On a class of semi-hypergeneralized quasi-Einstein manifolds

Matilal Sen
Department of Mathematics
Saldiha College
P.O. Saldiha, 722173, Bankura
West Bengal
India
matilal_sen@yahoo.co.in

Abstract. The object of the present paper is to introduce semi-hypergeneralized quasi-Einstein manifolds and the manifolds with semi- hypergeneralized quasi-constant curvature. Ricci recurrent semi-hypergeneralized quasi-Einstein manifolds have also been studied.
Keywords: generalized quasi-Einstein manifold, semi-hypergeneralized quasi-Einstein manifold, semi-hypergeneralized quasi-constant curvature, Ricci recurrent.

1. Introduction

A Riemannian or semi-Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the condition

$$S = \frac{r}{n}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to A. L. Besse [4], the above condition is called Einstein metric condition. Einstein manifolds play an important role in Riemannian geometry as well as in general theory of relativity. Einstein manifolds have been studied by several authors. For more details see [4]. Tamassay and Binh [25] studied symmetry of Einstein manifolds. Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [4]. For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1}$$

where a, b are real numbers and A is a non-zero $1-$form defined by $g(X,U) = A(X)$, for all vector fields X and the unit vector field U.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and for any vector fields X, Y it satisfies the equation (1) (see [14]). In the year 2000, Chaki and Maity [5] defined and studied quasi-Einstein manifolds by taking a, b as scalars. Such a quasi-Einstein manifold of dimension n is denoted by $(QE)_n$. In the paper [15] Ghosh, De and Binh studied certain curvature restrictions on quasi-Einstein manifolds. Quasi-Einstein manifold arose during the study of exact solution of Einstein's field equations as well as the study of quasi umbilical hypersurfaces of Euclidean space.

According to U. C. De and G. C. Ghosh [9] a non-flat Riemannian manifold (M^n, g) $(n > 2)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and for any vector fields X, Y it satisfies

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b, c are scalars and A, B are non-zero $1-$forms defined respectively by $g(X,U) = A(X)$ and $g(X,V) = B(X)$. The unit vector fields U and V corresponding to the $1-$forms A and B are orthogonal, i.e., $g(U,V) = 0$. U, V are known as the generators of the manifold. Such an $n-$dimensional manifold is denoted by $G(QE)_n$. Quasi-Einstein manifolds have been generalized by several authors in several ways. For instance we mention [6], [15], [17], [19], [20] [22]. Some generalizations of quasi Einstein manifolds appear in the study of hypersurfaces of some manifolds [24].

The notion of hypergeneralized quasi-Einstein manifold was introduced by A. A. Shaikh, C. Ozgur and A Patra [23]. An n-dimensional Riemannian manifold (M^n, g), $(n > 2)$ is called a hypergeneralized quasi-Einstein manifold if its Ricci tensor of type (0, 2) is non zero and satisfies the following condition

$$\begin{aligned} S(X,Y) = ag(X,Y) &+ bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \\ &+ d[A(X)D(Y) + A(Y)D(X)], \end{aligned}$$

for all $X, Y \in T(M)$, where a, b, c and d are real valued, non-zero scalars functions on (M^n, g).

Later hypergeneralized quasi-Einstein manifolds has been studied by D. Debnath [12].

In the present paper, we like to introduce the notion of semi-hypergeneralized quasi-Einstein manifold by ensuring its existence and physical relevance of study. The present paper is organized as follows:

Section 2 contains some preliminaries. In Section 3, we establish the existence of semi-hypergeneralized quasi-Einstein manifolds. Section 4 is devoted to study the nature of a manifold with semi-hypergeneralized quasi-constant curvature and to prove that such a manifold is semi-hypergeneralized quasi-Einstein manifold. Finally, in Section 5 a necessary and sufficient condition for a semi-hypergeneralized quasi-Einstein manifold to be Ricci recurrent is obtained.

