Fuzzy stability of sextic functional equation in normed spaces (direct method)

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Abstract. Instituted stability result with regard to the sextic functional equation

$$f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)=(n^4+n^2)[f(x+y)+f(x-y)] \\ +2(n^6-n^4-n^2+1)[f(x)+f(y)]$$

in Fuzzy normed space.
Keywords: generalized stability, sextic functional equation (S.Fun.E), fuzzy normed spaces (F.N. spaces), fuzzy Banach spaces (F.B. spaces).

1. Introduction and preliminaries

The theory of Fuzzy space has much advance . We used the concept of F.N space mentioned in the references [1, 5] to test a fuzzy version of generalized stability of the functional equation (this equation is from the reference [3, 4])

$$\begin{aligned} & f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny) \\ & =(n^4+n^2)[f(x+y)+f(x-y)] \\ & +2(n^6-n^4-n^2+1)[f(x)+f(y)] \end{aligned} \tag{1}$$

in the F.N vector space setting.

Definition 1.1 ([2]). *Let be a real linear space. A function* $N: X\times\mathbb{R}\longrightarrow[0,1]$ (*the so -called fuzzy subset*) *is said to be a fuzzy norm on* X *if for all* $x,y\in X$ *and all* $s,t\in\mathbb{R}$*;*

($N1$) $N(x,c)=0$, *for* $c\leq 0$*;*

($N2$) $x=0$ *if and only if* $N(x,t)=1$, *for all* $c>0$*;*

($N3$) $N(cx,t)=N(x,\frac{t}{|c|})$, *if* $c\neq 0$*;*

($N4$) $N(x+y,s+t)\geq\min\{(x,s),(y,t)\}$*;*

($N5$) $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty}N(x,t)=1$*;*

(N6) for $x = 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

The pair $N(X, N)$ is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement the norm of x is less than or equal to the real numbert.

Example 1.1 ([2]). Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,t) := \begin{cases} \dfrac{t}{t+\|.\|}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

is a Fuzzy norm on X.

Definition 1.2 ([5]). *Let (X, N) be a fuzzy normed vector space. A sequence x_n in X is said to be convergent or converge if there exists an x in X such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In this case x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 1.3 ([1]). *Let (X, N) be a fuzzy normed vector space. A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

2. Fuzzy stability

Theorem 2.1. *Let X be linear space and let $(Z, \aleph')$ be F.N space, let $\psi : X \times X \to Z$ a function such that for some $0 < x < n^6$*

$$\aleph'(\psi(nx, 0), t) \geq \aleph'(\alpha\psi(x, 0), t) \tag{2}$$

and

$$\lim_{\imath\to\infty} \aleph'(\psi(n^\imath x, n^\imath y), n^{6\imath} t) = 1, \tag{3}$$

$\forall x, y \in X$ and $t \geq 0$. Let $(Y, \aleph)$ be a fuzzy Banach space and $f : X \longrightarrow Y$ be a ψ-approximately sextic mapping in the sense that

$$\aleph(D_s f(x, y), t) \geq \aleph'(\psi(x, 0), t) \tag{4}$$

$\forall x, y \in X$, $t > 0$ where $D_s f(x, y) = f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny) - (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)]$. Then exists sextic mapping $A : X \longrightarrow Y$ s.t

$$\aleph(A(x) - f(x), t) \geq \aleph'(\psi(x, 0), 2t(n^6 - \alpha)) \tag{5}$$

$\forall x \in X$, $t > 0$.

Proof. Impose $y = 0$ in (4)

$$\aleph(2f(nx) - 2n^6 f(x), t) \geq \aleph'(\psi(x,0), t), \tag{6}$$

$\forall x \in X$, $t > 0$. Replacement x by $n^k x$ in (6) and employ (2) we obtain

$$\aleph(2f(n^{k+1}x) - 2n^6 f(n^k x), t) \geq \aleph'(\psi(n^k x, 0), t) \geq \aleph'(\alpha^k \psi(x,0), t) = \aleph'(\psi(x,0), \frac{t}{\alpha^k}),$$

$k \geq 0$. Replacement t by $\alpha^k t$ we see that

$$\aleph\left(\frac{f(n^{k+1}x)}{n^{6(k+1)}} - \frac{f(n^k x)}{n^{6k}}, \frac{\alpha^k t}{2n^{6(k+1)}}\right) \geq \aleph'(\psi(x,0), t), \tag{7}$$

