# Ordinary differential equations of the probability functions of the Benktander kind II distribution with $b = 1$ and its properties

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**Abstract.** The idea of convolution is the sum of independent and identically distributed (iid) random variables and the structure of linear combination of random variables. The cases of ordinary differential equation (ODE) of the convolution of probability distributions of mixture of Benktander distribution of the second kind have been studied. Moreover, the ODE of quantile function (QF), survival function (SF), hazard function (HF) and reversed hazard function (RHF) of convoluted probability distributions has been considered. We obtain explicit forms for the densities and distribution functions for Benktander distribution of the second kind, as well as their moments and related parameters. We derive basic properties of these laws and illustrate their modeling possible using a simulated data.

**Keywords:** convolution, Benktander distribution, ordinary differential equations.

## 1. Introduction

A differential equation is an equation with unknown function, the function and its derivatives may possibly appear in the equation. Differential equations are crucial for a mathematical explanation of nature. They are positioned at the center of many physical theories. A classic application of differential equations proceeds along the solution of mathematical model and interpreting the model. The ordinary differential equations (ODE) according to whether or not they have partial derivatives. The order of a differential equation is the highest order derivative happening. A solution (or particular solution) of a differential equation of order n consists of a function defined and n times differentiable on a domain D having the property that the functional equation achieved by substi-

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tuting the function and its n derivatives into the differential equation holds for all points in D [1]. The idea of convolution is the sum of iid random variables and the structure of linear combination of random variables. The primary motivation of convolution is to derive probability density function (PDF) of a sum of random variables. Moreover, the motivation is to conclude if the convolution random variables are more flexible than the base random variables. The general formula for the sum $Z = X + Y$ of continuous probability distribution is defined as

$$f(Z) = \int_{-\infty}^{z} f(z-x)f(x)dx = \int_{-\infty}^{z} f(z-y)f(y)dy. \tag{1}$$

In statistical distributions theory, numerous convolutions are studied for both discrete and continuous random variables. For examples, the sum of exponential random variables is gamma random variables; the sum of Bernoulli random variables is Binomial random variable, the sum of geometric random variables is Negative Binomial random variable and many others. In mathematical modeling uncertainties is very useful in many areas such as; ecology, engineering, meteorology, astronomy, medicine, geology, finance, ecology, economics and so on. There are many researchers introduced many convoluted distributions for examples; [2] introduced a convoluted Poisson Distributions with applications to Estimation, F distribution and Beta Prime distribution or Inverted Beta distribution pioneered in the text books by [3] and [4] respectively. [5] studed the products and convolutions of Gaussian probability density functions. The convoluted Beta-Exponential Distribution proposed by [6]. Moreover, [7] introduced Quantile approximation of the Chi–Square Distribution using the Quantile Mechanics. Again they are found the solutions of Chis-Quare quantile differential equation [8]. [9] studed the ODEs of probability functions of convoluted distributions. The ODE of Lomax distribution introduced by [10]. The beta distribution initiated by[11]. Raised Cosine distribution recovered by [12] and [13] performed the convoluted exponential distribution. ODE has supportive in statistical and probabilistic representation. It can also be seen as a tool for measuring uncertainties and forecasting. The cases of ODE of the convolution of probability distributions of mixture of Benktander distribution of the second kind have not been studied in literature. Moreover, the ODE of quantile function (QF), survival function (SF), hazard function (HF) and reversed hazard function (RHF) of convoluted probability distributions has not been considered.

## 2. The convolution Benktander distribution of the second kind

The Benktander distribution of the second kind is one of two distributions introduced by [14] to model heavy-tailed losses commonly found in non-life Acturial sciences, using various forms of mean excess functions. The Benktander distribution of the first kind is "close" to the Weibull distribution [15].

The pdf and cdf of this distribution defined as follows respectively

(2) $$f(x) = e^{\frac{a}{b}(1-x)}x^{b-2}(ax^b - b + 1);\;\; a > 0, 0 < b < 1 \text{ and } x > 0,$$

(3) $$F(x) = 1 - x^{b-1}e^{\frac{a}{b}(1-x^b)};\;\; a > 0, 0 < b < 1 \text{ and } x > 0.$$

By assuming $b = 1$ the benktander distribution becomes

(4) $$f(x) = ae^{a(1-x)};\;\; a > 0 \text{ and } x > 0,$$

(5) $$F(x) = 1 - e^{a(1-x)};\;\; a > 0 \text{ and } x > 0.$$

**Definition 2.1.** *Let $X$ and $Y$ be independent random variable belong to benktander distribution with constant parameter, for $x > 1$, $b = 1$ and $a > 0$. Then the PDF of the convolution of benktander distribution (CBD) is given by:*

