{12}$$

Corollary 2.1. *Let the function W be defined by (12). If $W \in S(\gamma, \tau, \delta, p)$, then*

$$a_{p+k-1}b_{p+k-1} \leq \frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}. \tag{13}$$

The result is sharp for functions W_{p+k-1} given by (12).

3. Distortion theorem

In this Section, a Hadamard product applied to the distortion property, in the class $W \in S(\gamma, \tau, \delta, p)$ by the following Theorem:

Theorem 3.1. *If the function W defined by (5) is in the class $W \in S(\gamma, \tau, \delta, p)$, then for $0 < |z| = r < 1$, we have*

$$\begin{aligned} r^{-p} &- \frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{p[1-\tau\gamma-\delta(p+1)]+\gamma(1-\tau)}r^p \\ &\leq |(f_1 * f_2)(z)| \leq r^{-p}+\frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{p[1-\tau\gamma-\delta(p+1)]+\gamma(1-\tau)}r^p, \end{aligned}$$

with equality holds for

$$W_1 = z^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{p[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)} z^p (z = \pm ir, \pm r),$$

and

$$pr^{-p-1} - \frac{p[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]}{p[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)} r^{p-1},$$

with equality holds for

$$W_1 = z^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{p[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)} z^p (z = \pm ir, \pm r).$$

Proof. Since $(f_1 * f_2)(z) \in S(\gamma, \tau, \delta, p)$, Theorem 5.2 yields the inequality

$$\sum a_{p+k-1} b_{p+k-1} \leq \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+k)] + \gamma(1-\tau)}, \tag{14}$$

thus, for $0 < |z| = r < 1$, and making use of (14) we have

$$\begin{aligned} \left|(f_1 * f_2)(z)\right| &= \left|z^{-p} + \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} z^{p+k-1}\right| \\ &\leq |z|^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+k)] + \gamma(1-\tau)} |z|^{p+k-1}. \end{aligned}$$

(Note: we substitute in (14) when k=1)

$$\begin{aligned} &\leq r^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+k)] + \gamma(1-\tau)} \\ &= r^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+k)] + \gamma(1-\tau)} \end{aligned}$$

and

$$\begin{aligned} \left|(f_1 * f_2)(z)\right| &= \left|z^{-p} + \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} z^{p+k-1}\right| \\ &\geq |z|^{-p} - \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} |z|^{p+k-1} \\ &\geq r^{-p} - \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+k)] + \gamma(1-\tau)} \end{aligned}$$

(Note: we substitute in (14)) when k=1)

$$r^{-p} - \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{p[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)}.$$

Also from Theorem 2.1, it follows that

$$\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1} \leq \frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}, \tag{15}$$

thus

$$\begin{aligned} |(f_1*f_2)'|(z) &= \left| -pz^{-p-1}+\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2}\right| \\ &\leq \left|-pz^{-p-1}\right| + \left|\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2}\right. \\ &\leq pr^{-p-1}+\frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{1-\tau\gamma-\delta(p+k)+\gamma(1-\tau)}. \end{aligned}$$

(Note: we substitute in 15 when k=1)

$$\leq pr^{-p-1}+\frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{1-\tau\gamma-\delta(p+k)+\gamma(1-\tau)}$$

and

$$\begin{aligned} |(f_1*f_2)'|(z) &= \left| -pz^{-p-1}+\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2}\right| \\ &\geq \left|-pz^{-p-1}\right| + \left|-\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2}\right| \\ &\geq pr^{-p-1}-\frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{1-\tau\gamma-\delta(p+k)+\gamma(1-\tau)}. \end{aligned}$$

(Note: we substitute in 15 when k=1)

$$\geq pr^{-p-1}+\frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{1-\tau\gamma-\delta(p+1)+\gamma(1-\mu)}.$$

Hence, completes the proof of theorem.

4. Radii of starlikeness and convexity

In this Section, we show that the radii of starlikeness and convexity for the class $S(\gamma,\tau,\delta,p)$ is given by the following theorem:

Theorem 4.1. *If the function $(f_1*f_2)(z)$ defined by (3) is in the class $S(\gamma,\tau,\delta,p)$, then $(f_1*f_2)(z)$ is starlike of order $\rho(0\leq\rho<p)$ in the disk $|z|\leq r_1(\gamma,\tau,\delta,p,\rho)$ where $r_1(\gamma,\tau,\delta,p,\rho)$ is the largest value for which*

$$\begin{aligned} r_1 &= r_1(\gamma,\tau,\delta,p,\rho) \\ &= \inf_{k\geq 1}\left(\frac{(p-\rho)(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}{(p+k-1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}\right)^{\frac{1}{2p+k-1}}. \end{aligned}$$

The result is sharp for functions W_{p+k-1} given by (12).