$\forall x \in X$, $t > 0$, $k \geq 0$. From

$$\frac{f(n^j x)}{n^{6j}} - f(x) = \sum_{\imath=0}^{j=1} \left(\frac{f(n^{\imath+1}x)}{n^{6(\imath+1)}} - \frac{f(n^\imath x)}{n^{6\imath}}\right)$$

and (7) perform

$$\begin{aligned} &\aleph\left(\frac{f(n^j x)}{n^{6j}} - f(x), \sum_{\imath=0}^{j-1} \frac{\alpha^\imath t}{2n^{6(\imath+1)}}\right) \\ &\geq \min \bigcup_{\imath=0}^{j-1} \left(\aleph\left(\frac{f(n^{\imath+1}x)}{n^{6(\imath+1)}} - \frac{f(n^\imath x)}{n^{6\imath}}, \frac{\alpha^\imath t}{2n^{6(\imath+1)}}\right)\right) \\ &\geq \aleph'(\alpha\psi(x,0), t), \end{aligned} \tag{8}$$

$\forall x \in X$, $t > 0$, $j > 0$. By replacement x by $n^m x$ in (8) we observe that

$$\begin{aligned} &\aleph\left(\frac{f(n^{j+m}x)}{n^{6j+6m}} - \frac{f(n^m x)}{n^{6m}}, \sum_{\imath=0}^{j-1} \frac{\alpha^\imath t}{2n^{6(\imath+6m)}}\right) \\ &\geq \aleph'(\psi(n^m x, 0), t) \geq \aleph'(\alpha^m \psi(x,0), t) = \aleph'\left(\psi(x,0), \frac{t}{\alpha^m}\right) \end{aligned} \tag{9}$$

whence

$$\aleph\left(\frac{f(n^{j+m}x)}{n^{6j+6m}} - \frac{f(n^m x)}{n^{6m}}, \sum_{\imath=m}^{j-1+m} \frac{\alpha^\imath t}{2n^{6(\imath+1)}}\right) \geq \aleph'(\psi(x,0), t),$$

$\forall x \in X$, $t > 0$, $j \geq 0$,$m > 0$. Hence

$$\aleph\left(\frac{f(n^{j+m}x)}{n^{6j+6m}} - \frac{f(n^m x)}{n^{6m}}, t\right) \geq \aleph'\left(\psi(x,0), \frac{t}{\sum_{\imath=m}^{j-1+m} \frac{\alpha^\imath t}{2n^{6(\imath+1)}}}\right), \tag{10}$$

$\forall x \in X$, $t > 0$, $j \geq 0$,$m \geq 0$. Since $0 < \alpha < n^6$ and $\sum_{\imath=0}^{\infty} (\frac{\alpha}{n^6})^\imath < \infty$.

The Cauchy concept for convergence and (N5) show that $\{\frac{f(n^m x)}{n^{6m}}\}$ is a Cauchy sequence in $(Y, \aleph)$. Since $(Y, \aleph)$ is a F.B. space, this sequence converges to some point $A(x) \in Y$.Set$x \in X$ and impose $m = 0$ in (10) to gain

$$\aleph(\frac{f(n^j x)}{n^{6j}} - f(x), t) \geq \aleph' \left(\psi(x, 0), \frac{t}{\sum_{\imath=0}^{j-1} \frac{\alpha^\imath t}{2n^{6(\imath+1)}}} \right),$$

$t > 0$, $j > 0$. From which we obtain

$$\begin{aligned}\aleph(A(x) - f(x), t) &\geq \min\{\aleph \left(C(x) - \frac{f(n^j x)}{n^{6j}}, \frac{t}{2} \right), \aleph \left(\frac{f(n^j x)}{n^{6j}} - f(x), \frac{t}{2} \right)\} \\ &\geq \aleph' \left(\psi(x, 0), \frac{t}{\sum_{\imath=0}^{j-1} \frac{\alpha^\imath t}{2n^{6(\imath+1)}}} \right),\end{aligned}$$

for $\imath$ large enough. Taking the limit an $j \longrightarrow \infty$ and using (N6) we gain

$$\aleph(A(x) - f(x), t) \geq \aleph'(\psi(x, 0), 2t(n^6 - \alpha))$$

replacement x, y by $n^\imath x, n^\imath y$ respectively in (4) to gain

$$\aleph \left(\frac{D_s f(n^\imath x, n^\imath y)}{n^{6\imath}}, \frac{t}{n^{6\imath}} \right) \geq \aleph'(\psi(n^{6\imath} x, n^{6\imath} y), t).$$