2. Preliminaries

Let M be a submanifold immersed in a $(2n+1)-$dimensional Riemannian manifold $\tilde{M}$, we denote by the same symbol g the induced metric on M. Let TM be the set of all vector fields tangent to M and $T^{\perp}M$ is the set of all vector fields normal to M. Then Gauss and Weingarten formula are given by [8]

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{2}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X{}^{\perp} N, \tag{3}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in $T^{\perp}M$. The second fundamental form B and A_N are related by

$$g(A_N X, Y) = g(B(X, Y), N). \tag{4}$$

For a unit normal vector field N of the submanifold we get

$$B(X, Y) = H(X, Y)N, \tag{5}$$

where $H(X, Y)$ is the second fundamental tensor and $B(X, Y)$ is the second fundamental form of the submanifold. The Gauss equation is given by

$$\begin{aligned}\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) &- g(B(X, W), B(Y, Z)) \\ &+ g(B(Y, W), B(X, Z)),\end{aligned} \tag{6}$$

where R is the Riemannian curvature tensor of the submanifold and $\tilde{R}$ is the Riemannian curvature tensor of the manifold.

Definition 2.1. A hypersurface M^{2n} immersed isometrically in a generalized Sasakian-space-form $\tilde{M}^{2n+1}(c)$ is said to be 2−quasi-umbilical [16] if the second fundamental tensor $H(X, Y)$ satisfies the the equation

$$H(X, Y) = \alpha g(X, Y) + \beta\omega(X)\omega(Y) + \gamma T(X)T(Y), \tag{7}$$

where α, β, γ are scalars and ω, T are 1−forms defined by $\omega(X) = g(X, U)$ and $T(X) = g(X, \rho)$, and U, ρ are two unit vectors such that $g(U, \rho) = 0$.

If $\gamma = 0$, the hypersurface is called quasi-umbilical and for $\beta = \gamma = 0$, it is known as umbilical.

Recently, P. Alegre, D. Blair and A. Carriazo [1] introduced and studied generalized Sasakian-space-forms. These space-forms are defined as follows:

Given an almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$, we say that $\tilde{M}$ is generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on $\tilde{M}$ such that the curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by

$$\begin{aligned}\tilde{R}(X, Y)Z \;=\;& f_1\{g(Y, Z)X - g(X, Z)Y\} \\ +\;& f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ +\;& f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ +\;& g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},\end{aligned} \tag{8}$$

for any vector fields X, Y, Z on $\tilde{M}$. In such a case we denote the manifold as $\tilde{M}(f_1, f_2, f_3)$. These kind of manifolds appear as a generalization of the well known Sasakian-space-forms, which can be obtained as a particular case of generalized Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. But, it is to be noted that generalized Sasakian-space-forms are not merely generalization of Sasakian-space-forms. It also contains a large class of almost contact manifolds. For example, it is known that [2] any three-dimensional trans Sasakian manifold of type (α, β) with α, β depending on ξ is a generalized Sasakian-space-form. However, we can find generalized Sasakian-space-forms with non-constant functions and arbitrary dimensions. In [1], the authors cited several examples of generalized Sasakian-space-forms in terms of warped product spaces. Again generalized Sasakian-space-forms have been studied in the papers [10], [11], [21].

3. Existence of semi-hypergeneralized quasi-Einstein manifolds

Definition 3.1. A non-flat Riemannian manifold of dimension m $(m > 2)$ will be called a semi-hypergeneralized quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) + dT(X)T(Y), \tag{9}$$

where, a, b, c, d are certain non-zero scalars and A, B and T are three non-zero $1-$forms defined by $A(X) = g(U, X)$ and $B(X) = g(X, V)$, and $T(X) = g(X, \rho)$. U, V, ρ are three mutually orthogonal unit vectors. A semi-hypergeneralized quasi-Einstein manifold of dimension m will be denoted by $H(GQE)_m$.

If $d = 0$, then the manifold becomes generalized quasi-Einstein manifold. If $c = d = 0$, the manifold is qusi-Einstein. The manifold is Einstein when $b = c = d = 0$.

Let us consider a $(2n+1)-$dimensional generalized Sasakian-space-form $\tilde{M}$. Now, from (8) we get

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= f_1\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &+ f_2\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) \\ &+ 2g(X, \phi Y)g(\phi Z, W)\} \\ &+ f_3\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ &+ g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W)\}. \end{aligned} \tag{10}$$

By (5) and (6) we get

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - H(Y, Z)H(X, W) + H(X, Z)H(Y, W). \tag{11}$$

Let us consider a $2-$quasi-umbilical hypersurface of $\tilde{M}$. In view of (7) the above equation yields

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W)$$

$$
\begin{aligned}
&- (\alpha g(Y,Z) + \beta\omega(Y)\omega(Z) + \gamma T(Y)T(Z))(\alpha g(X,W)\\
&+ \beta\omega(X)\omega(W) + \gamma T(X)T(W))\\
&+ (\alpha g(X,Z) + \beta\omega(X)\omega(Z) + \gamma T(X)T(Z))(\alpha g(Y,W)\\
&+ \beta\omega(Y)\omega(W) + \gamma T(Y)T(W)).
\end{aligned} \tag{12}
$$

