An hybrid block method for direct integration of first, second and third order IVPs

E.O. Adeyefa*
Department of Mathematics
Federal University Oye-Ekiti
P.M.B. 373, Oye-Ekiti, Ekiti State
Nigeria
adeoluman@yahoo.com

J.O. Kuboye
Federal University Oye-Ekiti
Pmb 373, Oye-Ekiti
kubbysholly2007@yahoo.com

O.A. Olajide
Department of Mathematics
Federal University Oye-Ekiti
P.M.B. 373, Oye-Ekiti, Ekiti State
Nigeria
holisticdapo4favor@gmail.com

S.A. Osikoya
Department of Mathematics
Federal University Oye-Ekiti
P.M.B. 373, Oye-Ekiti, Ekiti State
Nigeria
samuelabidemi2@gmail.com

Abstract. Development of numerical methods for the solution of initial value problems (IVPs) in ordinary differential equations (ODEs) has been considered overwhelmingly in literature. However, the use of a single numerical method for the integration of ODEs of different order has not been commonly reported.

In this paper, we focus on development of a numerical method capable of obtaining the numerical solution of first, second and third order IVPs. The method is formulated from continuous schemes obtained via collocation and interpolation techniques and applied in a block-by-block manner as numerical integrator for first, second and third order ODEs. The convergence properties of the method are discussed via zero-stability and consistency. Numerical examples are included and comparisons are made with existing methods in the literature.

Keywords: consistency, convergence, hybrid block method, initial value problems, zero-stability.

*. Corresponding author

1. Introduction

In science and engineering, many physical problems are modelled by ordinary differential equations of the form

$$y^m(x) = f(x, y(x), y'(x), \dots, y^{m-1}(x)), \quad y^{m-1}(x_0) = \gamma, \ \ , a \le x \le b. \tag{1}$$

Many of the equations resulting from these problems have no analytical solutions hence, the need for efficient numerical methods to address this difficulty as the quest for the solution is on the increase.

Several numerical methods have been proposed for the solution of (1) for m = 1, 2, 3, 4 with the ultimate goal of improving on the efficiency and convergence of the existing methods (see Butcher (2003), Yahaya *et al.* (2016), Adeniyi and Alabi (2011), Jator (2007, 2010a, 2010b), Kuboye (2015), Kuboye *et al.* (2018), Lambert (1973, 1991), Ismail (2009), Ramos *et al.* (2016), Simos (2002)). The advent of the self-starting method popularly called block method has been adopted greatly by researchers and discussed extensively in literature (see Sagir (2014), Vigo-Aguiar and Ramos (2006), Olabode (2009, 2013), Ramos *et al.* (2016), Adeyefa (2017)) among others. The formulation of block method to integrate (1) has been widely reported in the literature to be self starting and efficient. However, to use a formulated block method for integration of several order IVPs, say first, second and third order ODEs has not been commonly reported. Thus, the focus of this paper is to formulate a self-starting method for the numerical solution of the ODE (1) for m = 1, 2 and 3.

In what immediately follows in Section two, we derive the proposed method and discuss the basic properties of the method in Section three. Numerical examples are given to show the efficiency of the method in Section four while the discussion of results is given in Section five. Finally, the conclusion of the paper is discussed in Section six.

2. Materials and methods

In this section, we formulate new one-step hybrid method capable of solving first, second and third order ODEs.

Our approximate solution is formulated using constructed orthogonal polynomials $q_0(x) = 1$, $q_1(x) = \frac{1}{2}(3x-1)$, $q_2(x) = \frac{1}{2}(5x^2-2x-1)$, $q_3(x) = \frac{1}{8}(35x^3-15x^2-15x+3)$, $q_4(x) = \frac{1}{8}(63x^4-28x^3-42x^2+12x+3)$, $q_5(x) = \frac{1}{16}(231x^5-105x^4-210x^3+70x^2+35x-5)$, $q_6(x) = \frac{1}{16}(429x^6-198x^5-495x^4+180x^3+135x^2-30x-5)$ constructed in Adeyefa and Adeniyi (2015) as our basis function. Thus, we introduce

$$y(x) = \sum_{j=0}^{k+8} a_j q^j \tag{2}$$

as an approximate solution to first, second and third order ODEs of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \tag{3}$$

(4) $y''(x) = f(x, y(x), y'(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0,$

(5) $y'''(x) = f(x, y(x), y'(x), y''(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0,$

where a and q in (2) are constants to be determined and the constructed orthogonal polynomials respectively.

