Oscillation of fractional Emden-Fowler type neutral vector partial differential equations with mixed nonlinearities and deviating arguments

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Abstract. The main purpose of this paper is to extend and improve existing oscillation criteria of mixed fractional order Emden-Fowler type neutral vector partial differential equations with mixed nonlinearities subject to Robin and Dirichlet boundary conditions. Several sufficient conditions are obtained for oscillation of solutions of such class of equations by using the generalized Riccati substitution and integral average method. We support our results by an illustrative example.

Keywords: fractional differential equations, Riccati technique, oscillation, Emden-Fowler equation, neutral.

1. Introduction

The problem of oscillation and nonoscillation of third order originated by Hanan [19] in his monumental paper published in 1961. Since then a number of researches contributed to the subject investigating various classes of differential equations and applying variety of techniques[15-18,27,42-44]. Third order differential equations arise in modelling an entry flow phenomenon in a problem of hydro dynamics or of the propagation of electrical pulses in the nerve of a squid approximated by the famous Nagumous equation; see the papers [6,25] for more details regarding the applications of third order (neutral) differential equations. A systematic survey of the most significant efforts in this theory can be found

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in the excellent monographs of Swanson [45], Greguš and the very recent-one one of Padhi and Pati [38].

Fractional differential equations are now recognized as an excellent source of knowledge in modelling dynamical processes involving self similar and porous structures, probability and statistics, electrical networks, viscoelasticity, electro dynamics of complex medium, electro chemistry of corrosion, polymer rheology, industrial robotics, biotechnology, economics etc. For the theory and applications of fractional differential equations are refer the monographs and journals in the literature [1,4,20,21,33,42] and the reference cited there in.

In the middle of the nineteenth century, the Emden-Fowler equations emerged from theories deals with gaseous dynamics in astrophysics. The Emden-Fowler equations are considered to be one of the most important classical objects in the theory of differential equations. This type of equations has variety of interesting physical applications occuring in astrophysics in the form of Emden equation and in atomic physics in the form of Thomas-Fermi equation.

The Emden-Fowler type of equation has significant applications in many fields of scientific and technical world and this equation has been investigated by many researchers [2,3,5,9-12,24,28-31,40,48,50,52,54] and the reference cited there in.

The Emden-Fowler equations were first considered only for second order equations of the form $(p(t)u')' + q(t)u^{\gamma} = 0, t \geq 0$. By a mixed type Emden-Fowler equation we mean the equation contains a finite sum of powers of x and if there exists in sum exponents of which are both greater than and less than 1. These type of equations arises for instant in the growth of bacteria population with competitive species. As we have known, almost all existing oscillation criteria in the literature, see for the example [32,46,51] are established for Emden-Fowler type equations with mixed nonlinearies of second order.

Partial differential equations are used to model a number of real world problems arising in various branches of science and engineering [13,22,23,47,49,53]. In 1970, Domšlak introduced the concept of H-oscillation to study the oscillation of the solution of vector differential equations, where H is a unit vector in $\mathbb{R}^M$. We refer the reader to [7,8] for vector ordinary differential equations, [34-37] for vector partial differential equations, [26,14,39] for impulsive vector partial differential equations, [41] for fractional vector partial differential equations. There are essentially less results on oscillation of fractional order vector partial differential equations. Motivated by this, we initiate the oscillation criteria for fractional order Emden-Fowler type neutral vector partial differential equations with mixed nonlinearities and deviating arguments of the form

$$\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}\left(r_2(t)\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}\left(r_1(t)\left|D^{\alpha_3}_{+,t}Z(x,t)\right|^{\gamma-1}D^{\alpha_3}_{+,t}Z(x,t)\right)\right)$$
$$+p_1(x,t)|U(x,t-\delta_1)|^{\gamma_1-1}U(x,t-\delta_1)+p_2(x,t)|U(x,t-\delta_2)|^{\gamma_2-1}U(x,t-\delta_2)$$

$$= a(t)\Delta U(x,t) + \sum_{i=1}^{m} a_i(t)\Delta U(x,\rho_i(t))$$
$$(1) \qquad - \sum_{j=1}^{k} b_j(x,t) f_j\left(\int_0^t (t-s)^{-\alpha_3} ||U(x,\sigma_j(s))|| ds\right) U(x,\sigma_j(t)) + F(x,t),$$

$(x,t) \in \Omega \times \mathbb{R}_+ = G$ where $Z(x,t) = U(x,t) + q(t)U(x,t-\tau)$. $\mathbb{R}_+ = (0,\infty)$, where Ω is bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, $0 < \alpha_i \leq 1 (i = 1,2)$ is a constant. $\frac{\partial^{\alpha_i}}{\partial t^{\alpha_i}}$ is the conformable fractional partial derivative of order α_i with respect to t and $D^{\alpha_3}_{+,t}$ is the Riemann-Liouville fractional partial derivative of order α_3 with respect to t. $\alpha_3 \in (0,1)$. Δ is the Laplacian operator in the Euclidean N space $\mathbb{R}^N$, i.e., $\Delta u(x,t) = \sum_{r=1}^{N} \frac{\partial^2 u(x,t)}{\partial x_r{}^2}$. $||U(x,\sigma_j(t))||$ is the usual Euclidean norm in $\mathbb{R}^N$.

Equation (1) is supplemented with the Robin boundary condition with

$$(2) \qquad \frac{\partial U(x,t)}{\partial \mu} + g(x,t)U(x,t) = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}_+,$$

where μ is the unit exterior normal vector to $\partial\Omega$ and $g(x,t)$ is a non-negative continuous function on $\partial\Omega \times \mathbb{R}_+$. And the Dirichlet boundary condition

$$(3) \qquad U(x,t) = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}_+.$$

Throughout this paper, we will suppose that the following conditions hold:
(A_1) τ, δ_1 and δ_2 are positive constants, γ_1, γ_2 and γ are nonnegative constants with $0 < \gamma_1 < \gamma < \gamma_2$;
(A_2) $r_1(t) \in C^{\alpha_1+\alpha_2}(\mathbb{R}_+,\mathbb{R}_+), r_2(t) \in C^{\alpha_1}(\mathbb{R}_+,\mathbb{R}_+)$;
(A_3) $q(t) \in C^{\alpha_1+\alpha_2+\alpha_3}(\mathbb{R}_+,\mathbb{R})$; $0 \leq q(t) \leq 1$;
(A_4) $p_1(x,t), p_2(x,t) \in C(\bar{G},\mathbb{R}_+), p_i(t) = \min_{x\in\bar{\Omega}} p_i(x,t)$,where i=1,2, $p_i(t)$ is not identically zero on any ray from $[t_*,\infty)$ for any $t_* \geq 0$;
(A_5) $a(t), a_i(t) \in C(\mathbb{R}_+,\mathbb{R}_+), i = 1,2,\ldots,m$;
(A_6) $\sigma_j(t), \rho_j(t) \in C(\mathbb{R}_+,\mathbb{R}_+), \lim_{t\to\infty} \sigma_j(t) = \lim_{t\to\infty} \rho_i(t) = \infty, j = 1,2,\ldots,k$, $i = 1,2,\ldots,m$;
(A_7) $b_j(x,t) \in C(\bar{G},\mathbb{R})$ and $\min_{x\in\bar{\Omega}} b_j(x,t), j = 1,2,\ldots,k$;
(A_8) $F(x,t) \in C(\bar{G},\mathbb{R}^n), f_H(x,t) \in C(\bar{G},\mathbb{R})$ such that $\int_\Omega f_H(x,t)dx \leq 0$;
(A_9) $f_j \in C(\mathbb{R}_+,\mathbb{R})$ are convex and nondecreasing in $\mathbb{R}$ with $uf_j(u)$ for $u \neq 0$ and there exist positive constants ν_j such that $\frac{f_j(u)}{u} \geq \nu_j$ for all $u \neq 0, j = 1,2,\ldots,k$.

The following notations will be used for our convenience.

$$(4) \qquad u_H(x,t) =< U(x,t), H >,$$
$$(5) \qquad f_H(x,t) =< F(x,t), H >,$$
$$(6) \qquad V_H(t) = \frac{1}{|\Omega|}\int_\Omega u_H(x,t)dx, \ \text{ where } |\Omega| = \int_\Omega dx,$$

$$z_H(x,t) = u_H(x,t) + q(t)u_H(x,t-\tau), \tag{7}$$

$$Z_H(t) = V_H(t) + q(t)V_H(t-\tau). \tag{8}$$

This paper is organized as follows: In Section 2, we recall the basic definitions of the Riemann-Liouville fractional derivative and conformable fractional derivatives together with basic lemmas concerning the above set of operators. In Section 3, we present some new sufficient conditions for the H-oscillation of the solutions of (1),(2) and (1), (3). In Section 4, an example is provided to illustrate the main results.

2. Preliminaries

In this section, we present some definitions and review some note worthy results from the literature, which we will use throughout the paper.

Definition 2.1. *By a solution of* (1), (2) *or* (1), (3) *we mean a nontrivial function* $U(x,t) \in C^{\alpha_1+\alpha_2+\alpha_3}(\Omega\times[G,\mathbb{R}^M)\cap C^2(\Omega\times[t_1,\infty),\mathbb{R}^M)\cap C(\Omega\times[\hat{t_1},\infty),\mathbb{R}^M)$ *satisfying (1) on the domain G and the boundary condition (2), (3), where*

$$t_1 := \min\left\{0, \min_{1\le i\le m}\left\{\inf_{t\ge 0}\rho_i(t)\right\}\right\},$$
$$\hat{t_1} := \min\left\{0, \min_{1\le j\le m}\left\{\inf_{t\ge 0}\sigma_i(t)\right\}\right\}.$$

Denoted by $\mathbb{P}$ *the set of all solutions* $U(x,t)$ *of* $(1),(2)$ *or* $(1),(3)$*. The set* $\mathbb{P}$ *is said to be proper if there is a* $T_1 > t_0$ *such that*

$$\sup\left\{\sum_{k=1}^{n}|U_k(x,s)|, t\le s<\infty\right\} > 0,$$

for any $t \ge T$*, we make a standing hypothesis is that of (1) has such a solution. A proper solution* $U(x,t) \in \mathbb{P}$ *of* $(1),(2)$ *or* $(1),(3)$ *is said to be oscillatory in G if it is neither eventually positive nor eventually negative. Otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.*

Definition 2.2 ([53])**.** *Let H be a fixed unit vector in* $\mathbb{R}^n$*. A solution* $U(x,t)$ *of* (1) *is said to be H-oscillatory in G if the inner product* $< U(x,t), H >$ *has a zero in* $\Omega \times (t_0,\infty)$ *for* $t_0 > 0$*. Otherwise, it is H-nonoscillatory.*

Definition 2.3 ([33])**.** *The Riemann-Liouville fractional partial derivative of order* $0 < \alpha < 1$ *with respect to t of a function* $u(x,t)$ *is given by*

$$(D^{\alpha}_{+,t}u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\nu)^{-\alpha}u(x,\nu)d\nu \tag{9}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$*, where* Γ *is the gamma function.*

Definition 2.4 ([33]). *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by*

$$(I_+^\alpha y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} y(\nu) d\nu, \quad \text{for} \quad t > 0 \tag{10}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Definition 2.5 ([33]). *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by*

$$(D_+^\alpha y)(t) := \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \left(I_+^{\lceil \alpha \rceil - \alpha} y \right)(t), \quad \text{for} \quad t > 0 \tag{11}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where $\lceil \alpha \rceil$ is the ceiling function of α.