By virtue of (10) and (12), it follows that

$$
\begin{aligned}
&f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}\\
&+ f_2\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\}\\
&+ f_3\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)\\
&+ g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\}\\
&= R(X,Y,Z,W)\\
&- (\alpha g(Y,Z) + \beta\omega(Y)\omega(Z) + \gamma T(Y)T(Z))(\alpha g(X,W)\\
&+ \beta\omega(X)\omega(W) + \gamma T(X)T(W))\\
&+ (\alpha g(X,Z) + \beta\omega(X)\omega(Z) + \gamma T(X)T(Z))(\alpha g(Y,W)\\
&+ \beta\omega(Y)\omega(W) + \gamma T(Y)T(W)).
\end{aligned} \tag{13}
$$

In the above equation contracting Y and Z, we obtain

$$
\begin{aligned}
&(2n-1)f_1 g(X,W) + 3f_2 g(\phi X,\phi W)\\
&+ f_3\{(2-2n)\eta(X)\eta(W) - g(X,W)\}\\
&= S(X,W) - (2n\alpha + \beta + \gamma)(\alpha g(X,W) + \beta\omega(X)\omega(W) + \gamma T(X)T(W))\\
&+ (\alpha^2 g(X,W) + \alpha\beta\omega(X)\omega(W) + \alpha\gamma T(X)T(W) + \alpha\beta\omega(X)\omega(W)\\
&+ \beta^2\omega(X)\omega(W) + \alpha\gamma T(X)T(W) + \gamma^2 T(X)T(W)).
\end{aligned} \tag{14}
$$

After simplification the above equation yields

$$
\begin{aligned}
S(X,W) &= \{(2n-1)f_1 + 3f_2 - f_3 + \alpha(2n\alpha + \beta + \gamma) - \alpha^2\}g(X,W)\\
&+ \{(2-2n)f_3 - 3f_2\}\eta(X)\eta(W)\\
&+ \{\gamma(2n\alpha + \beta + \gamma) - 2\alpha\gamma - \gamma^2\}T(X)T(W)\\
&+ \{\beta(2n\alpha + \beta + \gamma) - 2\alpha\beta - \beta^2\}\omega(X)\omega(W).
\end{aligned} \tag{15}
$$

Therefore we can state the following:

Theorem 3.1. A $2-$quasi-umbilical hypersurface of a generalized Sasakian-space-form is a semi-hypergeneralized quasi-Einstein manifold.

If $\beta = 0$, the hypersurface becomes quasi-umbilical. Hence we obtain the following:

Corollary 3.1. A quasi-umbilical hypersurface of a generalized Sasakian-space-form is a generalized quasi-Einstein manifold.

If $\beta = \gamma = 0$, the hypersurface is known as umbilical. Hence we can state the following:

Corollary 3.2. An umbilical hypersurface of a generalized Sasakian-space-form is an Einstein manifold.

Example 3.1: physical example of semi-hypergeneralized quasi-Einstein manifolds. In [1], it is shown that the warped product space $\tilde{M} = \mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian-space-form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t), t \in \mathbb{R}$ and f' denotes derivative of f with respect to t. It is known that[3] the space near a massive star or black whole is warped product space. Hence, a $2-$quasi umbilical hypersurface of the manifold under consideration in this example gives a physical example of semi-hypergeneralized quasi-Einstein manifold by Theorem 3.1.

Example 3.2: example of non-trivial Riemannian manifold which is semi-hypergeneralized quasi-Einstein. Let us consider the $4-$dimensional real number space with the Riemennian metric g_{ij} given by

$$g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1 \sin x^2)^2(dx^3)^2 + (dx^4)^2,$$

$i, j = 1, 2, 3, 4$, $x^1 \neq 0$ and $0 < x^2 < \frac{\pi}{2}$. It can be easily calculated that the non-vanishing components of the Christoffel symbols are

$$\Gamma^1_{22} = -x^1, \Gamma^1_{33} = -x^1(\sin x^2)^2, \Gamma^2_{12} = \frac{1}{x^1} = \Gamma^3_{13}, \Gamma^2_{23} = \cot x^2, \Gamma^2_{33} = -\sin x^2 \cos x^2.$$