Then

$$\aleph \left(\frac{D_s f(n^\imath x, n^\imath y)}{n^{6\imath}}, t \right) \geq \aleph'(\psi(n^\imath x, n^\imath y), n^{6\imath} t),$$

$\forall x \in X$, $\forall t > 0$, since $\lim_{\imath \to \infty} \aleph'(\psi(n^\imath x, n^\imath y), n^{6\imath} t) = 1$ we infer that A complete (5). To state the uniqueness of the S. Fun C, presume exists a S. Fun $D : X \longrightarrow Y$ which satisfies (5) Set $x \in X$. Clearly $A(nx) = n^6 A(x)$ and $D(nx) = n^6 D(x)$ then $A(n^\imath x = n^{6\imath} A(x)$ and $D(n^\imath x = n^{6\imath} D(x)$ $\forall \imath \in \mathbb{N}$. Accompany from (5) that

$$\begin{aligned}\aleph(A(x) - D(x), t) &= \aleph \left(\frac{A(n^\imath x)}{n^{6\imath}} - \frac{D(n^\imath x)}{n^{6\imath}}, t \right) \\ &\geq \min\{\aleph \left(\frac{A(n^\imath x)}{n^{6\imath}} - \frac{f(n^\imath x)}{n^{6\imath}}, \frac{t}{2} \right), \aleph(\frac{A(n^\imath x)}{n^{6\imath}} - \frac{f(n^\imath x)}{n^{6\imath}}, \frac{t}{2}) \\ &\geq \aleph'(\psi(n^\imath x, 0), n^{6\imath} t(n^6 - \alpha)) \geq \aleph' \left(\psi(x, 0), \frac{n^{6\imath} t(n^6 - \alpha)}{\alpha^\imath} \right)\end{aligned}$$

since $\lim_{\imath \to \infty} \frac{n^{6\imath} t(n^6 - \alpha)}{\alpha^\imath} = \infty$, and $0 < \alpha < n^6$ we gain

$$\lim_{\imath \to \infty} \aleph' \left(\psi(x, 0), \frac{n^{6\imath} t(n^6 - \alpha)}{\alpha^\imath} \right) = 1,$$

therefore, $\aleph(A(x) - D(x), t) = 1 \forall t > 0$ whence $A(x) = D(x)$. □

Example 2.1. Let X be a Banach algebra space and Z be a normed space. Denote by $\aleph$ and $\aleph'$ the Fuzzy norms obtained as Example (1.2) on X and Z , respectively . $\psi : X \times X \longrightarrow Z$ realized by $\psi(x,y) = 8n^6(\| x \| + \| y \|)$.

Define $f : X \longrightarrow X$ by $f(x) = x^6 + \| x \| x_0$ for some unit vector $x_0 \in X$. Then

$$\begin{aligned} N(D_s f(x,y),t) &= \frac{t}{t+ \| N(D_s f(x,y) \|)} \\ &\geqslant \frac{t}{t+2(n^6+n+2)(\| x \| + \| x \|)} \\ &\geqslant \frac{t}{t+8n^6(\| x \| + \| x \|)} = \aleph'(\psi(x,y),t) \end{aligned}$$

and

$$\aleph'(\psi(nx,0),t) = \frac{t}{t+8n^7 \| x \|} \leqslant \frac{t}{8n^7 \| x \|} = \aleph'(\propto \psi(x,0),t),$$

$$\lim_{\imath \longrightarrow \infty} \frac{n^{6\imath}t}{n^{6\imath}t + 8n^{6\imath}(\| x \| + \| y \|)} = 1.$$

Hence, conditions of theorem (2) for $\alpha = n$ are complete. Therefore, exists sextic mapping $A : X \longrightarrow X$ s.t

$$\aleph(A(x) - f(x),t) \geqslant \aleph'(\psi(x,0), 2t(n^6 - n)).$$

References

[1] A. K. Mirmostafaee, M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theoerm*, Fuzzy Sets and Systems, (2008), 720-729.

[2] A. K. Mirmostafaee, M. Mirzavaziri, M S. Moslehian, *Fuzzy stability of the Jensen functiona equation l*, Fuzzy sets and Systems, (2008), 730-738.

[3] SH. Alshybani, S. Mansour Vaezpour, R. Saadati, *Stability of the sextic functional equation in various spaces*, J. Inequalities and Special Functions, (2018), 8-27.

[4] SH. Alshybani, S. Mansour Vaezpour, R. Saadati, *Generalized Hyers-Ulam stability of sextic functional equation in random normed spaces*, J. Computational Analysis and Applications, 2018, 172-196.

[5] T. Bag, K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., (2003), 687-705.

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