Equation (2) is interpolated at $x = x_n$, its first and second derivatives are collocated at $x = x_{n+v}, v = 0, \frac{1}{3}, 1$ while its third derivative is collocated at $x = x_{n+r}, r = 0, \frac{1}{3}$.

As a result, we have

$$\begin{aligned} &\sum_{j=0}^{k+8} \alpha_j q_n^j = y_n, \\ &\sum_{j=1}^{k+8} j\alpha_j q_{n+r}^{j-1} = f_{n+r}, \\ &\sum_{j=2}^{k+8} j(j-1)\alpha_j q_{n+r}^{j-2} = g_{n+r}, \\ &\sum_{j=3}^{k+8} j(j-1)(j-2)\alpha_j q_{n+r}^{j-3} = k_{n+r}. \end{aligned} \tag{6}$$

Solving equation (6) using Gaussian elimination approach in order to get the unknown variables $a's$ which are substituted into equation (2). This yields a continuous implicit scheme of the form:

$$\begin{aligned} \alpha_0(t) y_n = {} & h\left(\sum_{j=0}^{k} \beta_j(t) f_{n+j} + \beta_{\frac{1}{3}}(t) f_{n+\frac{1}{3}}\right) \\ & + h^2\left(\sum_{j=0}^{k} \lambda_j(t) g_{n+j} + \lambda_{\frac{1}{3}}(t) g_{n+\frac{1}{3}}\right) + h^3\left(\delta_0(t) k_n + \delta_{\frac{1}{3}}(t) k_{n+\frac{1}{3}}\right) \end{aligned} \tag{7}$$

where

$$\begin{aligned} t &= \frac{2x - 2x_n - h}{h}, (\alpha_0(t) = 1, \\ \beta_0 &= -\frac{1}{71680} h(t+1)(70875t^7 + 33885t^6 - 186345t^5 - 97743t^4 + 127633t^3 \\ &+ 86007t^2 + 37053t - 8773), \\ \beta_{\frac{1}{3}} &= \frac{243}{1146880} h(4725t^4 - 11820t^3 + 8990t^2 - 2924t + 1781)(t+1)^4, \\ \beta_1 &= -\frac{1}{1146880} h(14175t^4 - 12420t^3 - 11430t^2 - 4164t - 289)(t+1)^4, \\ \lambda_0 &= -\frac{1}{71680} h^2(10395t^6 - 6750t^5 - 19995t^4 + 9052t^3 + 7701t^2 \\ &+ 4386t - 373)(t+1)^2, \end{aligned}$$

$$
\begin{aligned}
\lambda_{\frac{1}{3}} &= -\frac{81}{143360}h^2(315t^4 - 780t^3 + 530t^2 - 44t + 11)(t+1)^4,\\
\lambda_1 &= \frac{1}{114688}h^1(189t^4 - 108t^3 - 114t^2 - 44t - 3)(t+1)^4,\\
\delta_0 &= -\frac{1}{430080}h^3(2835t^5 - 5265t^4 + 990t^3 + 1086t^2 + 927t - 29)(t+1)^3,\\
\delta_{\frac{1}{3}} &= \frac{9}{573440}h^3(945t^4 - 1980t^3 + 710t^2 + 388t + 133)(t+1)^4.
\end{aligned}
$$

Equation (7) is evaluated at $x = x_{n+1}$ (i.e. $t = 1$) and $x = x_{n+\frac{1}{3}}$ (i.e. $t = -\frac{1}{3}$) to obtain the following schemes

$$
\begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_n + hD \begin{pmatrix} f_n \\ f_{n+\frac{1}{3}} \\ f_{n+1} \end{pmatrix} + h^2E \begin{pmatrix} g_n \\ g_{n+\frac{1}{3}} \\ g_{n+1} \end{pmatrix} + h^3F \begin{pmatrix} k_n \\ k_{n+\frac{1}{3}} \\ k_{n+1} \end{pmatrix} \tag{8}
$$

where the values of D, E and F are

$$
D = \begin{pmatrix} \frac{3527}{22680} & \frac{797}{4480} & -\frac{29}{362880} \\ -\frac{489}{280} & \frac{11421}{4480} & \frac{883}{4480} \end{pmatrix}, E = \begin{pmatrix} \frac{641}{68040} & -\frac{11}{840} & \frac{1}{108864} \\ -\frac{69}{280} & -\frac{81}{280} & -\frac{5}{448} \end{pmatrix},
$$

$$
F = \begin{pmatrix} \frac{31}{136080} & \frac{83}{181440} \\ -\frac{17}{1680} & \frac{99}{2240} \end{pmatrix}.
$$

Equation (8) is our proposed first, second and third order IVPs solver.