Lemma 2.1. *Let*

$$K(t) := \int_0^t (t-\nu)^{-\alpha} y(\nu) d\nu \quad \text{for} \quad \alpha \in (0,1) \quad \text{and} \quad t > 0. \tag{12}$$

Then, $K'(t) = \Gamma(1-\alpha)(D_+^\alpha y)(t)$.

Definition 2.6 ([4]). *Let f be a function of m variables $x_1, x_2, \ldots, x_m$. Then the conformable partial derivative of f of order α, $0 < \alpha \leq 1$ in x_j is defined as follows*

$$\begin{aligned} &\frac{\partial^\alpha}{\partial x_j^\alpha} f(x_1, x_2, \ldots, x_m) \\ &= \lim_{\epsilon \to 0} \frac{f(x_1, x_2, \ldots, x_{j-1}, x_j + \epsilon x_j^{1-\alpha}, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)}{\epsilon}. \end{aligned}$$

Definition 2.7 ([20]). *Given a function $f : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α, is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

for all $t > 0, \alpha \in (0,1]$. If f is α- differentiable in some $(0,a), a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

We will sometimes write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α.

Some properties of conformable fractional derivative:
Let $\alpha \in (0,1]$ and f and g be α - differentiable at a point $t > 0$. Then:
(P_1) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
(P_2) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
(P_3) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(P_4) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
(P_5) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Definition 2.8 ([44]). *Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)d_\alpha x := \int_a^b f(x)x^{\alpha-1}dx$$

exists and is finite.

Lemma 2.2 ([1]). *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a). \tag{13}$$

3. Main results

In this section, we will give the sufficient conditions for the H-oscillation of all solutions of problem (1), (2). In the sequel, we need the following lemmas which will be used to prove our main theorems.

Lemma 3.1. *Assume that $(A_1)-(A_9)$ hold. Let H be a fixed unit vector in $\mathbb{R}^n$ and $U(x,t)$ be a solution of (1), (2).*

(i) If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies the scalar mixed fractional partial inequality

$$\begin{aligned}
&\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}\left(r_2(t)\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}\left(r_1(t)\left|D_{+,t}^{\alpha_3}z_H(x,t)\right|^{\gamma-1}D_{+,t}^{\alpha_3}z_H(x,t)\right)\right)\\
&+p_1(t)|u_H(x,t-\delta_1)|^{\gamma_1-1}u_H(x,t-\delta_1)+p_2(t)|u_H(x,t-\delta_2)|^{\gamma_2-1}u_H(x,t-\delta_2)\\
&-a(t)\Delta u_H(x,t)-\sum_{i=1}^m a_i(t)\Delta u_H(x,\rho_i(t))\\
&+\sum_{j=1}^k b_j(t)f_j\left(\int_0^t (t-s)^{-\alpha_3}u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t)) \le f_H(x,t).
\end{aligned} \tag{14}$$

(ii) If $u_H(x,t)$ is eventually negative, then $u_H(x,t)$ satisfies the scalar mixed fractional partial inequality

$$\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}\left(r_2(t)\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}\left(r_1(t)\left|D^{\alpha_3}_{+,t}z_H(x,t)\right|^{\gamma-1}D^{\alpha_3}_{+,t}z_H(x,t)\right)\right)$$
$$+p_1(t)|u_H(x,t-\delta_1)|^{\gamma_1-1}u_H(x,t-\delta_1)+p_2(t)|u_H(x,t-\delta_2)|^{\gamma_2-1}u_H(x,t-\delta_2)$$
$$-a(t)\Delta u_H(x,t)-\sum_{i=1}^{m}a_i(t)\Delta u_H(x,\rho_i(t))$$
$$+\sum_{j=1}^{k}b_j(t)f_j\left(\int_0^t(t-s)^{-\alpha_3}u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t))\geq f_H(x,t). \tag{15}$$

Proof. Assume that $u_H(x,t)$ is an eventually positive solution. Taking the inner product (1) and H, we get

$$\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}\left(r_2(t)\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}\left(r_1(t)\left|D^{\alpha_3}_{+,t}<Z(x,t),H>\right|^{\gamma-1}D^{\alpha_3}_{+,t}<Z(x,t),H>\right)\right)$$
$$+p_1(x,t)|<U(x,t-\delta_1),H>|^{\gamma_1-1}<U(x,t-\delta_1),H>$$
$$+p_2(x,t)|<U(x,t-\delta_2),H>|^{\gamma_2-1}<U(x,t-\delta_2),H>$$
$$=a(t)\Delta<U(x,t),H>+\sum_{i=1}^{m}a_i(t)\Delta<U(x,\rho_i(t)),H>$$
$$-\sum_{j=1}^{k}b_j(x,t)f_j\left(\int_0^t(t-s)^{-\alpha_3}||U(x,\sigma_j(s))||ds\right)<U(x,\sigma_j(t)),H>$$
$$+<F(x,t),H>. \tag{16}$$

Using $(4),(5)$ and (7), we have

$$\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}\left(r_2(t)\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}\left(r_1(t)\left|D^{\alpha_3}_{+,t}z_H(x,t)\right|^{\gamma-1}D^{\alpha_3}_{+,t}z_H(x,t)\right)\right)$$
$$+p_1(x,t)|u_H(x,t-\delta_1)|^{\gamma_1-1}u_H(x,t-\delta_1)$$
$$+p_2(x,t)|u_H(x,t-\delta_2)|^{\gamma_2-1}u_H(x,t-\delta_2)$$
$$=a(t)\Delta u_H(x,t)+\sum_{i=1}^{m}a_i(t)\Delta u_H(x,\rho_i(t))$$
$$-\sum_{j=1}^{k}b_j(x,t)f_j\left(\int_0^t(t-s)^{-\alpha_3}||U(x,\sigma_j(s))||ds\right)u_H(x,\sigma_j(t))+f_H(x,t). \tag{17}$$

Since $f_j \in C(\mathbb{R}_+, \mathbb{R}), j = 1, 2, ...k$. We have $u_H(x, \sigma_j(t)) \leq ||U(x, \sigma_j(t))||$ and using (A_7), we get

$$b_j(x,t) f_j \left(\int_0^t (t-s)^{-\alpha_3} ||U(x, \sigma_j(s))|| ds \right) u_H(x, \sigma_j(t))$$
$$\geq b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)). \tag{18}$$

Applying (A_4),

$$p_1(x,t)|u_H(x, t-\delta_1)|^{\gamma_1 - 1} u_H(x, t-\delta_1)$$
$$\geq p_1(t)|u_H(x, t-\delta_1)|^{\gamma_1 - 1} u_H(x, t-\delta_1) \tag{19}$$
$$p_2(x,t)|u_H(x, t-\delta_2)|^{\gamma_2 - 1} u_H(x, t-\delta_2)$$
$$\geq p_2(t)|u_H(x, t-\delta_2)|^{\gamma_2 - 1} u_H(x, t-\delta_2). \tag{20}$$

Applying the above results (19), (20) and (21) in (18), we get the inequality

$$\begin{aligned}
&\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) \left| D^{\alpha_3}_{+,t} z_H(x,t) \right|^{\gamma - 1} D^{\alpha_3}_{+,t} z_H(x,t) \right) \right) \\
&+ p_1(t)|u_H(x, t-\delta_1)|^{\gamma_1 - 1} u_H(x, t-\delta_1) + p_2(t)|u_H(x, t-\delta_2)|^{\gamma_2 - 1} u_H(x, t-\delta_2) \\
&- a(t)\Delta u_H(x,t) - \sum_{i=1}^{m} a_i(t) \Delta u_H(x, \rho_i(t)) \\
&+ \sum_{j=1}^{k} b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)) \leq f_H(x,t).
\end{aligned}$$

Similarly, letting $u_H(x,t)$ be eventually negative, we easily obtain (16). The proof is complete. □

The inner product of (2), (3) with H yield the following boundary conditions

$$\frac{\partial u_H(x,t)}{\partial \mu} + g(x,t) u_H(x,t) = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{21}$$

$$u_H(x,t) = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{22}$$

Lemma 3.2. *Assume that (A_1)-(A_9) hold. Let H be a fixed unit vector in $\mathbb{R}^n$. If the scalar mixed fractional partial inequality (15) has no eventually positive solutions and the scalar mixed fractional partial inequality (16) has no eventually negative solutions satisfying the boundary conditions (22) or (23), then every solution $U(x,t)$ of the problem $(1), (2)$ or $(1), (3)$ is H-oscillatory in G.*

Theorem 3.1. *Assume that (A_1)-(A_9) and*
(A_{10}) $\min_{j \in 1,2,\ldots,k} \sigma_j(t) = \sigma(t) \geq t$;
(A_{11}) $u_H(x,t) \geq L$ *hold.*

If the mixed fractional differential inequality

$$
\begin{aligned}
&T_{\alpha_1}\left(r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_{+}^{\alpha_3}Z_H(t)\right)^{\gamma}\right)\right)\\
&+p_1(t)(V_H(t-\delta_1))^{\gamma_1}+p_2(t)(V_H(t-\delta_2))^{\gamma_2}\\
&+L\sum_{j=1}^{k}b_j(t)f_j(K_H(t))\le 0
\end{aligned}
\tag{23}
$$

has no eventually positive solutions and the mixed fractional inequality

$$
\begin{aligned}
&T_{\alpha_1}\left(r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_{+}^{\alpha_3}Z_H(t)\right)^{\gamma}\right)\right)\\
&+p_1(t)(V_H(t-\delta_1))^{\gamma_1}+p_2(t)(V_H(t-\delta_2))^{\gamma_2}\\
&+L\sum_{j=1}^{k}b_j(t)f_j(K_H(t))\ge 0
\end{aligned}
\tag{24}
$$

has no eventually negative solutions, then the solution $U(x,t)$ of equations (1), (2) *is H-oscillatory in G.*

