Coefficient estimates for subclasses $B_{\Sigma}^{m}(\alpha,\lambda)$ and $B_{\Sigma}^{m}(\beta,\lambda)$ of analytic and bi-univalent functions defined by a differential operator

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Abstract. In this paper two subclasses $B_{\Sigma}(\alpha,\lambda)$ and $B_{\Sigma}(\beta,\lambda)$ are studied and a new estimation for the initial coefficients $|a_2|$ and $|a_3|$ in these subclasses are found.
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1. Introduction

Let $\mathbb{R} = (-\infty,\infty)$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and $\mathbb{N} = \{1,2,3\ldots\} = \mathbb{N}_0\backslash\{0\}$ the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

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We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [8]; see also some of the recent investigations [2, 3, 4, 5, 11, 15]).

For $f \in \mathcal{A}$, we utilize the following differential operator studied by Darus and Ibrahim [9].

$$T^0 f(z) = f(z), \tag{2}$$

$$T^1_{\sigma,\delta,\gamma} = [1-\delta(\gamma-\sigma)]f(z) + \delta(\gamma-\sigma)zf'(z), \tag{3}$$

$$T^m_{\sigma,\delta,\gamma} f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\delta(\gamma-\sigma)]^m a_n z^n, \tag{4}$$

for $\sigma \geq 0, \delta \geq 0, \gamma \geq 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $T^m_{\sigma,\delta,\gamma} f(0) = 0$.

When $\delta = \sigma = 1$ and $\gamma = 2$, we get Sălăgean's differential operator [13].

Univalent functions are injective they do not require to be defined on the entire unit disk $\mathbb{U}$ since they are reversible, indeed, the image of $\mathbb{U}$ under every univalent function $f \in S$ contains a disk of radius $1/4$, which verified by Koebe one-quarter theorem [8]. According to this, every function $f \in \mathcal{S}$ has an inverse map f^{-1} that satisfies the following conditions:

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{5}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The class Σ has been investigated and the Taylor-Maclaurin coefficient inequalities for various classes of bi-univalent functions have been studied, see Srivastava et al. [14], see also [1, 6, 7, 10, 16, 17, 18, 19].

In this paper two new subclasses of the class Σ are introduced and the estimation for the coeffcients $|a_2|$ and $|a_3|$ for the functions in the above new subclasses of the class Σ are found.

Firstly, to achieve our main objectives, we need the following lemma.

Lemma 1.1 ([12])**.** *If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $\Re e(p(z)) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ for $z \in \mathbb{U}$.*

2. Coefficient bounds for the class $B^m_\Sigma(\alpha,\lambda)$

Definition 2.1 ([20]). *A function f given by (1) is said to be in the class $B^m_\Sigma(\alpha,\lambda)$ if the following conditions are satisfied:*

$$\left|\arg\left((1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}f(z)}{z}+\lambda\left(T^m_{\sigma,\delta,\gamma}f(z)\right)'\right)\right|<\frac{\alpha\pi}{2}, \tag{6}$$
$$(0<\alpha\le 1, \lambda\ge 1 \text{ and } \sigma,\delta,\gamma\ge 0, z\in\mathbb{U})$$

and

$$\left|\arg\left((1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}g(w)}{w}+\lambda\left(T^m_{\sigma,\delta,\gamma}g(w)\right)'\right)\right|<\frac{\alpha\pi}{2}, \tag{7}$$
$$(0<\alpha\le 1, \lambda\ge 1 \text{ and } \sigma,\delta,\gamma\ge 0, w\in\mathbb{U}),$$

where the function g is given by (5).