3. Basic properties of the method

We shall consider in this section, the analysis of basic properties of this method such as order, error constant, zero stability and consistency is investigated.

3.1 Order and error constant

Equation (8) derived is a discrete scheme belonging to the class of LMMs of the form

$$
\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \omega_j f_{n+j} + h^2\sum_{j=0}^{k} \beta_j g_{n+j} + h^3\sum_{j=0}^{k} \gamma_j G_{n+j}. \tag{9}
$$

Following, Fatunla (1988) and Lambert (1991), we define the local truncation error associated with equation (9) by the difference operator

$$
\begin{aligned}
L[y(x):h] = \sum_{j=0}^{k}[&\alpha_j y(x_n + jh) - h\omega_j f(x_n + jh) - h^2\beta_j g(x_n + jh)\\
&- h^3\gamma_j G(x_n + jh)],
\end{aligned} \tag{10}
$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (10) in Taylor series about the point x, we obtain the expression

$$L[y(x);h] = C_0y(x) + C_1hy'(x) + C_2y''(x) + \cdots + C_{p+2}h^{p+2}y^{p+2}(x),$$

where the C_0, C_1, C_2, ..., C_p, ..., C_{p+2} are obtained as

$$C_0 = \sum_{j=0}^{k} \alpha_j, C_1 = \sum_{j=1}^{k} j\alpha_j,$$

$$C_2 = \frac{1}{2}\sum_{j=0}^{k} j^2\alpha_j,$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)\sum_{j=1}^{k}\beta_j j^{q-2} - q(q-1)(q-2)\sum_{j=1}^{k}\gamma_j j^{q-3}\right].$$

In the spirit of Lambert (1991), equation (10) is of order p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1}$ and $C_{p+2} \neq 0$. The $C_{p+2} \neq 0$ is called the error constant and $C_{p+2}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.

Thus, the block (8) is of order $p = 6$ and error constant

$$C_{p+3} = \left[\frac{113}{1999918771200}, \frac{13}{914457600}\right]^T.$$

3.2 Zero stability of the method

The linear multistep method (9) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (9) in vector notation form of column vectors $e = (e_1, \ldots, e_\gamma)^T$, $d = (d_1, \ldots, d_\gamma)^T$, $y_m = (y_{n+1}, \ldots, y_{n+\gamma})^T$, $F(y_m) = (f_{n+1}, \ldots, f_{n+\gamma})^T$, $G(y_m) = (g_{n+1}, \ldots, g_{n+\gamma})^T$, $W(y_m) = (k_{n+1}, \ldots, k_{n+\gamma})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$. Thus, equation (8) forms the block formula

(11)
$$A_0y_m = hBF(y_m) + A^1y_n + hbf_n + h^2DG(y_m) + h^2dg_n + h^3VW(y_m) + h^3uT_n,$$

where h is a fixed mesh size within a block.

In line with equation (11), $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. The first characteristic polynomial of the block hybrid method is given by

(12) $$\rho(R) = det(RA^0 - A^1).$$

Substituting $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ in equation (12) and solving for R , the values of R are obtained as 0 and 1.

According to Fatunla in (1988, 1991), the block method represented by equation (8) are zero-stable, since from equation (12), $\rho(R) = 0$, satisfy $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

3.3 Consistency and convergence of the method

The linear multistep method (9) is said to be consistent if it has order $p \geq 1$. The method is consistent being of order 6.

According to the theorem of Dahlquist in Dahlquist (1979), the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

4. Numerical experiment

We consider in this section, four test problems which includes first, second and third order ordinary differential equations to test the effectiveness of this new scheme.

Problem 1. We consider the IVP $y'' = y'$, $y(0) = 0$, $y'(0) = -1$, $h = 0.1$ with exact solution, $y(x) = 1 - e^x$ which has been solved in Mohammed and Adeniyi (2014) with step number k =5 and in Kuboye (2015) with step number k = 5.