Proof. Let us suppose to the contrary that there exists a solution $U(x,t)$ of Equations (1), (2) which is not H-oscillatory. Integrating (1) with respect to x over Ω, we have

$$
\begin{aligned}
&\int_\Omega \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}\left(r_2(t)\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}\left(r_1(t)\left|D_{+,t}^{\alpha_3}z_H(x,t)\right|^{\gamma-1}D_{+,t}^{\alpha_3}z_H(x,t)\right)\right)dx\\
&+p_1(t)\int_\Omega |u_H(x,t-\delta_1)|^{\gamma_1-1}u_H(x,t-\delta_1)dx\\
&+p_2(t)\int_\Omega |u_H(x,t-\delta_2)|^{\gamma_2-1}u_H(x,t-\delta_2)dx-a(t)\int_\Omega \Delta u_H(x,t)dx\\
&-\sum_{i=1}^{m}a_i(t)\int_\Omega \Delta u_H(x,\rho_i(t))dx\\
&+\sum_{j=1}^{k}b_j(t)\int_\Omega f_j\left(\int_0^t (t-s)^{-\alpha_3}u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t))dx\\
&\le \int_\Omega f_H(x,t)dx,\quad t\ge t_0.
\end{aligned}
\tag{25}
$$

Using Green's formula and boundary condition (22) yield that

$$
\int_\Omega \Delta u_H(x,t)dx=\int_{\partial\Omega}\frac{\partial u_H(x,t)}{\partial\mu}dS=-\int_{\partial\Omega}g(x,t)u_H(x,t)dS\le 0, t\ge t_0 \tag{26}
$$

and

$$
\begin{aligned}
\int_\Omega \Delta u_H(x,\rho_i(t))dx&=\int_{\partial\Omega}\frac{\partial u_H(x,\rho_i(t))}{\partial\mu}dS\\
&=-\int_{\partial\Omega}g(x,t)u_H(x,\rho_i(t))dS\le 0, t\ge t_0, i=1,2,\dots,m.
\end{aligned}
\tag{27}
$$

Applying Jensen's inequality, it follows that

$$
\begin{aligned}
p_1(t)\int_\Omega |u_H(x,t-\delta_1)|^{\gamma_1-1}u_H(x,t-\delta_1)dx &\geq p_1(t)\int_\Omega (u_H(x,t-\delta_1))^{\gamma_1}\,dx\\
&\geq p_1(t)\left(\int_\Omega u_H(x,t-\delta_1)dx\right)^{\gamma_1}\\
&\geq p_1(t)\left(V_H(t-\delta_1)\right)^{\gamma_1},\ t\geq t_0, \qquad (28)\\
p_2(t)\int_\Omega |u_H(x,t-\delta_2)|^{\gamma_2-1}u_H(x,t-\delta_2)dx &\geq p_2(t)\int_\Omega (u_H(x,t-\delta_2))^{\gamma_2}\,dx\\
&\geq p_2(t)\left(\int_\Omega u_H(x,t-\delta_2)dx\right)^{\gamma_2}\\
&\geq p_2(t)\left(V_H(t-\delta_2)\right)^{\gamma_2},\ t\geq t_0. \qquad (29)
\end{aligned}
$$

Using $(A_{10}),(A_{11})$ and Jensen's inequality, we get

$$
\begin{aligned}
&\int_\Omega f_j\left(\int_0^t (t-s)^{-\alpha_3}u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t))dx\\
&\geq Lf_j\left(\int_\Omega\left(\int_0^t (t-s)^{-\alpha_3}u_H(x,\sigma_j(s))ds\right)dx\right)\\
&\geq Lf_j\left(\int_0^t (t-s)^{-\alpha_3}\left(\int_\Omega u_H(x,\sigma_j(s))dx\right)ds\right)\\
&\geq L\int_\Omega dx f_j\left(\int_0^t (t-s)^{-\alpha_3}\left(\int_\Omega u_H(x,\sigma_j(s))dx\right)\left(\int_\Omega dx\right)^{-1}ds\right)\\
&\geq L\int_\Omega dx f_j\left(\int_0^t (t-s)^{-\alpha_3}\left(V_H(\sigma_j(s))ds\right)\right)\\
&\geq L\int_\Omega dx f_j(K_H(t)),\ t\geq t_0. \qquad (30)
\end{aligned}
$$

Also, by (A_8),

$$
\int_\Omega f_H(x,t)dx\leq 0. \qquad (31)
$$

In view of (6), (8) and (26)-(32) yield

$$
\begin{aligned}
&T_{\alpha_1}\left(r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^{\gamma}\right)\right)+p_1(t)(V_H(t-\delta_1))^{\gamma_1}\\
&+p_2(t)(V_H(t-\delta_2))^{\gamma_2}+L\sum_{j=1}^k b_j(t)f_j(K_H(t))\leq 0. \qquad (32)
\end{aligned}
$$

Therefore, $V_H(t)$ is an eventually positive solution of (24). This contradicts the hypothesis. The case where $u_H(x,t)<0$ in $\Omega\times[t_0,\infty)$ can be treated similarly and we are also getting a contradiction. This completes the proof. □

Lemma 3.3. *Assume that* (A_1)*-*(A_{11}) *hold and* $u_H(x,t) > 0$*. If*

$$\int_0^\infty \frac{1}{r_2(s)} d_{\alpha_2}s = \infty, \tag{33}$$

$$\int_0^\infty \frac{1}{r_1(s)} ds = \infty, \tag{34}$$

$$\int_0^\infty \left(\frac{1}{r_1(\theta)} \int_\theta^\infty \frac{1}{r_2(\xi)} \left(\int_\xi^\infty P(s) d_{\alpha_1}s \right) d_{\alpha_2}\xi \right)^{\frac{1}{\gamma}} d\theta = \infty, \tag{35}$$

where

$$\begin{aligned} P(t) = {} & p_1(t)(1-q(t-\delta_1))^{\gamma_1} M^{\gamma_1} + p_2(t)(1-q(t-\delta_2))^{\gamma_2} M^{\gamma_2} \\ & + L \sum_{j=1}^{k} b_j(t)\nu_j \beta, \end{aligned} \tag{36}$$

then there exists a sufficiently large T *such that* $T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma\right) > 0$ *on* $[T,\infty)$ *and either* $D_+^{\alpha_3} Z_H(t) > 0$ *on* $[T,\infty)$ *or* $\lim_{t\to\infty} Z_H(t) = 0$*.*

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of (1),(2) which is not H-oscillatory in G. Without loss of generality we may assume that $u_H(x,t)$ in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. That is, $V_H(t)$ is an eventually positive solution of (24). There exists a $t_1 \geq t_0$ such that $V_H(t) > 0, V_H(t-\tau) > 0, V_H(t-\delta_1) > 0, V_H(t-\delta_2) > 0$ for $t \geq t_1$ and from (24) we have,

$$\begin{aligned} T_{\alpha_1}\left(r_2(t) T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma\right)\right) \leq & -p_1(t)(V_H(t-\delta_1))^{\gamma_1} - p_2(t)(V_H(t-\delta_2))^{\gamma_2} \\ & - L \sum_{j=1}^{k} b_j(t) f_j(K_H(t)) < 0, \quad t \geq t_1. \end{aligned} \tag{37}$$

Then, $T_{\alpha_1}(r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma))$ is strictly decreasing on $[t_1,\infty)$, and thus $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)$ is eventually of one sign. For $t_2 > t_1$ is sufficiently large on $[t_1,\infty)$, we claim $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) > 0$ on $[t_2,\infty)$. Otherwise, assume that there exists a sufficiently large $t_3 > t_2$ such that $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) < 0$ on $[t_3,\infty)$. Then, $r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma$ is decreasing on $[t_3,\infty)$, and we have

$$\begin{aligned} r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma - r_1(t_3)(D_+^{\alpha_3} Z_H(t_3))^\gamma &= \int_{t_3}^t \frac{r_2(s)}{r_2(s)} T_{\alpha_2}\left(r_1(s)(D_+^{\alpha_3} Z_H(s))^\gamma\right) d_{\alpha_2}s \\ \leq r_2(t_3) T_{\alpha_2}\left(r_1(t_3)(D_+^{\alpha_3} Z_H(t_3))^\gamma\right) \int_{t_3}^t \frac{1}{r_2(s)} d_{\alpha_2}s. \end{aligned}$$

By (34), we have $\lim_{t\to\infty} r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma = -\infty$. So, there exists a sufficiently large t_4 with $t_4 > t_3$ such that $D_+^{\alpha_3} Z_H(t) < 0, t \in [t_4,\infty)$. Further more

$$D_+^{\alpha_3} Z_H(t) = \frac{K_H'(t)}{\Gamma(1-\alpha_3)} < 0,$$

$$\frac{1}{\Gamma(1-\alpha_3)}[K_H(t)-K_H(t_4)] = \frac{1}{\Gamma(1-\alpha_3)}\int_{t_4}^{t} K'_H(s)ds$$

$$\le r_1(t_4)K'_H(t_4)\int_{t_4}^{t}\frac{1}{r_1(s)}ds. \tag{38}$$

By (35), we deduce that $\lim_{t\to\infty} K_H(t) = -\infty$, which contradicts the fact that $K_H(t)$ is an eventually positive solution of (35). So $T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma\right) > 0$ on $[t_2,\infty)$. Thus $D_+^{\alpha_3}Z_H(t)$ is eventually of one sign. Now, we assume that $D_+^{\alpha_3}Z_H(t) < 0, t \in [t_5,\infty)$, for sufficiently large $t_5 > t_4$. Since $K'_H(t) > 0$, further more, we have $\lim_{t\to\infty} K_H(t) = \beta \ge 0$. We claim that $\beta = 0$. Otherwise assume that $\beta > 0$. Then $K_H(t) \ge \beta$ on $[t_5,\infty)$ and for $t \in [t_5,\infty)$ by (24)and (A_9), we have

$$T_{\alpha_1}\left(r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right)\right) + p_1(t)(V_H(t-\delta_1))^{\gamma_1} + p_2(t)(V_H(t-\delta_2))^{\gamma_2}$$

$$+ L\sum_{j=1}^{k} b_j(t)\nu_j\beta \le 0. \tag{39}$$

By (8), we get

$$V_H(t) \ge Z_H(t) - q(t)V_H(t-\tau) \ge Z_H(t)(1-q(t)). \tag{40}$$

Then, for all $t \ge t_5$,

$$V_H(t-\delta_1) \ge (1-q(t-\delta_1))Z_H(t-\delta_1), \tag{41}$$

$$V_H(t-\delta_2) \ge (1-q(t-\delta_2))Z_H(t-\delta_2). \tag{42}$$

Using (41)-(43) in (40), implies that,

$$T_{\alpha_1}\left(r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right)\right) + p_1(t)(1-q(t-\delta_1))^{\gamma_1}Z_H^{\gamma_1}(t-\delta_1)$$

$$+ p_2(t)(1-q(t-\delta_2))^{\gamma_2}Z_H^{\gamma_2}(t-\delta_2) + L\sum_{j=1}^{k} b_j(t)\nu_j\beta \le 0. \tag{43}$$