Using the above results we find the non-vanishing components of the curvature tensor and Ricci tensor as follows:

$$R_{2332} = -(x^1 \sin x^2)^2, S_{22} = -1, S_{33} = -(\sin x^2)^2.$$

Let

$$a = -\frac{1}{(x^1)^2}, \ b = \frac{1}{(x^1)^2}, \ c = \frac{1}{2(x^1)^2}, \ d = \frac{1}{2}$$

and

$$\begin{aligned} A_i &= 1, \text{for } i = 1 \\ &= 0, \text{otherwise}, \end{aligned}$$

$$\begin{aligned} B_i &= 1, \text{for } i = 4 \\ &= 0, \text{otherwise}. \end{aligned}$$

$$\begin{aligned} T_i &= \tfrac{1}{x}, \text{for } i = 4 \\ &= 0, \text{otherwise}. \end{aligned}$$

From the above results we can verify the following:

$$\begin{array}{lclcl}
S_{11} & = & 0 & = & ag_{11}+bA_1A_1+cB_1B_1+dT_1T_1, \\
S_{22} & = & -1 & = & ag_{22}+bA_2A_2+cB_2B_2+dT_2T_2, \\
S_{33} & = & -(\sin x^2)^2 & = & ag_{33}+bA_3A_3+cB_3B_3+dT_3T_3, \\
S_{44} & = & 0 & = & ag_{44}+bA_4A_4+cB_4B_4+dT_4T_4, \\
S_{12} & = & 0 & = & ag_{12}+bA_1A_2+cB_1B_2+dT_1T_2, \\
S_{13} & = & 0 & = & ag_{13}+bA_1A_3+cB_1B_3+dT_1T_3, \\
S_{14} & = & 0 & = & ag_{14}+bA_1A_4+cB_1B_4+dT_1T_4, \\
S_{23} & = & 0 & = & ag_{23}+bA_2A_3+cB_2B_3+dT_2T_3, \\
S_{24} & = & 0 & = & ag_{24}+bA_2A_4+cB_2B_4+dT_2T_4, \\
S_{34} & = & 0 & = & ag_{34}+bA_3A_4+cB_3B_4+dT_3T_4.
\end{array}$$

From the above expressions it is clear that the manifold under consideration is a semi-hypergeneralized quasi-Einstein manifold.

4. Manifold of semi-hypergeneralized quasi-constant curvature

Definition 4.1. A Riemannian manifold is called a manifold of quasi-constant curvature if it is conformally flat and if the curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}
\tilde{R}(X,Y,Z,W) &= \alpha[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
&+ \beta[g(X,W)T(Y)T(Z)-g(X,Z)T(Y)T(W) \\
&+ g(Y,Z)T(X)T(W)-g(Y,W)T(X)T(Z)], \qquad (16)
\end{aligned}$$

where $\tilde{R}(X,Y,Z,W)=g(R(X,Y)Z,W)$, R is the curvature tensor of type (1,3), α,β are scalar functions and ρ is a unit vector field defined by $g(X,\rho)=T(X)$. The notion of quasi-constant curvature for Riemannian manifolds were given by Chen and Yano [7]. Again, in the paper [9], the authors have given the notion of manifolds of generalized quasi-constant curvature. In this section we like to give the idea of another generalization, viz., the manifolds of semi-hypergeneralized quasi constant curvature. We also show that such manifolds are semi-hypergeneralized quasi-Einstein manifolds.

Definition 4.2. A Riemannian manifold will be called a manifold of semi-hypergeneralized quasi-constant curvature if it is conformally flat and if the curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}
\tilde{R}(X,Y,Z,W) &= \alpha[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
&+ \beta[g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W) \\
&+ g(X,W)A(Y)A(Z)-g(Y,W)A(X)A(Z)] \\
&+ \gamma[g(Y,Z)B(X)B(W)-g(X,Z)B(Y)B(W) \\
&+ g(X,W)B(Y)B(Z)-g(Y,W)B(X)B(Z)] \\
&+ \delta[g(Y,Z)T(X)T(W)-g(X,Z)T(Y)T(W) \\
&+ g(X,W)T(Y)T(Z)-g(Y,W)T(X)T(Z)], \qquad (17)
\end{aligned}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type (1,3), $\alpha,\beta,\gamma,\delta$ are scalar functions and the $1-$forms have the same meanings as in Definition 3.1.