Problem 2. We consider the IVP $y''' = e^x$, y(0) = 3, $y'(0) = 1$, $y''(0) = 5$, h = 0.1 for which the exact solution is given by $y(x) = 2 + 2x^2 + e^x$.

This example was solved in Abdullahi *et al.* (2016) and in Olabode (2009) using step number $k = 5$ methods.

Problem 3. Here we consider another IVP $y''' = 3\sin x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $h = 0.1$ solved by Olabode (2013) and Kuboye (2015) with step number $k = 6$ and for which the exact solution is given by $y(x) = 3\cos x + \frac{x^2}{2} - 2$.

Problem 4. We consider first order IVP $y' = 0.5(1 - y)$, $y(0) = 0.5$, $h = 0.1$ with the exact solution $y(x) = 1 - 0.5e^{-0.5x}$. This IVP was solved by Ajileye *et al.* (2018) and Sunday *et al.* (2013).

Problem 5. We consider non-linear IVP $y'' - x(y')^2 = 0$, y(0) = 1, $y'(0) = 0.5$, h = 0.003125 whose exact solution is $1 + \frac{1}{2}\ln(\frac{2+x}{2-x})$.

Details of the numerical results for Problems 1, 2, 3, 4 and 5 are shown in Figures 1, 2, 3, 4 and 5 respectively and discussed in Section five.

5. Discussion of results

Figures 1, 2, 3 and 4 summarize the results obtained for the four test problems considered. In Figures 1a, 2a, 3a and 4a, comparisons of the exact and approximate solutions are displayed while the Errors are compared in Figures 1b, 2b,