Since $z(t) \ge M$ on $[t_5,\infty)$, we get

$$T_{\alpha_1}\left(r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right)\right) + p_1(t)(1-q(t-\delta_1))^{\gamma_1}M^{\gamma_1}$$

$$+ p_2(t)(1-q(t-\delta_2))^{\gamma_2}M^{\gamma_2} + L\sum_{j=1}^{k} b_j(t)\nu_j\beta \le 0, \quad t \ge t_5. \tag{44}$$

Using (37), we have

$$T_{\alpha_1}\left(r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right)\right) + P(t) \le 0, \quad t \ge t_5. \tag{45}$$

α_1 - integrating from t to ∞ yields,

$$\int_t^\infty T_{\alpha_1}\left(r_2(s)T_{\alpha_2}\left(r_1(s)\left(D_+^{\alpha_3}Z_H(s)\right)^\gamma\right)\right)d_{\alpha_1}s \le -\int_t^\infty P(s)d_{\alpha_1}s,$$

$$T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right) \ge \frac{1}{r_2(t)}\int_t^\infty P(s)d_{\alpha_1}s$$

α_2 - integrating from t to ∞ yields,

$$\int_t^\infty T_{\alpha_2}\left(r_1(s)\left(D_+^{\alpha_3}Z_H(s)\right)^\gamma\right)d_{\alpha_2}s \ge \int_t^\infty \frac{1}{r_2(\xi)}\int_\xi^\infty P(s)d_{\alpha_1}sd_{\alpha_2}\xi,$$

$$r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma \le -\int_t^\infty \frac{1}{r_2(\xi)}\int_\xi^\infty P(s)d_{\alpha_1}sd_{\alpha_2}\xi,$$

$$D_+^{\alpha_3}Z_H(t) \le \left(-\frac{1}{r_1(t)}\int_t^\infty \frac{1}{r_2(\xi)}\int_\xi^\infty P(s)d_{\alpha_1}sd_{\alpha_2}\xi\right)^{\frac{1}{\gamma}}.$$

Using Lemma 2.1, we have

$$K_H'(t) \le \Gamma(1-\alpha_3)\left(-\frac{1}{r_1(t)}\int_t^\infty \frac{1}{r_2(\xi)}\int_\xi^\infty P(s)d_{\alpha_1}sd_{\alpha_2}\xi\right)^{\frac{1}{\gamma}}.$$

Once again, integrating with respect to s from t_5 to ∞ yields,

$$K_H(t) \le K_H(t_5) - \int_{t_5}^t \left[\frac{1}{r_1(\theta)}\int_\theta^\infty \frac{1}{r_2(\xi)}\int_\xi^\infty P(s)d_{\alpha_1}sd_{\alpha_2}\xi\right]^{\frac{1}{\gamma}}d\theta.$$

Letting $t\to\infty$, from (36) we get $\lim_{t\to\infty}K_H(t) = -\infty$, which causes a contradiction. So the proof is complete. □

Lemma 3.4. *Assume that* (A_1)-(A_{11}) *hold and* $V(t)$ *is an eventually positive solution of Eq.(1) such that* $T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right) > 0, D_+^{\alpha_3}Z_H(t) > 0$ *on* $[t_1,\infty)$, *where* $t_1 \ge t_0$ *is sufficiently large. Then one has*

$$K_H'(t) \ge \left\{\frac{1}{r_1(t)r_2(t)}T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right)R_1(t_1,t)\right\}^{\frac{1}{\gamma}} \tag{46}$$

or

$$K_H(t) \ge \left[T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right)\right]^{\frac{1}{\gamma}}R_2(t_1,t), \tag{47}$$

where $R_1(t_1,t) = \int_{t_1}^t \frac{1}{r_2(s)}d_{\alpha_2}s, R_2(t_1,t) = \int_{t_1}^t \left(\frac{R_1(t_1,s)}{r_1(s)r_2(s)}\right)^{\frac{1}{\gamma}}ds.$

Proof. By (45), we obtain that $r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^\gamma\right)$ is strictly decreasing on $[t_1,\infty)$. So,

$$\int_{t_1}^t T_{\alpha_2}\left(r_1(s)\left(D_+^{\alpha_3}Z_H(s)\right)^\gamma\right)d_{\alpha_2}s = r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma - r_1(t_1)(D_+^{\alpha_3}Z_H(t_1))^\gamma.$$

Multiplying and divided by $r_2(t)$,

$$r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma \geq r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma - r_1(t_1)(D_+^{\alpha_3}Z_H(t_1))^\gamma$$
$$= \int_{t_1}^{t} \frac{r_2(s)}{r_2(s)} T_{\alpha_2}\left(r_1(s)(D_+^{\alpha_3}Z_H(s))^\gamma\right) d_{\alpha_2}s,$$
$$r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma \geq \frac{1}{r_2(t)} T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma\right) \int_{t_1}^{t} \frac{1}{r_2(s)} d_{\alpha_2}s,$$
$$(D_+^{\alpha_3}Z_H(t))^\gamma \geq \frac{1}{r_1(t)r_2(t)} T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma\right) R_1(t_1,t),$$
$$D_+^{\alpha_3}Z_H(t) \geq \left\{\frac{1}{r_1(t)r_2(t)} T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma\right) R_1(t_1,t)\right\}^{\frac{1}{\gamma}}.$$

Integrating with respect to s from t_1 to t we obtain,

$$K'_H(t) \geq \Gamma(1-\alpha_3)\left\{\frac{1}{r_1(t)r_2(t)} T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma\right) R_1(t_1,t)\right\}^{\frac{1}{\gamma}},$$
$$\int_{t_1}^{t} K'_H(s)ds \geq \Gamma(1-\alpha_3)\int_{t_1}^{t}\left\{\frac{1}{r_1(s)r_2(s)} T_{\alpha_2}\left(r_1(s)(D_+^{\alpha_3}Z_H(s))^\gamma\right) R_1(t_1,s)\right\}^{\frac{1}{\gamma}} ds,$$
$$K_H(t) \geq \Gamma(1-\alpha_3)[T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma\right)]^{\frac{1}{\gamma}} \int_{t_1}^{t}\left(\frac{R_1(t_1,s)}{r_1(s)r_2(s)}\right)^{\frac{1}{\gamma}} ds,$$
$$K_H(t) \geq \Gamma(1-\alpha_3)[T_{\alpha_2}\left(r_1(t)(D_+^{\alpha_3}Z_H(t))^\gamma\right)]^{\frac{1}{\gamma}} R_2(t_1,t).$$

This completes the proof. □

In this section, we will obtain Philos - type oscillation criteria for (1),(2) under the case when $0 \leq q(t) \leq 1$. The following notations are used in the sequel.

Denote

$$\lambda = \min\left\{\frac{\gamma_2-\gamma_1}{\gamma_2-\gamma}, \frac{\gamma_2-\gamma_1}{\gamma-\gamma_1}\right\},$$

$$P_1(t) = \lambda\left[\left(\frac{p_1(t)}{r_1(t)}\right)^{\gamma_2-\gamma}\left(\frac{p_2(t)}{r_1(t)}\right)^{\gamma-\gamma_1}(1-q(t-\delta_1))^{\gamma_1(\gamma_2-\gamma)}(1-q(t-\delta_2))^{\gamma_2(\gamma-\gamma_1)}\right]^{\frac{1}{\gamma_2-\gamma_1}}.$$

Let us define the following Philos functions $\mathbb{J}$.

Let $\mathbb{D}_0 = \{(t,s) : t > s \geq t_0\}$ and $\mathbb{D} = \{(t,s) : t \geq s \geq t_0\}$. We say that the functions $B \in C(\mathbb{D},\mathbb{R})$ belongs to the class $\mathbb{J}$, denoted by $B \in \mathbb{J}$, if

(H_1) $B(t,t) = 0$ for $t \geq t_0$. $B(t,s) > 0$ on $(t,s) \in \mathbb{D}_0$;

(H_2) B has a continuous and nonpositive partial derivative on $\mathbb{D}_0$ with respect to the second variable, such that

$$\frac{\partial}{\partial s}B(t,s) = -h(t,s)B(t,s) \quad \text{for} \quad (t,s) \in \mathbb{D}_0,$$

where $h \in C(\mathbb{D}, \mathbb{R})$.