Theorem 2.1. *Let f given by (1) be in the class $B^m_\Sigma(\alpha,\lambda)$, $0<\alpha\le 1$ and $\lambda\ge 1$. Then*

$$|a_2|\le\frac{2\alpha}{\sqrt{2\alpha(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m-(\alpha-1)(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}}}, \tag{8}$$

and

$$|a_3|\le\frac{4\alpha^2}{(\lambda+1)^2[1+2\delta(\gamma-\sigma)]^{2m}}+\frac{2\alpha}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}. \tag{9}$$

Proof. It follows from (6) and (7) that

$$(1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}f(z)}{z}+\lambda T^m_{\sigma,\delta,\gamma}f'(z)=[p(z)]^\alpha, \tag{10}$$
$$(1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}g(w)}{w}+\lambda T^m_{\sigma,\delta,\gamma}g'(w)=[q(w)]^\alpha, \tag{11}$$

where $p(z)$ and $q(w)$ are in $\mathcal{P}$ have the forms

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\dots, \tag{12}$$
$$q(w)=1+q_1w+q_2w^2+q_3w^3+\dots \tag{13}$$

Now, equating the coefficients in (10) and (11), we get

$$(\lambda+1)[1+\delta(\gamma-\sigma)]^m a_2=\alpha p_1, \tag{14}$$
$$(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{15}$$
$$-(\lambda+1)[1+\delta(\gamma-\sigma)]^m a_2=\alpha q_1 \tag{16}$$

and

$$(2\lambda+1)(2a_2^2-a_3)[1+2\delta(\gamma-\sigma)]^m=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{17}$$

From (14) and (16), we get

$$p_1=-q_1 \tag{18}$$

and

$$2(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}a_2^2=\alpha^2(p_1^2+q_1^2). \tag{19}$$

Now, from (15), (17) and (19), we obtain

$$\begin{aligned}2(\lambda+1)[1+\delta(\gamma-\sigma)]^m a_2^2&=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)\\&=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\frac{2(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}}{\alpha^2}a_2^2.\end{aligned} \tag{20}$$

Therefore, we have

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{2\alpha(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m-(\alpha-1)(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}}. \tag{21}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|\leq\frac{2\alpha}{\sqrt{2\alpha(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m-(\alpha-1)(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}}}. \tag{22}$$

This gives the bound of $|a_2|$ as asserted in (8).

Next, in order to find the bound of $|a_3|$, by subtracting (17) from (15), we get

$$\begin{aligned}&2(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m a_3-2(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m a_2^2\\&=\alpha\left(p_2-q_2\right)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right).\end{aligned} \tag{23}$$

It follows from (17)-(23) that

$$|a_3|=\frac{4\alpha^2(p_1^2+q_1^2)}{(\lambda+1)^2[1+2\delta(\gamma-\sigma)]^{2m}}+\frac{2\alpha(p_2+q_2)}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}. \tag{24}$$

Applying Lemma 1.1 for the coefficients p_1,p_2,q_1 and q_2, we get

$$|a_3|\leq\frac{4\alpha^2}{(\lambda+1)^2[1+2\delta(\gamma-\sigma)]^{2m}}+\frac{2\alpha}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}. \tag{25}$$

This gives the required condition. Hence, the theorem follows. □

If we take $\lambda = 1$ in Theorem 2.1, then we have the following corollary.

Corollary 2.1. *Let the function f given by* (1) *be in the class $B_\Sigma(\alpha, \lambda)$, $(0 < \alpha \leq 1)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{6\alpha[1+2\delta(\gamma-\sigma)]^m - 4(\alpha-1)[1+\delta(\gamma-\sigma)]^{2m}}} \tag{26}$$

and

$$|a_3| \leq \frac{4\alpha^2}{4[1+2\delta(\gamma-\sigma)]^{2m}} + \frac{2\alpha}{3[1+2\delta(\gamma-\sigma)]^m}. \tag{27}$$

If we take $\alpha = 1$ in Theorem 2.1, then we have the following corollary.

Corollary 2.2. *Let the function f given by* (1) *be in the class $B_\Sigma(\alpha, \lambda)$, $(\lambda \geq 1)$, then*

$$|a_2| \leq \frac{2}{\sqrt{2(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}} \tag{28}$$

and

$$|a_3| \leq \frac{4}{(\lambda+1)^2[1+2\delta(\gamma-\sigma)]^{2m}} + \frac{2}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}. \tag{29}$$