(6) $$f(z) = \int_1^{z-1} f(z-x)f(x)dx,$$

(7) $$f(z) = a^2\int_1^{z-1} e^{a(1-z+x)}e^{a(1-x)}dx,$$

(8) $$f(z) = a^2e^{a(2-z)}(z-2);\;\; a > 0 \textit{ and } z \geq 2.$$

**Definition 2.2.** *Let $X$ and $Y$ be independent distributed random variables belong to benktander distribution with constant parameter, for $x \geq 1$, $y \geq 1$, $b = 1$ and $a > 0$. Then the CDF of the convolution of benktander distribution (CBD) is given by:*

(9) $$F(z) = \int_2^z a^2e^{a(2-t)}(t-2)dt = 1 + (2a - az - 1)e^{2a-az};\;\; a > 0 \textit{ and } z \geq 2.$$

The graphs of the pdf and the cdf of conveluted benktander distribution with b=1 as defined in (9) and (10) for different values of $a$ are shown in Figures 1 and 2 respectively.

## 3. Constant parameter convoluted benktander distribution (CBD)

The differential calculus used to find the various ODEs. The quantile mechanics approach used to find the nonlinear second order ODEs [16], [17] and [18]. It is used to find the quantile functions of the constant parameter convoluted benktander distribution. In this section, we derived various ODEs for the CBD as in the following propositions:

**Proposition 1.** The first ordinary differential equation whose solution is the pdf of CBD is defined as

(10) $$f'(x)(z-2) - f(z)[1 - a(z-2)] = 0.$$

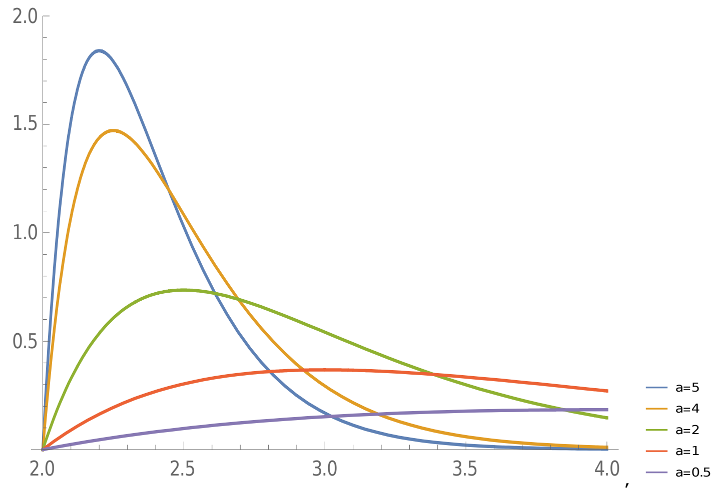


Figure 1: Graphs of pdf of the convoluted benktander distribution with b=1 for selected values of parameter

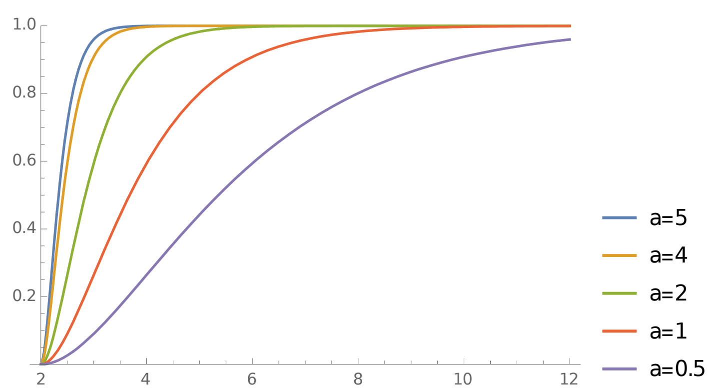


Figure 2: Graphs of cdf. of the convoluted benktander distribution with b=1 for selected values of parameter

**Proof.** The first derivative of CBD is given as

$$f'(x) = a^2 e^{a(2-z)}[1 - a(z-2)] \tag{11}$$

by using equation (8) the term $e^{a(2-z)}$ can be written as

$$e^{a(2-z)} = \frac{f(z)}{a^2(z-2)} \tag{12}$$

substitute equation (12) in equation (11) and simplify we have the first ODE whose solution is the PDF of CBD. □

**Proposition 2.** The second ordinary differential equation whose solution is the pdf of CBD is defined as

$$(z-2)f''(z)-(a[a(z-2)-2])f(z)=0. \tag{13}$$

**Proof.** The second derivative of CBD is given as

$$f''(z)=a^3[a(z-2)-2]e^{a(2-z)}. \tag{14}$$

Simplify and substitute Eq (14) in Eq (7), we have the second ODE whose solution is the PDF of CBD. □