For given functions $h \in C(\mathbb{D}, \mathbb{R}), \phi \in C'([\mathbb{R}_+, \infty), \mathbb{R}_+)$ and $\eta \in C'([\mathbb{R}_+, \infty), \mathbb{R})$ we set

$$\psi(t,s) = h(t,s) - \frac{\phi'(s)}{\phi(s)},$$
$$Q_1(t,s) = P_1(s) - \eta'(s) + \psi(t,s)\eta(s).$$

Theorem 3.2. *Let* $(A_1) - (A_{13})$ *hold. Then* $(1),(2)$ *is H-oscillatory provided that the following condition holds*

$$\limsup_{t\to\infty} \frac{1}{B(t,t_0)} \times \int_{t_0}^{t} B(t,s)\phi(s) \Big\{ Q_1(t,s) - |\Lambda(s)||\psi(t,s)|^{\gamma+1} - \Big|\frac{s^{\alpha_1 - 1} L \sum_{j=1}^{k} b_j(s)\nu_j}{r_1(s) Z_H^{\gamma}(t-\delta_1)}\Big| \Big\} ds = \infty, \tag{48}$$

where

$$\Lambda(t) = \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \frac{(r_2(t))^2}{\gamma R_1(t_1,t) Z_H(t-\delta_1) K_H(t)}. \tag{49}$$

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of (1),(2) which is not H-oscillatory in G. Without loss of generality we may assume that $u_H(x,t)$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. That is, $V_H(t)$ is an eventually positive solution of (24). There exists a $t_1 \geq t_0$ such that $V_H(t) > 0, V_H(t-\tau) > 0, V_H(t-\delta_1) > 0, V_H(t-\delta_2) > 0$ for $t \geq t_1$(The Case $U(x,t) < 0$ can be considered by the same method). Therefore, we get (24). Now, define

$$W(t) = \phi(t) \left(\frac{r_2(t) T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3} Z_H(t)\right)^{\gamma}\right)}{r_1(t) Z_H^{\gamma}(t-\delta_1) K_H(t)} + \eta(t) \right) \quad \text{for} \quad t \geq t_1. \tag{50}$$

α_1-differentiating (51) and (38), we have

$$\begin{aligned}
T_{\alpha_1} W(t) \leq{}& \frac{T_{\alpha_1}\phi(t)}{\phi(t)} W(t) - \phi(t) \left[\frac{p_1(t)(1-q(t-\delta_1))^{\gamma_1} Z_H^{\gamma_1}(t-\delta_1)}{r_1(t) Z_H^{\gamma}(t-\delta_1) K_H(t)} \right] \\
&- \phi(t) \left[\frac{p_2(t)(1-q(t-\delta_2))^{\gamma_2} Z_H^{\gamma_2}(t-\delta_2)}{r_1(t) Z_H^{\gamma}(t-\delta_1) K_H(t)} \right] - \phi(t) \left[\frac{L\sum_{j=1}^{k} b_j(t)\nu_j}{r_1(t) Z_H^{\gamma}(t-\delta_1)} \right] \\
&- \frac{\phi(t) r_2(t) T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3} Z_H(t)\right)^{\gamma}\right) T_{\alpha_1}\left(r_1(t) Z_H^{\gamma}(t-\delta_1) K_H(t)\right)}{\left(r_1(t) Z_H^{\gamma}(t-\delta_1) K_H(t)\right)^2} + \phi(t) T_{\alpha_1}\eta(t).
\end{aligned}$$

Applying (p_5), we get

$$W'(t) \leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t) t^{\alpha_1 - 1} \left[\frac{p_1(t)(1-q(t-\delta_1))^{\gamma_1} Z_H^{\gamma_1}(t-\delta_1)}{r_1(t) Z_H^{\gamma}(t-\delta_1) K_H(t)} \right]$$

$$- \phi(t)t^{\alpha_1-1}\left[\frac{p_2(t)(1-q(t-\delta_2))^{\gamma_2}Z_H^{\gamma_2}(t-\delta_2)}{r_1(t)Z_H^{\gamma}(t-\delta_1)K_H(t)}\right] - \phi(t)t^{\alpha_1-1}\left[\frac{L\sum_{j=1}^{k} b_j(t)\nu_j}{r_1(t)Z_H^{\gamma}(t-\delta_1)}\right]$$
$$- \frac{\phi(t)r_2(t)T_{\alpha_2}\left(r_1(t)\left(D_+^{\alpha_3}Z_H(t)\right)^{\gamma}\right)\left(r_1(t)(Z_H^{\gamma}(t-\delta_1))K_H'(t)\right)}{\left(r_1(t)Z_H^{\gamma}(t-\delta_1)K_H(t)\right)^2}$$
$$+ \phi(t)\eta'(t).$$

Applying Lemma 3.4, we have

$$W'(t) \le \frac{\phi'(t)}{\phi(t)}W(t) - \phi(t)t^{\alpha_1-1}\left[\frac{p_1(t)(1-q(t-\delta_1))^{\gamma_1}Z_H^{\gamma_1}(t-\delta_1)}{r_1(t)Z_H^{\gamma}(t-\delta_1)K_H(t)}\right]$$
$$- \phi(t)t^{\alpha_1-1}\left[\frac{p_2(t)(1-q(t-\delta_2))^{\gamma_2}Z_H^{\gamma_2}(t-\delta_2)}{r_1(t)Z_H^{\gamma}(t-\delta_1)K_H(t)}\right] - \phi(t)t^{\alpha_1-1}\left[\frac{L\sum_{j=1}^{k} b_j(t)\nu_j}{r_1(t)Z_H^{\gamma}(t-\delta_1)}\right]$$
$$- \frac{\begin{array}{c}\phi(t)r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3}Z_H(t))^{\gamma})(r_1(t)(Z_H^{\gamma}(t-\delta_1))\\ \{\frac{1}{r_1(t)r_2(t)}T_{\alpha_2}(r_1(t)(D_+^{\alpha_3}Z_H(t))^{\gamma})R_1(t_1,t)\}^{\frac{1}{\gamma}})\end{array}}{(r_1(t)Z_H^{\gamma}(t-\delta_1)K_H(t))^2}$$
$$+ \phi(t)\eta'(t).$$

Therefore,

$$W'(t) \le \frac{\phi'(t)}{\phi(t)}W(t) - \phi(t)t^{\alpha_1-1}\left[\frac{p_1(t)(1-q(t-\delta_1))^{\gamma_1}Z_H^{\gamma_1}(t-\delta_1)}{r_1(t)Z_H^{\gamma}(t-\delta_1)K_H(t)}\right]$$
$$- \phi(t)t^{\alpha_1-1}\left[\frac{p_2(t)(1-q(t-\delta_2))^{\gamma_2}Z_H^{\gamma_2}(t-\delta_2)}{r_1(t)Z_H^{\gamma}(t-\delta_1)K_H(t)}\right] - \phi(t)t^{\alpha_1-1}\left[\frac{L\sum_{j=1}^{k} b_j(t)\nu_j}{r_1(t)Z_H^{\gamma}(t-\delta_1)}\right]$$
$$- \frac{\phi(t)R_1^{\frac{1}{\gamma}}(t_1,t)Z_H^{\frac{1}{\gamma}}(t-\delta_1)K_H^{\frac{1}{\gamma}}(t)}{r_2^{\frac{2}{\gamma}}(t)}\left(\frac{W(t)}{\phi(t)} - \eta(t)\right)^{1+\frac{1}{\gamma}} + \phi(t)\eta'(t). \tag{51}$$

For simplicity $\delta_1 \ge \delta_1$ (a similar argument holds for $\delta_1 < \delta_1$) then $V_H(t-\delta_1) \le V_H(t-\delta_1)$ and using Young's inequality ($\frac{l}{p} + \frac{m}{q} \ge l^{\frac{1}{p}}m^{\frac{1}{q}}$), we obtain that

$$\frac{\gamma_2-\gamma}{\gamma_2-\gamma_1}\left[\frac{p_1(t)}{r_1(t)}(1-q(t-\delta_1))^{\gamma_1}z^{\gamma_1-\gamma}(t-\delta_1)\right]$$
$$+ \frac{\gamma-\gamma_1}{\gamma_2-\gamma_1}\left[\frac{p_2(t)}{r_1(t)}(1-q(t-\delta_2))^{\gamma_2}z^{\gamma_2-\gamma}(t-\delta_1)\right]$$
$$\ge \left\{\left(\frac{p_1(t)}{r_1(t)}\right)^{\gamma_2-\gamma}\left(\frac{p_2(t)}{r_1(t)}\right)^{\gamma-\gamma_1}(1-q(t-\delta_1))^{\gamma_1(\gamma_2-\gamma)}(1 - q(t-\delta_2))^{\gamma_2(\gamma-\gamma_1)}\right\}^{\frac{1}{\gamma_2-\gamma_1}} = P_1(t). \tag{52}$$

Combining (52) and (53) for $t \geq T_0$, we have

$$W'(t) \leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t) t^{\alpha_1 - 1} \left[\frac{L \sum_{j=1}^{k} b_j(t) \nu_j}{r_1(t) Z_H^{\gamma}(t-\delta_1)} \right]$$

$$- \phi(t) \left[P_1(t) - \eta'(t) \right] - \frac{\phi(t) R_1^{\frac{1}{\gamma}}(t_1,t) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(t)}{r_2^{\frac{2}{\gamma}}(t)} \left| \frac{W(t)}{\phi(t)} - \eta(t) \right|^{1+\frac{1}{\gamma}}. \tag{53}$$

Replacing t in (54) by s, then multiplying (54) by $B(t,s)$ and integrating on $[T,t]$, it follows from (H_2) that for all $t \geq T \geq T_0$,

$$\begin{aligned}
&\int_T^t B(t,s)\phi(s) \left[P_1(s) - \eta'(s) \right] ds \leq - \int_T^t B(t,s) W'(s) ds \\
&+ \int_T^t B(t,s) \frac{\phi'(s)}{\phi(s)} W(s) ds \\
&- \int_T^t \frac{\phi(s) R_1^{\frac{1}{\gamma}}(t_1,s) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} B(t,s) \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \\
&- \int_T^t \phi(s) s^{\alpha_1 - 1} B(t,s) \left[\frac{L \sum_{j=1}^{k} b_j(s) \nu_j}{r_1(s) Z_H^{\gamma}(t-\delta_1)} \right] ds \\
&= B(t,T) W(t) - \int_T^t B(t,s) \psi(t,s) W(s) ds \\
&- \int_T^t \frac{\phi(s) R_1^{\frac{1}{\gamma}}(t_1,s) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} B(t,s) \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \\
&- \int_T^t \phi(s) s^{\alpha_1 - 1} B(t,s) \left[\frac{L \sum_{j=1}^{k} b_j(s) \nu_j}{r_1(s) Z_H^{\gamma}(t-\delta_1)} \right] ds
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\int_T^t B(t,s)\phi(s) Q_1(t,s) ds \leq B(t,T) W(T) \\
&+ \int_T^t B(t,s)\phi(s) |\psi(t,s)| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right| ds \\
&- \int_T^t B(t,s)\phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1,s) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \\
&- \int_T^t \phi(s) s^{\alpha_1 - 1} B(t,s) \left[\frac{L \sum_{j=1}^{k} b_j(s) \nu_j}{r_1(s) Z_H^{\gamma}(t-\delta_1)} \right] ds.
\end{aligned}$$

For given t and s, $t \neq s$, set

$$G(\omega) := |\psi||\omega| - \left| \frac{R_1^{\frac{1}{\gamma}}(t_1,t) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(t)}{r_2^{\frac{2}{\gamma}}(t)} \right| |\omega|^{\frac{\gamma+1}{\gamma}}, \quad \omega > 0.$$