Proof. It suffices to show that

$$\left|\frac{zW'(z)}{W(z)}+1\right| \le (1-\rho)\text{for } |z| \le r_1.$$

We have

$$\begin{aligned} &\left|\frac{zW'(z)}{W(z)}+1\right| \\ (4.1)\qquad &= \frac{(-p-1)+\sum_{k=1}^{\infty}(p+k)a_{p+k-1}b_{p+k-1}z^{2p+k-1}}{1+\sum_{k=1}^{\infty}a_{p+k-1}b_{p+k-1}z^{2p+k-1}} \\ &\le \frac{(-p+1)+\sum\frac{(p+k)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}{(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}|z|^{2p+k-1}}{1-\sum\frac{[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}{(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}|z|^{2p+k-1}} \le 1-\rho, \end{aligned}$$

hence (4.1) holds true if

$$\sum_{k=1}^{\infty}\left(\frac{(p+k-1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}{(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}\right)|z|^{2p+k-1} \le p-\rho,$$

and it follows that

$$|z| \le \left(\frac{(p-\rho)(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}{(p+k-1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}\right)^{\frac{1}{2p+k-1}}, \qquad k \ge 1$$

then

$$r_1 = \inf_{k\ge 1}\left(\frac{(p-\rho)(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}{(p+k+1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}\right)^{\frac{1}{2p+k-1}}$$

as required.

Theorem 4.2. *If the function $(f_1 * f_2)(z)$ be defined by* (3) *is in the class $S(\gamma,\tau,\delta,p)$, then $(f_1 * f_2)(z)$ is convex of order $\rho(0 \le \rho < p)$ in the disk $|z| \le r_2(\gamma,\tau,\delta,p,\rho)$ where $r_2(\gamma,\tau,\delta,p,\rho)$ is the largest value for which r_2,*

$$\begin{aligned} &r_2(\gamma,\tau,\delta,p,\rho) \\ (16)\qquad &= \inf_{k\ge 1}\left(\frac{p(p-\rho)(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}{(p+k-1)(p+k-1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}\right)^{\frac{1}{2p+k-1}}. \end{aligned}$$

The result is sharp for functions W_{p+k-1} given by (12).

Proof. It suffices to show that $\left|\frac{z(f_1*f_2)''(z)}{(f_1*f_2)'(z)}+2\right| \le 1-\rho, for|z| \le r_2$. We have

$$\begin{aligned} &\left|\frac{z(f_1*f_2)''(z)}{(f_1*f_2)'(z)}+2\right| \\ &= \left|\frac{(p^2-p)+\sum_{k=1}^{\infty}(p+k-1)(p+k)a_{p+k-1}b_{p+k-1}z^{2p+k-1}}{-p+\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}z^{2p+k-1}}\right|, \\ &\le \frac{(p-p^2)+\sum_{k=1}^{\infty}(p+k-1)(p+k)a_{p+k-1}b_{p+k-1}|z|^{2p+k-1}}{p-\sum_{k=1}^{\infty}(p+k-1)a_{p+k-1}b_{p+k-1}|z|^{2p+k-1}}, \end{aligned} \tag{4.2}$$

hence (4.2) holds true if

$$\sum_{k=1}^{\infty}\frac{(p+k-1)(p+k+1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}{(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)} \le p(p-\rho), k \ge 1,$$

$$|z| \le \left(\frac{p(p-\rho)(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}{(p+k-1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}\right)^{\frac{1}{2p+k-1}}, k \ge 1.$$

Then,

$$r_2 = \inf_{k\ge 1}\left(\frac{p(p-\rho)(p+k-1)[1-\tau\gamma-\delta(p+k)]+\gamma(1-\tau)}{(p+k-1)(p+k+1-\rho)[\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2]}\right)^{\frac{1}{2p+k-1}}$$

as required.

5. Convex linear combination

Our next result involves a linear combination of function W of the type (12).