**Proposition 3.** The mixture of the first and second ordinary differential equation whose solution is the pdf of CBD is defined as

$$(z-2)f''(z)+a(z-2)f'(z)+af(z)=0. \tag{15}$$

**Proof.** The first and second derivatives of CBD are given by equations (11) and (14) respectively. Simplify and substitute equation (11) in Eq. (14), we have the mixture first and second ODE whose solution is the PDF of CBD. □

**Proposition 4.** The ODE whose solution is the QF $w(u)$ of CBD is given by:

$$(w-2)\frac{d^2w}{du^2}-\left(\frac{dw}{du}\right)^2[a(w-2)-1]=0. \tag{16}$$

**Proof.** Regarding to the definition of quantile functions (for more details see [16], [17]and [18]) the quantile function defined as $\frac{dw}{du}=\frac{1}{f(w)}$. Moreover, the first derivative of $w(u)$ can be evaluated as follows

$$\frac{dw}{du}=\frac{1}{a^2e^{a(2-w)}(w-2)}=\frac{e^{aw}}{a^2e^{2a}(w-2)}. \tag{17}$$

Now, we differentiate to obtain a second order of the nonlinear ODE:

$$\begin{aligned}\frac{d^2w}{du^2}&=\frac{1}{a^2e^{2a}}\left[\frac{(w-2)ae^{aw}-e^{aw}}{(w-2)^2}\right]\frac{dw}{du}\\&=\left[\frac{(w-2)ae^{aw}}{(a^2e^{2a})(w-2)^2}-\frac{e^{aw}}{(a^2e^{2a})(w-2)^2}\right]\frac{dw}{du}.\end{aligned} \tag{18}$$

Simplify Eq (18) as follows

$$\frac{d^2w}{du^2}=\left(\frac{dw}{du}\right)^2\left[\frac{a(w-2)-1}{w-2}\right]. \tag{19}$$

and rearrange Eq (19), we have the ODE whose solution is the QF of CBD. □

**Proposition 5.** The ODE whose solution is the survivor function (SF) of CBD is given by:

$$(1+a(z-2))S'(z)+a^2(z-2)S(z)=0. \tag{20}$$

**Proof.** The survival function of CBD is:

$$S(z)=1-F(z)=(1+a(z-2))e^{a(2-a)}. \tag{21}$$

From Eq. (21) we have

$$e^{a(2-a)}=\frac{S(z)}{1+a(z-2)}. \tag{22}$$

Differentiate Eq. (21), we have

$$S'(z)=(-a^2(z-2))e^{a(2-a)} \tag{23}$$

and substitute Eq. (22) in Eq (23), we have

$$S'(z)=\frac{-a^2(z-2)S(z)}{1+a(z-2)}. \tag{24}$$

Rearrange Eq. (24) we get the ODE whose solution is the (SF) of CBD. □

**Proposition 6.** The ODE whose solution is the hazard function (HF) of CBD is given by

$$a^2(z-2)^2h'(z)-(h(z))^2=0. \tag{25}$$

**Proof.** The hazard function of CBD is:

$$h(z)=\frac{f(z)}{S(z)}=\frac{a^2(z-2)}{1+a(z-2)}. \tag{26}$$

Differentiate Eq. (26) we get

$$h'(z)=\frac{a^2}{(az-2a+1)^2}. \tag{27}$$

Substitute Eq. (26) in Eq. (27), we have $a^2(z-2)^2h'(z)-(h(z))^2=0$ with the condition $h(2)=0$. □

## 4. Order statistics of CBD

In this section, we drive the single order statistics for CBD. The probability density function of $f_{(j)}(z)$ is given as:

$$f_{(j)}(z) = \frac{n!}{(j-1)!(n-j)!} g(z)[G(z)]^{j-1}[1-G(z)]^{n-j}; i = 1, 2, \ldots \tag{28}$$

$$\begin{aligned} f_{(j)}(z) &= \frac{n!}{(j-1)!(n-j)!}[a^2 e^{a(2-z)}(z-2)] \\ &\cdot [1+(2a-az-1)e^{a(2-z)}]^{j-1}[1-(1-(2a-az-1)e^{a(2-z)}]^{n-j} \\ &= \frac{n!}{(j-1)!(n-j)!}[a^2 e^{a(2-z)[1+(n-j)}(z-2)] \\ &\cdot [1+(2a-az-1)e^{a(2-z)}]^{j-1}[az-2a+1)]^{n-j}. \end{aligned} \tag{29}$$