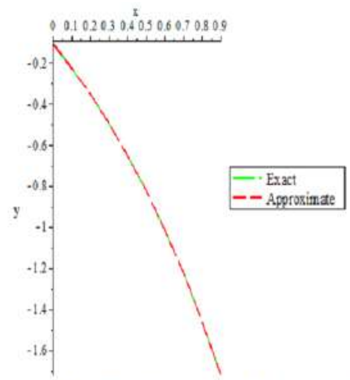


Fig 1a : Comparing exact and approximate solutions of Problem 1

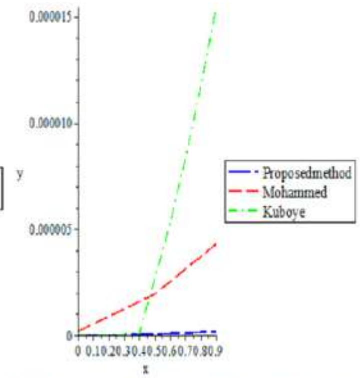


Fig 1b: Error comparison of existing methods and proposed method for Problem 1

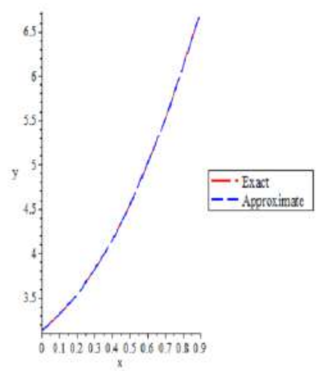


Fig 2a: Comparing exact and approximate solutions of Problem 2

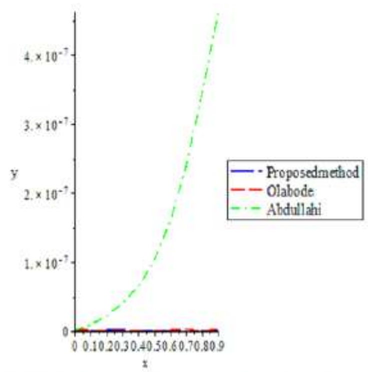


Fig 2b: Error comparison of existing methods and proposed method for Problem 2

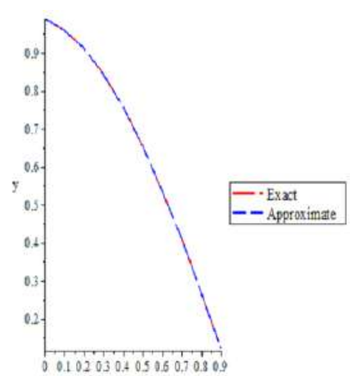


Fig 3a: Comparing exact and approximate solutions of Problem 3

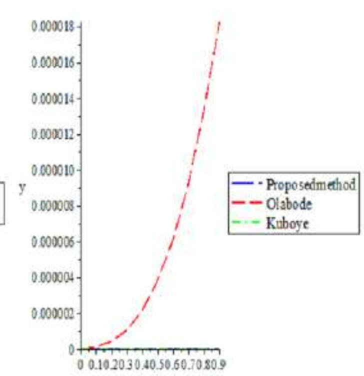


Fig 3b: Error comparison of existing methods and proposed method for Problem 3

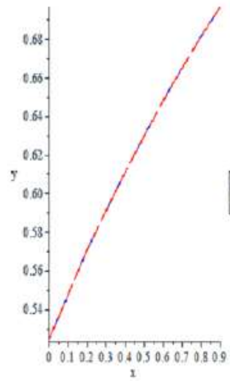


Fig 4a: Comparing exact and approximate solutions of Problem 4

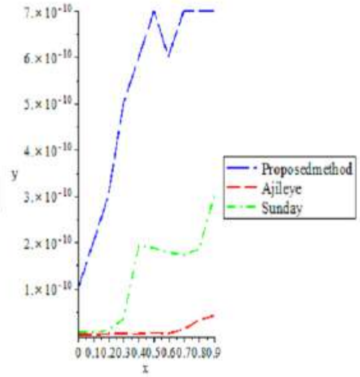
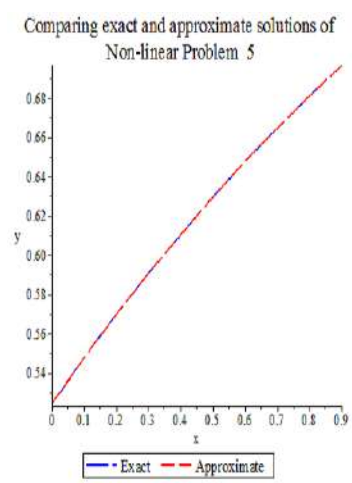
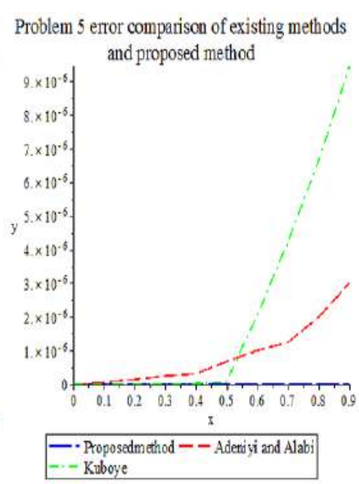


Fig 4b: Error comparison of existing methods and proposed method for Problem 4





3b and 4b. The proposed method is of step number $k = 1$ and it compared favourably with existing methods despite their $k > 1$ methods. In Problem 1, our step length is $h = 0.1$ against $h = 0.01$ used in Mohammed and Adeniyi (2014) and Kuboye (2015). The proposed method still gives better accuracy even with larger h. In Figure 4, the methods developed by Ajileye *et al.* (2018) and Sunday *et al.* (2013) performed better than the new method in terms of accuracy but their methods do not have the ability to solve higher order ordinary differential equations.

Figure 5 displays the comparison of the exact and the approximate solutions and the error between the existing methods and the proposed method. It is obvious that the new method compares favourably with the existing methods.

6. Conclusion

A block method has been applied to solve first, second and third order ordinary differential equations directly without construction of additional schemes or employing existing predictors for implementation. Numerical experiments performed using this method show that the method is consistent, efficient and accurate. We therefore recommend the method for direct integration of first, second and third order ordinary differential equations.

References

[1] Y.A. Abdullahi, O. Zurni, J.O. Kuboye, *Derivation of block predictor block corrector method for direct solution of third order ordinary differential equations*, Global Journal of Pure and Applied Mathematics, 12 (2016), 343-350.

[2] R.B. Adeniyi, M.O. Alabi, *A collocation method for direct numerical integration of initial value problems in higher order ordinary differential equations* Analele Ştiintifice ale Universităţii ,,AL. I. Cuza" din Iaşi (SN), Matematica, 2 (2011), 311-321.

[3] E.O. Adeyefa, R.B. Adeniyi, *Construction of orthogonal basis function and formulation of continuous hybrid schemes for the solution of third order ODEs*, Journal of the Nigerian Association of Mathematical Physics, 29 (2015), 21-28.