Theorem 5.1. *Let*

$$W_p = z^{-p} + \frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{p[(1-\tau\gamma-\delta(p+1))]+\gamma(1-\tau)}z^p, \tag{17}$$

and

$$W_{p+k-1} = z^{-p} + \frac{\gamma(\tau+1-\tau p)-p(1-\delta)-\delta p^2}{(p+k-1)[(1-\tau\gamma-\delta(p+1))]+\gamma(1-\tau)}z^{p+k-1}, k \ge 1 \tag{18}$$

then $(f_1*f_2)(z) \in S(\gamma,\tau,\delta,P)$, if and only if, it can be expressed in the form

$$(f_1*f_2)(z) = \sum_{k=1}^{\infty}\vartheta_{p+k-1}W_{p+k-1(z)},\ \text{where } \vartheta_{p+k-1} \ge 0 \text{ and } \sum_{k=1}^{\infty}\vartheta_{p+k-1} \le 1.$$

Proof. From (17) and (18), it is easily seen that

$$\begin{aligned}(f_1 * f_2)(z) &= \sum_{k=1}^{\infty}(\vartheta_{p+k-1}W_{p+k-1(z)} \\ &= z^{-p} + \sum_{k=1}^{\infty} \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2 \vartheta_{p+k-1}}{(p+k-1)[(1-\tau\gamma-\delta(p+k))] + \gamma(1-\tau)} z^{p+k-1}. \qquad (19)\end{aligned}$$

Since

$$\begin{aligned}\Rightarrow \quad & \sum_{k=1}^{\infty} \frac{(p+k-1)[(1-\tau\gamma-\delta(p+k))] + \gamma(1-\tau)}{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2} \\ \times \quad & \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2 \vartheta_{p+k-1}}{(p+k-1)[(1-\tau\gamma-\delta(p+k))] + \gamma(1-\tau)} \\ = \quad & \sum_{k=1}^{\infty} \vartheta_{p+k-1} \le 1,\end{aligned}$$

It follows from Theorem 2.1 that the function $f_1 \in S(\gamma, \tau, \delta, p)$.

Conversely, suppose that $(f_1 * f_2)(z) \in S(\gamma, \tau, \delta, p)$, Since

$$a_{p+k-1}b_{p+k-1} \le \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)}, \qquad k \ge 1$$

we can write

$$\begin{aligned}(f_1 * f_2)(z) &= \sum_{k=1}^{\infty}(\vartheta_{p+k-1}W_{p+k-1(z)} \\ &= z^{-p} + \sum_{k=1}^{\infty} \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2 \vartheta_{p+k-1}}{(p+k-1)[(1-\tau\gamma-\delta(p+1))] + \gamma(1-\tau)} z^{p+k-1},\end{aligned}$$

then

$$a_{p+k-1}b_{p+k-1} = \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)} \vartheta_{p+k-1}, k \ge 1.$$

Setting

$$\vartheta_{p+k-1} = \frac{(p+k-1)[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)}{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}, k \ge 1$$

it follows that $(f * f_1)(z) = \sum_{k=1}^{\infty} \vartheta a_{p+k-1} W_{p+k-1(z)}$, thus the theorem is complete.

Finally we prove the following theorem:

Theorem 5.2. *The class is closed under convex linear combinations.*

Proof. Suppose that the functions W_1 and W_2 defined by

$$W_i = z^{-p} + \sum_{k=1}^{\infty} a_{p+k-1,i} b_{p+k-1,i} z^{p+k-1} i = 1, 2; z \in U \tag{20}$$

are in the class $S(\gamma, \tau, \delta, P)$. Setting

$$W(z) = \sigma W_1(z) + (1-\sigma) W_2(z), 0 \leq \sigma \leq 1 \tag{21}$$

we want to show that $W \in S(\gamma, \tau, \delta, P)$, we find that from (20)

$$\begin{aligned} W(z) &= \sigma W_1(z) + (1-\sigma) W_2(z), \\ &= z^{-p} + \sum_{k=1}^{\infty} \sigma a_{p+k-1,1} b_{p+k-1,1} + (1-\sigma) a_{p+k-1,2} b_{p+k-1,2} z^{p+k-1}, (0 \leq \sigma \leq 1), \end{aligned}$$

where $z \in U$. In view of Theorem 2.1, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} [(p+k-1)(1-\tau\gamma - \delta(p+k) + \gamma(1-\tau))] \sigma a_{p+k-1,1} b_{p+k-1,1} \\ &+ (1-\sigma) a_{p+k-1,2} b_{p+k-1,2} \\ &\leq \sigma[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2] + (1-\sigma)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2] \\ &= \gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2, \end{aligned}$$

which shows that $W \in S(\gamma, \tau, \delta, P)$.

6. Conclusion

In this paper, we proved properties of the coefficient inequalities, distortion theorem and radii of starlikeness and convexity for the Hadamard product of the class $S(\gamma, \tau, \delta, p)$ of multivalent function. Also, it is shown that the class $S(\gamma, \tau, \delta, p)$ is closed under convex linear combination.

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