Let $Z_{[1]} = min(Z_1, Z_2, \ldots, Z_n)$ and $Z_{[n]} = max(Z_1, Z_2, \ldots, Z_n)$ with pdfs $f_{(1)}(z)$ and $f_{(n)}(z)$ defined as follows $f_{(1)}(z) = nf(z)[1-F(z)]^{n-1}$ and $f_{(n)}(z) = nf(z)[1-F(z)]^{n-1}$. As a special case of Eq. (28), the pdf of the minimum and maximum order statistics, respectively, are

$$f_{(1)}(z) = na^2 e^{an(2-z)}(az-2a+1)^{n-1}, \tag{30}$$

$$f_{(n)}(z) = na^2 e^{a(2-z)}(z-2)[1+(2a-az-1)e^{a(2-z)}]^{n-1} \tag{31}$$

## 5. Some asymptotic properties of the CBD

Here, we show some asymptotic properties of the random variable z, where $z\ CBD(a)$. We derive the asymptotic $r^{th}$ moment, asymptotic expected value and asymptotic standard deviation of $z$ as special cases. The $r^{th}$ moment of the CBD is given by the following theorem.

**Lemma 5.1.** *If $z \sim$ CBD(a), then the rth moment is given by*

$$E(Z^r) = \sum_{k=0}^{r} rCk2^{r-k}(\frac{1}{a})^k \gamma(k+2). \tag{32}$$

**Proof.** By using the definition of the rth moment as follows:

$$E(Z^r) = \lim_{c\to\infty} \int_2^c z^r f(z) dz. \tag{33}$$

By substitute $f(z)$ in Eq. 32, we get

$$E(Z^r) = \lim_{c\to\infty} \int_2^c z^r (a^2 e^{a(2-z)}(z-2)) dz. \tag{34}$$

Assume that $x = z - 2$ and we have $dx = dz$, at $z = 2$ then $x = 0$ and as $z$ goes to $\infty$ then $x$ goes to $\infty$. Now Eq. (34) becomes as

$$E(Z^r) = a^2 \lim_{c\to\infty} \int_2^c x(x+2)^r e^{-ax} dx. \tag{35}$$

By using binomial theorem we have

$$(x+2)^r = \sum_{k=0}^{r} rCkx^r 2^{r-k}. \tag{36}$$

Substitute Eq. (36) in Eq. (35) and some algebraic manipulation, we have

$$\begin{aligned} E(Z^r) &= a^2 \lim_{c\to\infty} \int_2^c x \sum_{k=0}^{r} rCkx^k 2^{r-k} e^{-ax} dx \\ &= a^2 \sum_{k=0}^{r} 2^{r-k} rCk \lim_{c\to\infty} \int_2^c x^{k+1} e^{-ax} dx. \end{aligned} \tag{37}$$

Let $ax = y$ and $x = \frac{y}{a}$, $dx = \frac{dy}{a}$ then we get

$$\begin{aligned} E(Z^r) &= a^2 \sum_{k=0}^{r} 2^{r-k} rCk \lim_{c\to\infty} \int_0^c (\frac{y}{a})^{k+1} e^{-y} dy \\ &= a^2 \sum_{k=0}^{r} 2^{r-k} rCk (\frac{1}{a})^{k+2} \lim_{c\to\infty} \int_0^c y^{k+1} e^{-y} dy \end{aligned} \tag{38}$$

since $\lim_{c\to\infty} \int_0^c y^{\alpha-1} e^{-y} = \Gamma(\alpha)$, so $\alpha - 1 = k + 1$, $\alpha = k + 2$ and substitute in Eq. (37) then the proof is complete. □

As a special case of the $r^{th}$ moments, we have the following remarks:

**Remark 1.** When r =1, we found the asymptotic expected value as follows:

$$\mu = E(Z) = (2 + \frac{2}{a}). \tag{39}$$

**Remark 2.** When r =2, we found the asymptotic expected value of $z^2$ as follows:

$$E(Z^2) = 4 + 8\frac{1}{a} + 6(\frac{1}{a})^2. \tag{40}$$

**Remark 3.** The asymptotic variance of Z can be derived by using the formula $var(Z) = \sigma^2 = E(Z^2) - [E(Z)]^2$ and substitute Eqs. (38) and (39) in it, we have the asymptotic variance of z as follows

$$var(z) = \sigma^2 = \frac{2}{a^2}. \tag{41}$$

**Remark 4.** The asymptotic skewness of z may be computed by

$$Skewness = E(\frac{Z-\mu}{\sigma})^3 = \sqrt{2}, \tag{42}$$

where $E(\frac{z-\mu}{\sigma})^3 = \frac{E[z^3]-3\mu\sigma^2-\mu^3}{\sigma^3}$ and $E[z^3] = 8 + \frac{24}{a} + 36(\frac{1}{a})^2 + +24(\frac{1}{a})^3$.