$G(\omega)$ attains its maximum at $|\psi|^{\gamma}\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma}}\left|\frac{r_2^2(t)}{R_1(t_1,t)Z_H(t-\delta_1)K_H(t)}\right|$ and

$$G(\omega) \leq G_{\max} \leq |\Lambda(t)||\psi|^{\gamma+1}. \tag{55}$$

Substituting (56) into (55), we have

$$\begin{aligned}\int_T^t B(t,s)\phi(s)Q_1(t,s)ds &\leq B(t,T)W(T)\\ &+\int_T^t B(t,s)\phi(s)|\Lambda(s)||\psi(t,s)|^{\gamma+1}ds\\ &-\int_T^t \phi(s)s^{\alpha_1-1}B(t,s)\left[\frac{L\sum_{j=1}^k b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}\right]ds\end{aligned} \tag{56}$$

Set $T=T_0$, so

$$\begin{aligned}&\int_{T_0}^t B(t,s)\phi(s)\left[Q_1(t,s)-|\Lambda(s)||\psi(t,s)|^{\gamma+1}-s^{\alpha_1-1}\left[\frac{L\sum_{j=1}^k b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}\right]\right]ds\\ &\leq H(t,T_0)W(T_0).\end{aligned}$$

Thus, by (A_{13}), we obtain

$$\begin{aligned}&\int_{T_0}^t B(t,s)\phi(s)\left[Q_1(t,s)-|\Lambda(s)||\psi(t,s)|^{\gamma+1}-s^{\alpha_1-1}\left[\frac{L\sum_{j=1}^k b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}\right]\right]ds\\ &=\left(\int_{t_0}^{T_0}+\int_{T_0}^t\right)B(t,s)\phi(s)\left[Q_1(t,s)-|\Lambda(s)||\psi(t,s)|^{\gamma+1}\right.\\ &\left.-s^{\alpha_1-1}\left[\frac{L\sum_{j=1}^k b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}\right]\right]ds\\ &\leq B(t,t_0)\left(\int_{t_0}^{T_0}\phi(s)\left[Q_1(t,s)-|\Lambda(s)||\rho(t,s)|^{\gamma+1}\right.\right.\\ &\left.\left.-s^{\alpha_1-1}\left[\frac{L\sum_{j=1}^k b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}\right]\right]ds+|W(T_0)|\right),\end{aligned} \tag{57}$$

we divide (58) through by $B(t,t_0)$ and take lim sup in it as $t\to\infty$. Eq. (49) gives a desired contradiction in (58). This proves the theorem. □

Theorem 3.3. *Let H,ϕ,η be as in Theorem 3.1. Suppose that*

$$0<\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{B(t,s)}{B(t,t_0)}\right\}\leq\infty \tag{58}$$

and

$$\limsup_{t\to\infty}\frac{1}{B(t,t_0)}\int_{t_0}^t B(t,s)\phi(s)|\psi(t,s)|^{\gamma+1}ds<\infty. \tag{59}$$

Then (1), (2) *is H-oscillatory provided the following condition holds. There exists* $\Theta \in C([t_0,\infty),\mathbb{R})$

$$\int^{\infty} \phi(s)\left(\frac{\Theta(s)}{\phi(s)}-\eta(s)\right)_{+}^{\frac{\gamma+1}{\gamma}} ds=\infty, \tag{60}$$

and for any $T \geq t_0$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{B(t,T)}\int_T^t B(t,s)\phi(s)\Big[Q_1(t,s)-|\Lambda(s)||\psi(t,s)|^{\gamma+1}\\ &-\left|s^{\alpha_1-1}\frac{L\sum_{j=1}^k b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}\right|\Big]ds\geq\Theta(t),\end{aligned} \tag{61}$$

where $\Theta_+(s)=\max\{\Theta(s),0\}$.

Proof. Proceeding as in the proof of Theorem 3.2, we have that (55) and (57) hold. Therefore, from (57), for all $t > T \geq T_0$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{B(t,T)}\int_T^t B(t,s)\phi(s)\Big[Q_1(t,s)-|\Lambda(s)||\psi(t,s)|^{\gamma+1}\\ &-|s^{\alpha_1-1}\frac{L\sum_{j=1}^k b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}|\Big]ds\leq W(T)\end{aligned}$$

Also, by (62), we have

$$\Theta(t)\leq W(T),\quad T\geq T_0. \tag{62}$$

Define

$$E_1(t)=\frac{1}{B(t,T_0)}\int_{T_0}^t B(t,s)\phi(s)|\rho(s)|\left|\frac{W(s)}{\phi(s)}-\eta(s)\right|ds$$

and

$$E_2(t)=\frac{1}{B(t,T_0)}\int_{T_0}^t B(t,s)\phi(s)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,s)Z_H^{\frac{1}{\gamma}}(t-\delta_1)K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)}\right|^{\frac{\gamma+1}{\gamma}}\left|\frac{W(s)}{\phi(s)}-\eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds.$$

Then, by (55) and (62), we see that

$$\begin{aligned}&\liminf_{t\to\infty}[E_2(t)-E_1(t)]\\ &\leq W(T_0)-\limsup_{t\to\infty}\frac{1}{B(t,t_0)}\int_{T_0}^t B(t,s)\phi(s)Q_1(t,s)ds\\ &\leq W(T_0)-\Theta(T_0)<\infty.\end{aligned} \tag{63}$$

Now, we claim that

$$\int_{T_0}^{\infty} \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds < \infty. \tag{64}$$

Suppose to the contrary that

$$\int_{T_0}^{\infty} \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds = \infty. \tag{65}$$

By (59), there exists a positive constant l_1 such that

$$\inf_{s \geq t_0} \left\{ \liminf_{t\to\infty} \frac{B(t,s)}{B(t,t_0)} \right\} \geq l_1. \tag{66}$$

Letting l_2 be an arbitrary positive number, then it follows from (66) that there exists a $T_1 \geq T_0$ such that

$$\int_{T_0}^{t} \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \geq \frac{l_2}{l_1}, \quad t \geq T_1. \tag{67}$$

Therefore,

$$\begin{aligned}
E_2(t) &= \frac{1}{B(t,T_0)} \int_{T_0}^{t} B(t,s) d \\
&\cdot \left(\int_{T_0}^{s} \phi(\xi) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, \xi) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(\xi)}{r_2^{\frac{2}{\gamma}}(\xi)} \right| \left| \frac{W(\xi)}{\phi(\xi)} - \eta(\xi) \right|^{\frac{\gamma+1}{\gamma}} d\xi \right) \\
&\geq \frac{1}{B(t,T_0)} \int_{T_1}^{t} \left(\frac{-\partial B(t,s)}{\partial s} \right) \cdot \\
&\cdot \int_{T_0}^{s} \phi(\xi) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, \xi) Z_H^{\frac{1}{\gamma}}(t-\delta_1) K_H^{\frac{1}{\gamma}}(\xi)}{r_2^{\frac{2}{\gamma}}(\xi)} \right| \left| \frac{W(\xi)}{\phi(\xi)} - \eta(\xi) \right|^{\frac{\gamma+1}{\gamma}} d\xi ds \\
&\geq \frac{l_2}{l_1} \frac{1}{B(t,T_0)} \int_{T_1}^{t} \left(\frac{-\partial B(t,s)}{\partial s} \right) ds \\
&\geq \frac{l_2}{l_1} \frac{1}{B(t,T_0)} B(t,T_1).
\end{aligned}$$

By (67), there exists a $T_2 \geq T_1$ such that $\frac{B(t,T_1)}{B(t,T_0)} \geq l_1$ for all $t \geq T_2$, which implies that $E_2(t) \geq l_2$. Since l_2 is arbitrary, then

$$\lim_{t\to\infty} E_2(t) = \infty. \tag{68}$$

Next, in view of (64), we may consider a sequence $\{T_n\}_{n=1}^{\infty}$ in $[t_0, \infty)$ satisfying

$$\lim_{n\to\infty}[E_2(T_n) - E_1(T_n)] = \liminf_{t\to\infty}[E_2(t) - E_1(t)] < \infty.$$

Then, there exists a constants M such that

$$E_2(T_n) - E_1(T_n) \leq M \tag{69}$$

for all sufficiently large $n \in \mathbb{N}$. Since (69) ensure that

$$\lim_{n\to\infty} E_2(T_n) = \infty \tag{70}$$

and we have (70) implies that

$$\lim_{n\to\infty} E_1(T_n) = \infty. \tag{71}$$

Further, (70) and (72) yield the inequalities

$$\frac{E_1(T_n)}{E_2(T_n)} - 1 \geq -\frac{M}{E_2(T_n)} > -\frac{1}{2} \quad or \quad \frac{E_1(T_n)}{E_2(T_n)} \geq \frac{1}{2}$$

hold for all sufficiently large $n \in \mathbb{N}$. In view of this and (72) we have

$$\lim_{n\to\infty} \frac{E_1^{\gamma+1}(T_n)}{E_2^{\gamma}(T_n)} = \infty \tag{72}$$

On the other hand, from the definition of E_1 we can obtain by Hölder's inequality

$$E_1(T_n) \leq \left(\frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s)\phi(s) \right.$$
$$\left. \left|\frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)}\right| \left|\frac{W(s)}{\phi(s)} - \eta(s)\right|^{\frac{\gamma+1}{\gamma}} ds\right)^{\frac{\gamma}{\gamma+1}}$$
$$\times \left(\frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s)\phi(s)|\psi(t, s)|^{\gamma+1} ds\right)^{\frac{1}{\gamma+1}}$$

and accordingly

$$\frac{E_1^{\gamma+1}(T_n)}{E_2^{\gamma}(T_n)} \leq \frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s)\phi(s)|\psi(t, s)|^{\gamma+1} ds.$$

So, because of (73), we have

$$\lim_{n\to\infty} \frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s)\phi(s)|\psi(t, s)|^{\gamma+1} ds = \infty$$

which gives that

$$\limsup_{n\to\infty} \frac{1}{B(t,T_0)} \int_{T_0}^{t} B(t,s)\phi(s)|\psi(t,s)|^{\gamma+1} ds = \infty.$$

Contradicting (60). Therefore (65) holds. Now, in view of (63) and (65) we obtain

$$\int_{T_0}^{\infty} \phi(s) \left(\frac{\Theta(s)}{\phi(s)} - \eta(s)\right)_{+}^{\frac{\gamma+1}{\gamma}} ds \leq \int_{T_0}^{\infty} \phi(s) \left|\frac{W(s)}{\phi(s)} - \eta(s)\right|^{\frac{\gamma+1}{\gamma}} ds < \infty,$$

which contradicts (61). This completes the proof. □

Theorem 3.4. *Let H, ϕ and η be as in Theorem 3.2, suppose that (59) holds and*

$$\liminf_{t\to\infty} \frac{1}{B(t,t_0)} \int_{t_0}^{t} B(t,s)\phi(s)|\psi(t,s)|^{\gamma+1} ds < \infty. \tag{73}$$

then (1) is H-oscillatory provided the following condition holds.