[4] E.O. Adeyefa, *Orthogonal-based hybrid block method for solving general second order initial value problems*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 659-672.

[5] G. Ajileye, S.A. Amoo, O.D. Ogwumu, *Hybrid block method algorithms for solution of first order initial value problems in ordinary differential equations*, J. Applied and Computational Math., 7 (2018), 1-4, DOI: 10.4172/2168-9679.1000390.

[6] J.C. Butcher, *Numerical methods for ordinary differential equations*, John Wiley and Sons Ltd., West Sussex, 2003.

[7] G. Dahlquist, *Some properties of linear multistep and one leg method for ordinary differential equations*, Department of Computer Science, Royal institute of Technology, Stockholm, 1979.

[8] S.O. Fatunla, *Numerical methods for initial value problems in ordinary differential equations*, Academic Press Inc. Harcourt Brace, Jovanovich Publishers, New York, 1988.

[9] S.O. Fatunla, *Block method for second order initial value problem (IVP)*, International Journal of Computer Mathematics, 41 (1991), 55-63.

[10] F. Ismail, Y.L. Ken, M. Othman, *Explicit and implicit 3-point block methods for solving special second order ordinary differential equations directly*, Int. J. Math. Anal., 3 (2009), 239-254.

[11] S.N. Jator, *On a class of hybrid methods for* $y = f(x, y, y')$, Int. J. Pure Appl. Math., 59 (2010), 381-395.

[12] S.N. Jator, *Solving second order initial value problems by a hybrid multistep method without predictors*, Appl. Math. Comput., 217 (2010), 4036-4046.

[13] S.N. Jator, *A sixth order linear multistep method for direct solution of* $y'' = f(x, y, y')$, International Journal of Pure and Applied Mathematics, 40 (2007), 407-472.

[14] J.O. Kuboye, *Block methods for direct solution of higher order ordinary differential equations using interpolation and collocation approach*, PhD. Thesis, Universiti Utara Malaysia, 2015.

[15] J.O. Kuboye, Z. Omar, O.E. Abolarin, R. Abdelrahim, *Generalized hybrid block method for solving second order ordinary differential equations directly*, Res. Rep. Math., 2 (2018).

[16] J.D. Lambert, *Numerical methods for ordinary differential systems*, New York John Willey and Sons, 1991.

[17] J.D. Lambert, *Computational methods for ordinary differential equations*, John Wiley, New York, 1973.

[18] U. Mohammed, R.B. Adeniyi, *Derivation of five step block Hybrid Backward Differential Formulas (HBDF) through the continuous multi-step collocation for solving second order differential equation*, Pacific Journal of Science and Technology, 15 (2014), 89-95.

[19] B.T. Olabode, *An accurate scheme by block method for the third order ordinary differential equation*, Pacific Journal of Science and Technology, 10 (2009), 136-142.

[20] B.T. Olabode, *Block multistep method for the direct solution of third order of ordinary differential equations*, Futa Journal of Research in Sciences, 9 (2013), 194-200.

[21] H. Ramos, S. Mehta, J. Vigo-Aguiar, *A unified approach for the development of k-Step block Falkner-type methods for solving general second-order initial-value problems in ODEs*, J. of Computational and Applied Mathematics, Article in Press, 2016.

[22] A.M. Sagir, *Numerical treatment of block method for the solution of ordinary differential equations*, International Journal of Mathematical, Computational, Natural and Physical Engineering, 8 (2016), 259 - 263.

[23] T.E. Simos, *Dissipative trigonometrically fitted methods for second order IVPs with oscillating solution*, Int. J. Mod. Phys., 13 (2002), 1333-1345.

[24] J. Sunday, M.R. Odekunle, A.O. Adesanya, *Order six block integrator for the solution of first-order ordinary differential equations*, Int. J. Math Soft. Comp., 3 (2013), 87-96.

[25] J. Vigo-Aguiar, H. Ramos, *Variable stepsize implementation of multistep methods for* $y'' = f(x, y, y')$, J. Comput. Appl. Math, 192 (2006), 114-131.

[26] O. Yahaya, Kuboye, *Derivation of block predictor-corrector method for direct solution of third order ordinary differential equations*, Global Journal of Pure and Applied Mathematics, 12 (2016), 343-350.

Accepted: December 24, 2020