**Remark 5.** The asymptotic kurtosis of z may be computed by

$$Kurtosis = E(\frac{Z-\mu}{\sigma})^4, \tag{43}$$

since $E(\frac{Z-\mu}{\sigma})^4 = \frac{E[Z^4]-4\mu E[z^3]+6(\mu)^2 E[Z^2]-4\mu^3 E[Z]+\mu^4}{\sigma^4}$ just we need to find the $E[Z^4]$ and it can be computed as $E[Z^4] = 16 + 32\frac{1}{a} + 24(\frac{1}{a})^2 + 8(\frac{1}{a})^3 + (\frac{1}{a})^4$ and we get the asymptotic kurtosis of $z$.

**Lemma 5.2.** *If $Z \sim$ CBD(a), then the moment generating function is given as*

$$E[e^{tz}] = \frac{a^2 e^{2t}}{(a-t)^2}; t < a. \tag{44}$$

**Proof.** The moment generating function can be defined as follows

$$E[e^{tz}] = \int_2^\infty e^{tz} f(z)dz = \int_2^\infty e^{tz}(a^2 e^{a(2-z)}(z-2))dz. \tag{45}$$

Let $x = z - 2$ then at $z = 2$, $x = 0$ and at $x$ goes to $\infty$ then $z$ goes to $\infty$. We have

$$E[e^{tz}] = a^2 e^{a(2-z)} \int_0^\infty x e^{-x(a-t)} dx. \tag{46}$$

Let $y = x(a-t)$, $dy = (a-t)dx$ and $x = \frac{y}{a-t}$. We computed the integral and simplified as follows

$$E[e^{tz}] = a^2 e^{2t} \frac{1}{(a-t)^2} \int_0^\infty y e^{-y} dy = a^2 e^{2t} (\frac{1}{(a-t)^2} \Gamma(2) \tag{47}$$

and the proof is completed. □

## 6. Maximum likelihood estimation of the parameter

The maximum likelihood estimator (MLE) is a well famous estimator. It is defined by take care of our parameters as unknown values and computing the joint density of all members of a data set, which are suggested to be independent and identically distributed (iid). Let $Z_1, Z_2, \ldots, Z_N$ be random sample of n (iid) random variables each from CBD. The likelihood function is given by

$$L(a, Z_1, Z_2, \ldots, Z_N) = \prod_{i=1}^{N} f(z; a) = \prod_{i=1}^{N} a^2 e^{a(2-z_i)}(z_i - 2). \tag{48}$$

By taking the natural logarithm of both sides, the log-likelihood function is defined to be:

$$l(a, Z_1, Z_2, \ldots, Z_N) = ln[a^{2N}(e^{a\sum_{i=1}^{N}(2-z_i)})^N \prod_{i=1}^{N}(z_i - 2)]. \tag{49}$$

After simplify we get

$$l(a, Z_1, Z_2, \ldots, Z_N) = 2N ln a + a\sum_{i=1}^{N}(2 - z_i) + \sum_{i=1}^{N}(z_i - 2). \tag{50}$$

Taking the partial derivative with respect to $a$ we have

$$\frac{\partial l(a, Z_1, Z_2, \ldots, Z_N)}{\partial \hat{a}} = \frac{2N}{a} + \sum_{i=1}^{N}(2 - z_i). \tag{51}$$

Set the derivative function to zero and then rearrange the equation to make the parameter of interest the subject of the equation as follows:

$$\frac{2N}{a} + \sum_{i=1}^{N}(2 - z_i) = 0. \tag{52}$$

Solve it for $\hat{a}$ we get

$$\hat{a} = \frac{2N}{\sum_{i=1}^{N}(z_i - 2)} = \frac{2}{\bar{Z} - 2}, \tag{53}$$

which is the exact solution for the estimated parameter $\hat{a}$.

## 7. Asymptotic behaviors of CBD

The asymptotic properties of the CBD when $Z \to 2$ and when $Z \to \infty$, the behavior as follows $\lim_{z\to 2} f(z) = \lim_{z\to 2} a^2 e^{a(2-z)}(z - 2) = 0$ and $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} a^2 e^{a(2-z)}(z - 2) = 0$. This confirms the unimodality of the distribution; this means there is only a single highest value.