There exists $\Theta \in C([t_0,\infty),\mathbb{R})$ such that (61) holds, and for any $T \geq t_0$,

$$\liminf_{t\to\infty} \frac{1}{B(t,T)} \int_{T}^{t} B(t,s)\phi(s) \left[Q_1(t,s) - |\Lambda(s)||\psi(t,s)|^{\gamma+1}\right] ds \geq \Theta(t). \tag{74}$$

Theorem 3.5. *Let H, ϕ and η be as in Theorem 3.1, suppose that (59) holds. Then (1) is H-oscillatory provided the following condition holds*

$$\liminf_{t\to\infty} \frac{1}{B(t,T)} \int_{T}^{t} B(t,s)\phi(s)Q_1(t,s) ds < \infty. \tag{75}$$

Further, suppose that there exists $\phi \in C([t_0,\infty),\mathbb{R})$ such that (61), (75) hold.

Corollary 3.1. *Let H, ϕ and η be as in Theorem 3.2. Then Eq. (1), (2) is H-oscillatory provided that the following condition holds:*

$$\limsup_{t\to\infty} \frac{1}{B(t,t_0)} \int_{T}^{t} B(t,s)\phi(s)Q_1(t,s) ds = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{B(t,t_0)} \int_{T}^{t} B(t,s) \left[|\Lambda(s)||\psi(t,s)| - |s^{\alpha_1 - 1} \frac{L\sum_{j=1}^{k} b_j(s)\nu_j}{r_1(s)Z_H^{\gamma}(t-\delta_1)}| \right] ds \leq \infty.$$

Corollary 3.2. *Eq. (1), (2) is H- oscillatory provided that the following condition holds:*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n Q_1(t,s) ds = \infty,$$

for some $n > \gamma$.

In this section we establish sufficient condition for the oscillation of all solutions of equations (1), (3). For this we need the following.

The smallest eigen value β_0 of the Dirichlet problem

$$\begin{aligned} \Delta\omega(x) + \beta\omega(x) = 0 \quad \text{in} \quad \Omega \\ \omega(x) = 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

is positive and the corresponding eigen function $\varphi(x)$ is positive in Ω.

Theorem 3.6. *Let all conditions of Theorem* 3.2 *hold. Assume that* $|\varphi(x)| \le M$ *for* $x \in \bar{\Omega}$. *Then every solution of equations* (1), (3) *H-oscillatory in* G.

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of (1),(3) which is not H-oscillatory in G. Without loss of generality we may assume that $u_H(x,t)$ in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. That is, $V_H(t)$ is an eventually positive solution of (24). There exists a $t_1 \ge t_0$ such that $V_H(t) > 0, V_H(t-\tau) > 0, V_H(t-\delta_1) > 0, V_H(t-\delta_2) > 0$ for $t \ge t_1$. Multiplying both sides of (15) by $\varphi > 0$ and integrating with respect to x over Ω, we obtain for $t \ge t_1$,

$$\begin{aligned} &\int_\Omega \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) \left| D^{\alpha_3}_{+,t} z_H(x,t) \right|^{\gamma-1} D^{\alpha_3}_{+,t} z_H(x,t) \right) \right) \varphi(x) dx \\ &+ \int_\Omega p_1(t) |u_H(x,t-\delta_1)|^{\gamma_1 - 1} u_H(x,t-\delta_1) \varphi(x) dx \\ &+ \int_\Omega p_2(t) |u_H(x,t-\delta_2)|^{\gamma_2 - 1} u_H(x,t-\delta_2) \varphi(x) dx = \int_\Omega a(t) \Delta u_H(x,t) \varphi(x) dx \\ &- \int_\Omega \sum_{i=1}^m a_i(t) \Delta u_H(x,\rho_i(t)) \varphi(x) dx \\ &+ \int_\Omega \sum_{j=1}^k b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x,\sigma_j(s)) \varphi(x) ds \right) u_H(x,\sigma_j(t)) \varphi(x) dx \\ &\le \int_\Omega f_H(x,t) \varphi(x) dx. \end{aligned} \tag{76}$$

Using Green's formula and boundary condition (23), it follows that

$$\begin{aligned} \int_\Omega \Delta u_H(x,t) \varphi(x) dx &= \int_\Omega u_H(x,t) \Delta\varphi(x) dx \\ &= -\beta_0 \int_\Omega u_H(x,t) \varphi(x) dx \le 0, \quad t \ge t_1. \end{aligned} \tag{77}$$

Also, from Jensen's inequality, it follows that

$$\begin{aligned} &\int_\Omega p_1(t) |u_H(x,t-\delta_1)|^{\gamma_1 - 1} u_H(x,t-\delta_1) \varphi(x) dx \\ &\ge p_1(t) \int_\Omega \left(u_H(x,t-\delta_1) \right)^{\gamma_1} \varphi(x) dx \\ &\ge p_1(t) \left(\int_\Omega u_H(x,t-\delta_1)^{\gamma_1} (\varphi(x))^{1-\gamma_1+\gamma_1} dx \right) \end{aligned}$$

$$\geq M^{1-\gamma_1} p_1(t) \left(V_H(t-\delta_1)\right)^{\gamma_1},\ t \geq T, \tag{78}$$

$$\int_\Omega p_2(t)|u_H(x,t-\delta_2)|^{\gamma_2-1} u_H(x,t-\delta_2)\varphi(x)dx$$
$$\geq M^{1-\gamma_2} p_2(t) \left(V_H(t-\delta_2)\right)^{\gamma_2},\ t \geq T. \tag{79}$$

Set

$$V_H(t) = \int_\Omega u_H(x,t)\varphi(x)dx, \quad t \geq t_1. \tag{80}$$

In view of (78)-(81) and (A_8), (77) yield

$$\left(r_2(t)\left(r_1(t)\left(z'(t)\right)^{\gamma}\right)'\right)'$$
$$+ M^{1-\gamma_1} p_1(t)(V(t-\delta_1))^{\gamma_1} + M^{1-\gamma_2} p_2(t)(V(t-\delta_2))^{\gamma_2} \leq 0,\ t \geq T. \tag{81}$$

Rest of the proof is similar to that of Theorem 2.2 and hence the details are omitted. □

Theorem 3.7. *Let the conditions of Theorem* 3.2 *hold, then every solution* $U(x,t)$ *of* (1), (3) *is H-oscillatory in G.*

Theorem 3.8. *Let the conditions of Theorem* 3.3 *hold, then every solution* $U(x,t)$ *of* (1), (3) *is H-oscillatory in G.*

Theorem 3.9. *Let the conditions of Theorem* 3.4 *hold, then every solution* $U(x,t)$ *of* (1), (3) *is H-oscillatory in G.*

Theorem 3.10. *Let the conditions of Theorem* 3.5 *hold, then every solution* $U(x,t)$ *of* (1), (3) *is H-oscillatory in G.*

4. Example

In this section, we give an example to illustrate our main results.

Example 4.1.

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left(\frac{\partial^{\frac{1}{4}}}{\partial t^{\frac{1}{4}}}\left(t^{\frac{2}{3}} D_{+,t}^{\frac{1}{3}} Z(x,t)\right)\right) + \frac{\sqrt{3}+1}{2}\frac{17}{12} t^{\frac{11}{12}} |U(x,t-\pi)|^{\frac{2}{3}-1} U(x,t-\pi)$$
$$+ \frac{\sqrt{3}+1}{2} t^{\frac{23}{12}} |U(x,t-\frac{\pi}{2})|^{\frac{5}{3}-1} U(x,t-\frac{\pi}{2}) + \frac{\sqrt{3}-1}{2} t^{\frac{23}{12}} \Delta U(x,t)$$
$$+ \frac{\sqrt{3}-1}{3}\frac{5}{12} t^{\frac{-1}{12}} \Delta U(x,t-\pi)$$
$$- \frac{2}{9} t^{\frac{1}{4}} \left(\int_0^t (t-s)^{-\frac{1}{3}} ||U(x,t-\frac{\pi}{2})||ds\right) U(x,t-\frac{\pi}{2})$$
$$+ F(x,t),\ (x,t) \in \Omega \times \mathbb{R}_+, \tag{82}$$

$(x,t)\in G$ where $G=(0,\pi)\times(0,\pi)\times(0,\infty)$, with the boundary condition

$$U(0,t)=\begin{pmatrix} u_1(0,t)\\ u_2(0,t)\end{pmatrix}=U(\pi,t)=\begin{pmatrix} u_1(\pi,t)\\ u_2(\pi,t)\end{pmatrix}=\begin{pmatrix} 0\\ 0\end{pmatrix},$$

$t\geq 0$. Here $\alpha_1=\frac{1}{2},\alpha_2=\frac{1}{4},\alpha_3=\frac{1}{3},r_1(t)=t^{\frac{2}{3}},r_2(t)=1,\gamma_1=\frac{2}{3},\gamma_2=1,q(t)=1,\tau=\frac{3\pi}{2},f_1(u)=u,a(t)=\frac{\sqrt{3}-1}{2}t^{\frac{23}{12}},a_1(t)=\frac{\sqrt{3}-1}{3}\frac{5}{12}t^{\frac{-1}{12}},b(t)=\frac{2}{9}t^{\frac{1}{4}},\delta_1,\rho_1=\pi,\delta_2,\sigma_1=t-\frac{\pi}{2},p_1(t)=\min_{x\in\bar{\Omega}}p_1(x,t)=\frac{\sqrt{3}+1}{2}\frac{17}{12}t^{\frac{11}{12}},p_2(t)=\min_{x\in\bar{\Omega}}p_2(x,t)=\frac{\sqrt{3}+1}{2}t^{\frac{23}{12}}$,

$$H=e_1=\begin{pmatrix} 0\\ 0\end{pmatrix}$$

we observe that

$$f_{e_1}(x,t)=\left[\frac{\sqrt{3}}{3}t^{\frac{11}{12}}+\frac{\sqrt{3}-1}{2}\frac{17}{12}t^{\frac{11}{12}}+\frac{\sqrt{3}+1}{3}\frac{5}{12}t^{\frac{-1}{12}}\right]\sin x\cos t-\frac{\sqrt{3}+1}{3}t^{\frac{11}{12}}\sin x\sin t,$$

$$\int_\Omega f_{e_1}(x,t)dx\leq\left[\frac{2\sqrt{3}}{3}t^{\frac{11}{12}}+\sqrt{3}-1\frac{17}{12}t^{\frac{11}{12}}+\frac{\sqrt{3}+1}{3}\frac{5}{6}t^{\frac{-1}{12}}\right]\cos t\leq 0,\ \frac{\pi}{2}\leq t\leq\frac{3\pi}{2},$$

$$F(x,t)=\begin{pmatrix}\left[\frac{\sqrt{3}}{3}t^{\frac{11}{12}}+\frac{\sqrt{3}-1}{2}\frac{17}{12}t^{\frac{11}{12}}+\frac{\sqrt{3}+1}{3}\frac{5}{12}t^{\frac{-1}{12}}\right]\sin x\cos t-\frac{\sqrt{3}+1}{3}t^{\frac{11}{12}}\sin x\sin t\\ \frac{\Gamma\frac{1}{3}}{12\pi\sqrt{3}}t^{\frac{-5}{12}}+3t^{\frac{2}{3}}\end{pmatrix}.$$