Let us consider a semi-hypergeneralized quasi-Einstein manifold. Then by definition the manifold is conformally flat. Therefore, we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)]\\ &+ \frac{r}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \qquad (18)\end{aligned}$$

Since the manifold is semi-hypergeneralized quasi-Einstein, by Definition 3.1 and equation (18) it follows that

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= (\frac{3a}{n-2} - \frac{r}{(n-1)(n-2)})[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &+ \frac{b}{n-2}[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)\\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)]\\ &+ \frac{c}{n-2}[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)\\ &+ g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)]\\ &+ \frac{d}{n-2}[g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W)\\ &+ g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z)]. \qquad (19)\end{aligned}$$

From above we get the following:

Theorem 4.1. A semi-hypergeneralized quasi-Einstein manifold is a manifold of semi-hypergeneralized quasi-constant curvature.

Conversely, consider a manifold of semi-hypergeneralized quasi-constant curvature. Then in (17), contracting Y and Z we get

$$\begin{aligned}S(X,W) &= ((n-1)\alpha + \beta + \gamma + \delta)g(X,W) + \beta(n-2)A(X)A(W)\\ &+ \gamma(n-2)B(X)B(Z) + \delta(n-2)T(X)T(Z). \qquad (20)\end{aligned}$$

In view of (20), we can easily state the following:

Theorem 4.2. A manifold of semi-hypergeneralized quasi-constant curvature is a semi-hypergeneralized quasi-Einstein manifold.

5. Ricci-recurrent $H(GQE)_n$

Definition 5.1. A Riemannian manifold is called Ricci recurrent if its Ricci tensor satisfies

$$(\nabla_Z S)(X,Y) = \lambda(Z)S(X,Y),$$

where λ is a non-zero $1-$form, for any vector fields X, Y, Z.

Let U, V, ρ are parallel vector fields. Then $\nabla_X U = 0$ for all X, which implies that $R(X,Y)U = 0$. Consequently, $S(X,U)$=0. Again, from (9) $S(X,U) = (a+b)A(X)$. Hence, $a+b=0$, because $A(X) \neq 0$ for all X other than V. Similarly, $a+c=0$ and $a+d=0$. In this way, it follows that $b=c=d=-a$. Hence, (9) takes the form

$$S(X,Y) = a[g(X,Y) - A(X)A(Y) - B(X)B(Y) - T(X)T(Y)]. \tag{21}$$

From the above equation we obtain by covariant differentiation

$$\begin{aligned}(\nabla_Z S)(X,Y) &= da(Z)[g(X,Y) - A(X)A(Y) - B(X)B(Y) - T(X)T(Y)] \\ &- a[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y) \\ &+ (\nabla_Z B)(X)B(Y) + B(X)(\nabla_Z B)(Y)].\end{aligned} \tag{22}$$

Since U, V, ρ are parallel vector fields, $(\nabla_Z A)(X) = 0$, $(\nabla_Z B)(X) = 0$ and $(\nabla_Z T)(X) = 0$. Therefore, the above equation reduces to

$$(\nabla_Z S)(X,Y) = \frac{da(Z)}{a} S(X,Y). \tag{23}$$

Thus, the manifold is Ricci recurrent.

Conversely, let a semi-hypergeneralized quasi-Einstein manifold is Ricci recurrent. Then by Definition 5.1 and equation (9), it follows that

$$\begin{aligned}&da(W)g(X,Y)+db(W)A(X)A(Y)+dc(W)B(X)B(Y)+dd(W)T(X)T(Y) \\ &+ b((\nabla_W A)(X)A(Y) + A(X)(\nabla_W A)Y) \\ &+ c((\nabla_W B)(X)B(Y) + B(X)(\nabla_W B)(Y)) \\ &+ d((\nabla_W T)(X)T(Y) + T(X)(\nabla_W T)(Y)) \\ &= \lambda(Z)(ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + dT(X)T(Y)).\end{aligned} \tag{24}$$

In the above equation putting $X = U$ and $Y = V$, where U, V are described in the Definition 3.1, we get

$$(\nabla_W A)(V) = 0.$$

By the definition of the $1-$form A the above equation yields $g(\nabla_W V, U) = 0$. Hence, either $\nabla_W V = 0$, or, $\nabla_W V$ is orthogonal to U. Since W is arbitrary, $\nabla_W V$ is not orthogonal to U, in general. Therefore, $\nabla_W V = 0$. In the similar way it can be proved that $\nabla_W U = 0$ and $\nabla_W \rho = 0$. Thus, we have the following:

Theorem 5.1. The generators of a semi-hypergeneralized quasi-Einstein manifold is parallel if and only if the manifold is Ricci recurrent.