## 8. Renyi entropy

There is wide literature on the applications of the Renyi entropy in many fields such as biology, medicine, and economics, computer science, chemistry and physics. The Rényi entropy has been widely used in the study of quantum systems. Especially, it used in the analysis of quantum correlations and quantum measurement and quantum statistical mechanics [19]. In the paper, in view of these numerous and successful applications, it seems valuable to formulate the Rényi entropies in the following theorem. Because the entropic measure of uncertainty and it is adequate in quantum mechanical measurements. The Rényi entropy is defined as

$$E_R = \frac{1}{1-\rho}\int_2^{\infty}[g(z)]^{\rho}dz. \tag{54}$$

**Lemma 8.1.** *If $Z \sim$ CBD(a), then the Rényi entropy function of CBD is given by*

$$E_R = \frac{1}{1-\rho}(a^2)^{\rho}(\frac{1}{a\rho})^{\rho+1}\Gamma(\rho+1). \tag{55}$$

**Proof.** Since

$$E_R = \frac{1}{1-\rho}\int_2^{\infty}(a^2e^{a(2-z)}(z-2))^{\rho}dz \tag{56}$$

let $x = z-2$, $dx = dz$ and as $z = 2 \to x = 0$, $z \to \infty$, $x \to \infty$ then the integral becomes

$$E_R = \frac{1}{1-\rho}(a^2)^{\rho}\int_0^{\infty}e^{-a\rho}x^{\rho}dx. \tag{57}$$

Now, assume that $y = a\rho x$, $x = \frac{y}{a\rho}$ and $dx = \frac{dy}{a\rho}$ then the integral becomes

$$E_R = \frac{1}{1-\rho}(a^2)^{\rho}\frac{1}{a\rho}\int_0^{\infty}e^{-y}y^{\rho}dy. \tag{58}$$

Simplify, rearrange and the integral $\int_0^{\infty}e^{-y}y^{\rho}dy = \Gamma(\rho+1)$ and the proof is completed. □

## 9. Simulation study

Simulation is a way to model random events, such that simulated outcomes closely match real-world outcomes. By observing simulated outcomes, researchers gain insight on the real world. An estimator is a numerical function of the data. There are many ways to specify the form of an estimator for a particular parameter of a given distribution, and many ways to evaluate the quality of an estimator. The following algorithm is used to generate data from the CBD:

- Generate U random numbers from $U(0,1)$ distribution.
- Since $Z$ has a continuous distribution function $F(z)$ belongs to CBD (i.e. $Pr(Z \leq z)$) then $F(Z) \sim Uniform(0,1)$.
- Set $F(z) = u$ provided the inverse as follows $z = F^{-1}(u)$ and

$$1 + (2a - az - 1)e^{2a-az} = u. \tag{59}$$

Since no exact solution for Eq. (59), so we used Newton Raphson method. beginning with some arbitrary $z_0$, calculate the equation of the line tangent to

Table 1: Simulated results of average, variance, skewness, kurtosis and bias of the estimator $a = 0.5$.

| $a = 0.5$ | | | | | | |
|---|---|---|---|---|---|---|
| n | average | variance | skewness | kurtosis | a | bias |
| 5 | 6.328125 | 6.095517 | -0.61743 | 0.087814 | 0.462094 | 0.037906 |
| 10 | 5.94627 | 10.45756 | 0.997421 | -0.11516 | 0.506808 | -0.00681 |
| 20 | 6.398513 | 7.867614 | 0.789465 | -0.10424 | 0.454699 | 0.045301 |
| 30 | 5.600391 | 5.925499 | 1.077468 | 1.203318 | 0.555495 | -0.0555 |
| 50 | 6.628457 | 8.511992 | 1.311712 | 2.292154 | 0.432109 | 0.067891 |
| 100 | 5.493748 | 5.268311 | 0.659865 | -0.64601 | 0.572451 | -0.07245 |
| 200 | 6.070159 | 9.222475 | 1.554787 | 4.649339 | 0.491381 | 0.008619 |
| 2000 | 5.999317 | 8.363794 | 1.502168 | 3.485188 | 0.500085 | -8.5E-05 |
| 5000 | 6.024985 | 7.929047 | 1.391688 | 3.020997 | 0.496896 | 0.003104 |

Table 2: Simulated results of average, variance, skewness, kurtosis and bias of the estimator $a = 1$.