3. Coefficient bounds for the class $B^m_\Sigma(\beta, \lambda)$

Definition 3.1. *A function f given by* (1) *is said to be in the class $B_\Sigma(\beta, \lambda)$ if the following conditions are satisfied:*

$$\Re e\left((1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}f(z)}{z} + \lambda T^m_{\sigma,\delta,\gamma}f'(z)\right) < \beta, (0 \leq \beta < 1, \quad \lambda \geq 1 \text{ and } \sigma, \delta, \gamma \geq 0, z \in \mathbb{U}) \tag{30}$$

and

$$\Re e\left((1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}g(w)}{w} + \lambda T^m_{\sigma,\delta,\gamma}g'(w)\right) < \beta, (0 \leq \beta < 1, \quad \lambda \geq 1 \text{ and } \sigma, \delta, \gamma \geq 0, w \in \mathbb{U}), \tag{31}$$

where the function g is defined by (5).

Theorem 3.1. *Let $f(z)$ given by* (1) *be in the class $B^m_\Sigma(\beta, \lambda)$, $0 \leq \beta < 1$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}} \tag{32}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}} + \frac{2(1-\beta)}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}. \tag{33}$$

Proof. It follows from (30) and (31) that there exist p and $q \in \mathcal{P}$ such that

$$(1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}f(z)}{z} + \lambda T^m_{\sigma,\delta,\gamma}f'(z) = \beta + (1-\beta)p(z) \tag{34}$$

and

$$(1-\lambda)\frac{T^m_{\sigma,\delta,\gamma}g(w)}{w} + \lambda T^m_{\sigma,\delta,\gamma}g'(w) = \beta + (1-\beta)q(w), \tag{35}$$

where $p(z)$ and $q(w)$ have the forms (12) and (13), respectively. Equating coefficients in (34) and (35) yields

$$(\lambda+1)[1+\delta(\gamma-\sigma)]^m a_2 = (1-\beta)p_1, \tag{36}$$

$$(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m a_3 = (1-\beta)p_2, \tag{37}$$

$$-(\lambda+1)[1+\delta(\gamma-\sigma)]^m a_2 = (1-\beta)q_1 \tag{38}$$

and

$$(2\lambda+1)(2a_2^2-a_3)[1+2\delta(\gamma-\sigma)]^m = (1-\beta)q_2. \tag{39}$$

From (36) and (38), we get

$$p_1 = -q_1 \tag{40}$$

and

$$2(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{41}$$

Also, from (37) and (39), we find that

$$2(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m a_2^2 = (1-\beta)(p_2+q_2). \tag{42}$$

Thus, we have

$$\begin{aligned} |a_2^2| &\le \frac{(1-\beta)}{2(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}(|p_1|+|q_2|) \\ &= \frac{2(1-\beta)}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}, \end{aligned} \tag{43}$$

which is the bound of $|a_2|$ as given in (32).

Next, in order to find the bound of $|a_3|$, by subtracting (39) from (37), we get

$$\begin{aligned} &(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m a_3 - (2\lambda+1)(2a_2^2-a_3)[1+2\delta(\gamma-\sigma)]^m \\ &= (1-\beta)(p_2-q_2) \end{aligned} \tag{44}$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\beta)(p_2 - q_2)}{2(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}. \tag{45}$$

Upon substituting the value of a_2^2 from (41), we obtain

$$a_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}} + \frac{(1-\beta)(p_2 - q_2)}{2(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}. \tag{46}$$

Applying Lemma 1.1 for the coefficients p_1,p_2,q_1 and q_2, we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2[1+\delta(\gamma-\sigma)]^{2m}} + \frac{2(1-\beta)}{(2\lambda+1)[1+2\delta(\gamma-\sigma)]^m}, \tag{47}$$

which is the bound on $|a_3|$ as asserted in (33). □

If we take $\lambda = 1$ in Theorem 3.1, then we get the following corollary.

Corollary 3.1. *Let the function* $f(z)$ *given by* (1) *be in the class* $B_\Sigma(\beta, \lambda)$ $(0 \leq \beta < 1)$. *Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3[1+2\delta(\gamma-\sigma)]^m}}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{[1+\delta(\gamma-\sigma)]^{2m}} + \frac{2(1-\beta)}{3[1+2\delta(\gamma-\sigma)]^m}.$$

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