Acknowledgement

The author is thankful to the referee for his valuable suggestions towards the improvement of the paper.

References

[1] P. Alegre, D. Blair, A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math., 14 (2004), 157-183.

[2] P. Alegre, A. Cariazo, *Structures on generalized Sasakian-space-forms*, Differential Geom. and its application, 26 (2008), 656-666.

[3] J.K. Beem, ,P. E. Ehrlich, Th. G. Powel, *Warped product manifolds in relativity*, Selected Studies- Physics, Astrophysics, Mathematics, History of Science- New York, North Holand, 1982.

[4] A.L. Besse, *Einstein manifolds*, Springer Verlag, 1987.

[5] M.C. Chaki, R. K. Maity, *On quasi-Einstein manifolds*, Publ. Math. Debreceen, 57 (2000), 297-306.

[6] M.C. Chaki, *On generalized quasi-Einstein manifolds*, Publ. Math. Debreceen, 58 (2001), 683-691.

[7] B.Y. Chen, K. Yano, *Hypersurfaces of conformally flat spaces*, Tensor(N.S.), 26 (1972), 318-322.

[8] B.Y. Chen, *Geometry of submanifolds*, Marcel Dekker, Inc. New York, 1973.

[9] U.C. De, G.C. Ghosh, *On generalized quasi-Einstein manifold*, Kyungpook Math. J., 44 (2004), 607-615.

[10] U.C. De, A. Sarkar, *Some results on generalized Sasakian-space-forms*, Thai. J. Math., 8 (2010), 1-10.

[11] U.C. De, A. Sarkar, *On the projective curvature tensor of generalized Sasakian-space-forms*, Quaest. Math., 33 (2010), 245-252.

[12] D. Debnath, *On hypergeneralized quasi-Einstein manifolds*, Int. J. of Math. Combin, 3 (2018), 25-31.

[13] R. Deszcz, L. Verstraelen, *Hypersurface of semi-Riemannian conformally flat manifolds*, Geom. Topol. Submanifolds, 3 (1991), 131-147.

[14] R. Deszcz, M. Glogowska, M. Hotlos, Z. Senturk, *On certain quasi-Einstein semisymmetric hypersurfaces*, Annales Univ. Sci. Budapest. Eotvos Sect. Math., 41 (1998), 151-164.

[15] G.C. Ghosh, U.C. De, T.Q. Binh, *Certain curvature restrictions on a quasi-Einstein manifold*, Publ. Math. Debrecee, 69 (2006), 209-217.

[16] W. De Muelemester, L. Verstraelen, *Codimension* 2 *submanifolds with a* $2-$*quasi-umbilical normal direction*, J. Korean Math. Soc. 15 (1979), 101-108.

[17] C. Özgür, *On some classes of super quasi-Einstein manifolds*, Chaos, Solitons and Fractals, 40 (2009), 1156-1161.

[18] C. Özgür, *$N(k)-$quasi-Einstein manifolds satisfying certain conditions*, Chaos, Solitons and Fractals, 32 (2008), 1373-1377.

[19] C. Özgür, M.M. Tripathi, *On the concircular curvature tensor of an $N(k)-$quasi-Einstein manifold*, Math. Panon, 18 (2007), 95-100.

[20] C. Özgür, S. Sular, *On $N(k)-$quasi-Einstein manifolds satisfying certain conditions*, Balkan J. Geom. and Its Appl., 13 (2008), 74-79.

[21] A. Sarkar, M. Sen, *Locally $\phi-$symmetric generalized Sasakian-space-forms*, Ukrainian Math. J., 65 (2014), 1588-1597.

[22] A. A. Shaikh, S. K. Jana, *On pseudo generalized quasi-Einstein manifold*, Tamkang J. Math., 39 (2008), 9-24.

[23] A. A. Shaikh, C. Özgür, A. Patra, *On hypergeneralized quasi-Einstein manifolds*, Int. J. Math Sci. And Eng. Appl., 5 (2011), 189-206.

[24] S. Sular, C. Özgür, *On some submanifolds of Kenmotsu manifolds*, Chaos, Solitons and Fractals, 42 (2009), 199-195.

[25] L. Tamassay, T. Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*, Tensor (N.S.), 53 (1993), 140-148.

Accepted: May 14, 2020