| $a = 1$ | | | | | | |
|---|---|---|---|---|---|---|
| n | average | variance | skewness | kurtosis | a | bias |
| 5 | 4.212014 | 1.858279 | -0.02157 | -0.55717 | 0.904154 | 0.095846 |
| 10 | 4.114667 | 0.956655 | -0.09912 | -0.05631 | 0.945776 | 0.054224 |
| 20 | 4.730873 | 3.809207 | 1.383953 | 1.99232 | 0.732367 | 0.267633 |
| 25 | 3.840373 | 1.290374 | 0.843757 | -0.32851 | 1.086736 | -0.08674 |
| 50 | 4.106315 | 1.567502 | 0.582919 | -0.15735 | 0.949525 | 0.050475 |
| 100 | 3.624332 | 1.322742 | 1.053599 | 0.607814 | 1.231276 | -0.23128 |
| 200 | 3.857998 | 1.707839 | 1.155441 | 1.745109 | 1.076427 | -0.07643 |
| 1000 | 4.071128 | 2.128726 | 1.581991 | 4.794034 | 0.965657 | 0.034343 |
| 2000 | 4.063872 | 2.022965 | 1.227163 | 1.771559 | 0.969052 | 0.030948 |
| 5000 | 4.032718 | 2.050063 | 1.367185 | 2.555873 | 0.983904 | 0.016096 |

$f$ at $z_0$, and see where that tangent line crosses 0. The equation of the line tangent is $g(z) = f(z_0) + f'(z_0)(z - z_0)$ also it is the first two terms Taylor approximation at $z_0$. Rearranging terms as $z = z_0 - \frac{f(z_0)}{f'(z_0)}$, then Now setting $z_0 = z$ and repeating until convergence is essentially Newton's method. The simulation study is based on generating $N = 10,000$ samples of size $n = 5, 10, 20, 50, 100, 200, 2000$ *and* $5000$ for $a = 0.5, 1, 2$ using above algorithm. Then we calculate the following measures; average, variance, skewness, kurtosis and bias of the estimator. All the results for different values of $a = 0.5, 1, 2$ are illustrated in Tables 1,2 and 3. As well as the result of the simulated data that derived from the new distribution CBD for the CDF are shown in Figures 3, 4 and 5 for $a = 0.5, 1, 2$.

Table 3: Simulated results of average, variance, skewness, kurtosis and bias of the estimator $a = 2$.

| $a = 2$ | | | | | | |
|---|---|---|---|---|---|---|
| n | average | variance | skewness | kurtosis | a | bias |
| 5 | 3.183778 | 0.835779 | 1.409327 | 2.15818 | 1.689505 | 0.310495 |
| 10 | 3.33264 | 0.535555 | 0.105309 | -1.30659 | 1.500781 | 0.499219 |
| 20 | 3.398707 | 0.90538 | 0.790657 | -0.49655 | 1.429892 | 0.570108 |
| 30 | 2.991535 | 0.36166 | 1.016148 | 0.361959 | 2.017075 | -0.01708 |
| 50 | 2.994086 | 0.515195 | 1.020777 | 1.482307 | 2.011898 | -0.0119 |
| 100 | 3.126318 | 0.65937 | 1.294906 | 2.017583 | 1.775698 | 0.224302 |
| 200 | 3.09809 | 0.57787 | 1.141619 | 1.174657 | 1.821344 | 0.178656 |
| 1000 | 2.993328 | 0.449259 | 1.184761 | 1.712884 | 2.013434 | -0.01343 |
| 2000 | 3.025623 | 0.525183 | 1.432676 | 2.906886 | 1.950035 | 0.049965 |
| 5000 | 2.974395 | 0.467879 | 1.368338 | 2.589046 | 2.052555 | -0.05255 |