Take $n=2,h(t,s)=(t-s),\phi(t)=1,\phi'(t)=0,\eta(t)=1$. It is clear that $(A_1)-(A_{11})hold$. Thus all the conditions of Corollary 3.2 are satisfied. Therefore, every solution $U(x,t)$ of (83) with the boundary condition is e_1- oscillatory in G. Infact,

$$U(x,t)=\begin{pmatrix}\sin x\sin t\\ 2\end{pmatrix}$$

is one such solution of the problem (82) with the boundary condition. We note that the above solution $U(x,t)$ is not e_2- oscillatory in G, where

$$e_2=\begin{pmatrix} 0\\ 1\end{pmatrix}.$$

References

[1] T. Abdeljawad, *On conformable fractional calculus*, Journal of Computational and Applied Mathematics, 279 (2015), 57-66.

[2] R. P. Agarwal, M. Bohner, W. T. Li, *Nonoscillation and oscillation: theory for functional differential equations*, vol. 267 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, NewYork, NY, USA, 2004.

[3] R. P. Agarwal, M. Bohner, T. Li, C. Zhang, *Oscillation of second-order Emden-Fowler neutral delay differential equations*, Annali di Matematica, 193 (2014), 1861–1875.

[4] A. Atangana, D. Baleanu, A. Alsaedi, *New properties of conformable derivative*, Open Math., (2015), 889-898.

[5] L. M. Berkovich, *The generalized Emden-Fowler equation*, Sym. Nonlinear Math. Phys, 1 (1997), 155-163.

[6] G. E. Chatzarakis, S. R. Grace, I. Jadlovská, T. Li, E. Tunç, *Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients*, Complexity, 2019 (2019), Article ID 5691758, 1-7.

[7] J.I. Domšlak, *On the oscillation of solutions of vector differential equations*, Soviet Math. Dokl., 11 (1970), 839-841.

[8] J.I. Domšlak, *The oscillatoriness and non-oscillatoriness of the solutions of vector differential equations*, Differ. Uravan, 7 (1971), 961-969.

[9] A. Domoshnitsky, R. Koplatadze, *On asymptotic behavior of solutions of generalized Emden-Fowler differential equations with delay argument*, Abstract and Applied Analysis, (2014), 1-13.

[10] Z. Došla, M. Marini, *On super-linear Emden-Fowler type differential equations*, Journal of Mathematical Analysis and Applications, 416 (2014), 497-510.

[11] J. Džurina, S. R. Grace, I. Jadlovská, T. Li, *Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term*, Math. Nachr., 293 (2020), 910-922.

[12] R. H. Fowler, *Further studies of Emden's and similar differential equations*, Quarterly Journal of Mathematics, 2 (1931), 259-288.

[13] S. Frassu, G. Viglialoro, *Boundedness for a fully parabolic Keller-Segel model with sublinear segregation and superlinear aggregation*, Acta Appl. Math., 171 (2021), Art. 19, 1-20.

[14] G. E. Chatzarakis, V. Sadhasivam, T. Raja, *On the oscillation of impulsive vector partial differential equations with distributed deviating arguments*, Analysis, 38 (2018) 101-114.

[15] G. E. Chatzarakis, M.Deepa, N. Nagajothi, V. Sadhasivam, *Oscillatory properties of a certain class of mixed fractional differential equations*, Appl. Math. Inf. Sci., 14 (2020).

[16] M. Gregus, *Third order linear differential equations*, Springer, 1987.

[17] I. Györi, G. Ladas, *Oscillation theory of delay differential equations*, Clarendon Press, Oxford, 1991.

[18] J. K. Hale, *Theory of functional differential equations*, Springer-Verlag, NewYork, NY, USA, 1953.

[19] M. Hanan, *Oscillation criteria for third-order linear differential equations*, Pacific Journal of Mathematics, 11 (1961) 919-944.

[20] R. Khalil, M. AlHorani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264 (2014), 65-70.

[21] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B. V., Amsterdam, The Netherlands, 204, 2006.

[22] T. Li, N. Pintus, G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys., 70 (2019), Art. 86, 1-18.

[23] T. Li, G. Viglialoro, *Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime*, Differential Integral Equations, 34 (2021), 315-336.

[24] T. Li, Z. Han, C. Zhang, S. Sun, *On the oscillation of second-order Emden-Fowler neutral differential equations*, J. Appl. Math. Compute, 37 (2011), 601-610.

[25] T. Li, Yu. V. Rogovchenko, *On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations*, Appl. Math. Lett., 105 (2020), Art. 106293, 1-7.

[26] W. N. Li, M. Han,*Oscillation of solutions for certain impulsive vector parabolic differential equations with delays*, J. Math. Anal. Appl., 326 (2007), 363-371.

[27] W. N. Li, M. Han, F. W. Meng, *H-Oscillation of solutions of certain vector hyperbolic differential equations with deviating arguments*, Appl. Math. Comput., 158 (2004), 637-653.

[28] T. Li, Yu. V. Rogovchenko, *Oscillation of second-order neutral differential equations*, Math. Nachr., 288 (2015), 1150-1162.

[29] T. Li, Yu. V. Rogovchenko, *Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations*, Monatsh. Math., 184 (2017), 489-500.

[30] T. Li, Yu. V. Rogovchenko, *On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations*, Appl. Math. Lett., 67 (2017), 53-59.

[31] H. Liu, F. Meng, P. Liu, *Oscillation and asymptotic analysis on a new generalized Emden-Fowler equation*, Applied Mathematics and Computation, 219 (2012), 2739-2748.

[32] X. Liu, Z. Xu, *Oscillation criteria for a forced mixed type Emden-Fowler equation with impulses*, Applied Mathematics and Computation, 215 (2009), 283-291.

[33] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, New York, 1993.

[34] E. Minchev, N. Yoshida, *Oscillations of solutions of vector differential equations of parabolic type with functional arguments*, J. Comput. Math. Appl., 151 (2003), 107-117.

[35] E. Minchev, N. Yoshida,*Oscillations of vector differential equations for hyperbolic type with functional arguments*, Math. J. Toyama Univ., 26 (2003), 75-84.

[36] E. S. Noussair, C. A. Swanson, *Oscillation theorems for vector differential equations*, Utilitas Math., 1 (1972), 97-109.

[37] E. S. Noussair, C. A. Swanson, *Oscillation of nonlinear vector differential equations*, Ann. Mat. Pura. Appl., 109 (1976), 305-315.

[38] S. Padhi, S. Pati, *Theory of third order differential equations,* Springer, New Delhi, 2014.

[39] P. Prakash, S. Harikrishnan, *Oscillation of solutions of impulsive vector hyperbolic differential equations with delays*, Appl. Anal., 91 (2012), 459-473.

[40] V. Sadhasivam, N. Nagajothi, M.Deepa, *Oscillation of third-order neutral delay mixed type Emden-Fowler differential equations*, International Journal of Research and Analytical Reviews, 3 (2018), 25-31.

[41] V. Sadhasivam, J. Kavitha, *On the oscillation of solutions of fractional vector partial differential equations with deviating arguments*, Amer. Rev. Math. Stat., 5 (2017), 45-57.

[42] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives, Theory and applications*, Gordan and Breach Science Publishers, Singapore, 1993.

[43] M. Sathish Kumar, V. Ganesan, *On the oscillatory behavior of solutions of third order nonlinear neutral differential equations*, Malaya Journal of Mathematic, (2019), 596-599.

[44] M. Z. Sarikaya, F. Usta, *On comparison theorems for conformable fractional differential equations*, International Journal of Analysis and Applications, 12 (2016), 207-214.

[45] Ch. A. Swanson, *Comparison and oscillation theory of linear differential equations*, Elsevier, 1968.

[46] Y. G. Sun, F. W. Meng, *Oscillation of second-order delay differential equations with mixed nonlinearities*, Applied Mathematics and Computation, 207 (2009), 135-139.

[47] E. Thandapani, R. Savithri, *On the oscillation of neutral partial functional differential equation*, Bulletin of the institute of mathematics, 31 (2003), 273-292.

[48] J. S. W. Wong, *On the generalized Emden-Fowler equation*, SIAM Rev., 17 (1975), 339-360.

[49] J. Wu, *Theory and applications of partial differential equations*, Springer, Newyork, 1996.

[50] Y. Wu, Y. Yu, J. Zhang, J. Xiao, *Oscillation criteria for second order Emden-Fowler functional differential equations of neutral type*, Journal of Inequalities and Applications, 328 (2016), 1-11.

[51] Z. Xu, X. Liu, *Philos-type oscillation criteria for Emden-Fowler neutral delay differential equations*, Journal of Computational and Applied Mathematics, 206 (2007), 1116-1126.

[52] R. Xu, F. Meng, *Some new oscillation for second order quasi-linear neutral delay differential equations*, Applied Mathematics and Computation, 182 (2006), 797-803.

[53] N. Yosida, *Oscillation theory of partial differential equations,* World Scientific, Singapore, 2008.

[54] C. Zhang, R. P. Agarwal, M. Bohner, T. Li, *New results for oscillatory behavior of even-order half-linear delay differential equations*, Appl. Math. Lett., 26 (2013), 179-183.

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