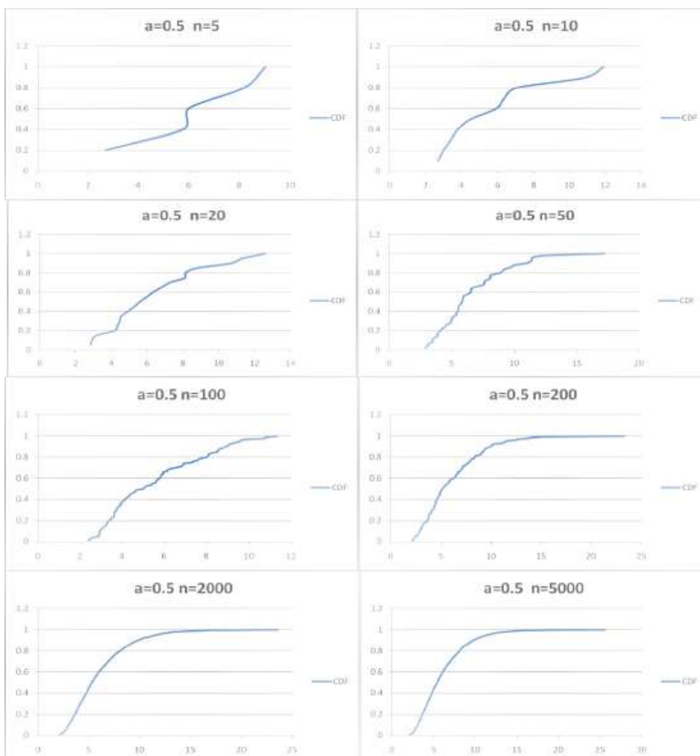


Figure 3: The CDF of CBD when $a = 0.5$

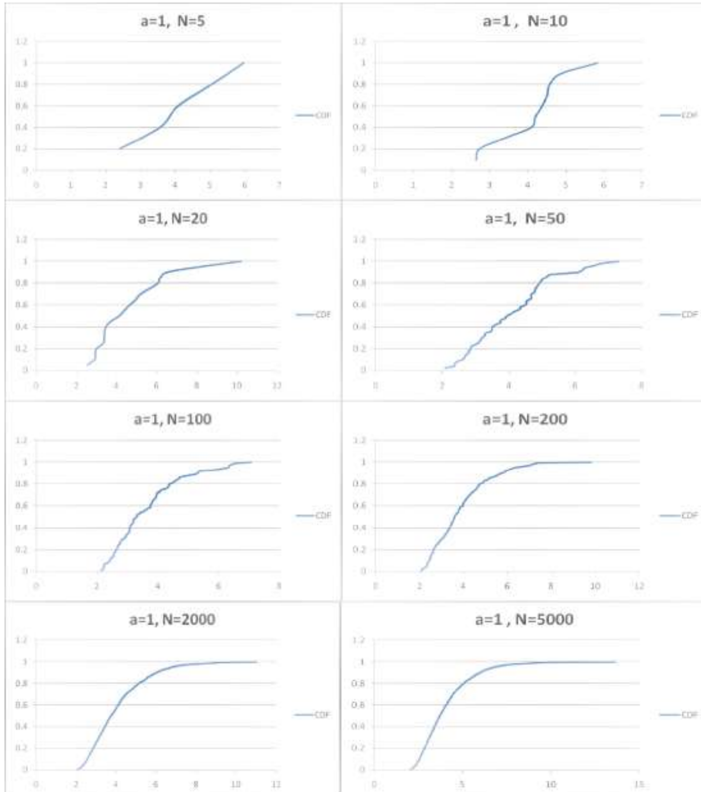


Figure 4: The CDF of CBD when $a = 1$

## 10. Conclusions

In this paper, we proposed the convolution $Z = X + Y$ of the benktander distribution, of which the parameter $b = 1$. Several properties of the CBD such as moments, skewness and kurtosis have been discussed. Moreover, the ODE of (QF), (SF), (HF) and (RHF) of convoluted probability distributions has been illustrated. The Graphs of CDF of the simulated data derived from CBD for selected values of parameter are shown. These graphs show the behavior of the distribution function more fit when the size of the sample increased. In addition, the biases are decreased when the size of the sample increased.

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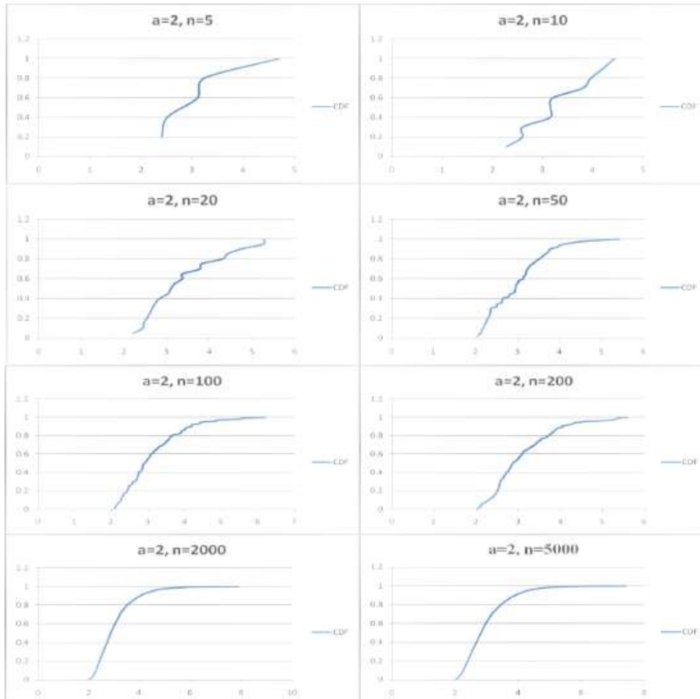


Figure 5: The CDF of CBD when $a = 2$

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Accepted: January 07, 2020