z-clean and strongly z-clean rings

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Abstract. In this article we introduce the concept of z-clean and strongly z-clean rings. The ring R is said to be a z-clean ring if every element of R is sum of a zero divisor of R and an idempotent element of R.

We present the necessary and sufficient condition when the clean rings, and z-clean rings become equivalent. We study various properties of the z-clean and strongly z-clean rings.
Keywords: clean ring, z-clean ring, strongly z-clean ring.

1. Introduction

Throughout this paper, R denotes an associative ring with identity, $U(R)$ the group of units, $Id(R)$ the set of idempotents, $Reg(R)$ the set of regulars, $M_n(R)$ the ring of all $n \times n$ matrices over R and

$$Z(R) = \{0 \neq x \in R | \exists 0 \neq y \in R, xy = 0\}.$$

Nicholson in [8] proved some important properties of clean rings. It is important to know when we can decompose some (or all) elements of a ring as a sum of an idempotent and an element with some special properties. For instance, an element x of a ring R is

- clean if $x = u + e$, where $e \in Id(R)$ and $u \in U(R)$ ([9]):
- $r-$clean if $x = r + e$, where $e \in Id(R)$ and $r \in Reg(R)$ ([2]);

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- nill-clean if $x = n + e$,where $e \in Id(R)$ and $n \in N(R)$ ([8]).

Several authors worked on clean rings and investigated properties of clean rings [1, 4, 5, 8].

Further in 1999, Nicholson [10] called an element of a ring R as strongly clean if it is the sum of a unit and an idempotent that commute with each other that means if $a = e + u$, where $e^2 = e$ and u is a unit of R such that $eu = ue$ and R is strongly clean if each of its element is strongly clean. Again clearly from [10], a strongly clean ring is clean, and the converse holds for an abelian ring (that is, all idempotents in the ring are central). Local rings and strongly $\pi-$regular rings are well-known examples of the strongly clean rings. In 1936, Von Neuman defined regular element and regular ring and some properties of regular rings has been studied in [7].

The clean rings were further extended to $r-$clean rings and the $r-$clean rings were introduced by Ashrafi and Nasibi [2, 3] and they defined that an element x of a ring R is $r-$clean if $x = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$. A ring R is said to be $r-$clean if each of its element is $r-$clean. In [3] they proved that every abelian $r-$clean ring is clean.

Garima Sharma and Amit B. Singh in [13] introduced the concept of new class of rings, strongly $r-$clean rings which is generalization of the class of strongly clean rings and stronger class of $r-$clean rings.

Motivated by all above studies, in this paper, we introduce the concept of new class of rings, namely z-clean and strongly z-clean rings.

We call an element x of a ring R is z-clean if $x = z + e$, where $z \in Z(R)$ and $e \in Id(R)$.

A ring R is z-clean if each of its element is z-clean. If $ze = ez$ then x of a ring R is strongly z-clean and a ring R is strongly z-clean.

We will give examples for z-clean ring and cheek the relationship between clean ring, $r-$clean ring with z-clean ring. Moreover, we give the necessary and sufficient condition when the clean rings, strongly clean rings, z-clean rings and strongly z-clean rings become equivalent.

We show that a directly finite of rings $R_i \neq 0$, $i = 1, 2, \cdots, n$ is z-clean if and only if every R_i is z-clean.

Finally we provide some properties of z-clean rings and prove that if R is an z-clean ring then the matrix ring $M_n(R)$ is z-clean for any $n \geq 1$.

2. $z-$clean and strongly z-clean rings

In this section first we define $z-$clean elements, strongly z-clean elements, $z-$clean rings and strongly z-clean rings and we investigate the properties of the $z-$clean ring.

Definition 2.1. *An element x of a ring R is $z-$ clean if $x = z + e$, where $z \in Z(R)$ and $e \in Id(R)$, a ring R is $z-$clean if each of its element is $z-$clean.*

If $ze = ez$ then x of a ring R is strongly z-clean and a ring R is strongly z-clean. From the above definition it is clear that every strongly z-clean ring is z-clean.

Now, in the following example we show that the converse need not be true.

Example 2.1. $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_2 \right\}$ is a z-clean ring but not strongly z-clean ring.

Proof. Let R: $M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus the set of idempotents $Id(R) = \{e_1 = M_1, e_2 = M_2, e_3 = M_3\}$. Also all the elements are z-clean. Now, we prowe that R has at least one element $x = z + e$ such that $ze \neq ez$ for $e \in Id(R)$ and $z \in Z(R)$. Take $M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$ so representations of M_4 with commutative condition are $M_4 = e_1 + M_2$ but $e_1 M_2 \neq M_2 e_1$. Thus there exists $M_4 \neq 0$ such that M_4 is z-clean but not strongly z-clean. □

Example 2.2. If p is a prime number, then $\mathbb{Z}_p$ is a field thence $\mathbb{Z}_p$ is not a $z-$clean ring. But for $n = 2k$, $k \in \mathbb{N}$, $\mathbb{Z}_n$ is a strongly z-clean.

Proof The set of zero divisors of $R = \mathbb{Z}_{2n} = \{1, 2, 3, \cdots, 2n-1, 2n\}$ is $Z(R) = \{z, (z, 2n) \neq 1\}$ and the set of idempotents is $Id(R) = \{e, e^2 \overset{2n}{\equiv} e\}$.

Cleary, each of the elements R, writes as follows $x = z+e$ such that $z \in Z(R)$, $e \in I(R)$ and $ze = ez$. Thus $\mathbb{Z}_{2n}$ is strongly z-clean.

Definition 2.2. *The ring R is called Abelian if every idempotent is central, that is, $ae = ea$ for any $e \in Id(R)$, $a \in R$.*

Example 2.3. Every Boolean ring is strongly z-clean.

Example 2.4. Let S be a subset of fixed set U,

$$S = \mathcal{A}(U) = \{X : X \subseteq U\}.$$

We define

$$\forall A, B \in S : \quad A + B = (A - B) \cup (B - A), \quad A \cdot B = A \cap B.$$

In this case $(S, +, \cdot)$ is a strongly z-clean ring.

Remark 2.1. Recall that an element $r \in R$ is regular if $r = ryr$ for some $y \in R$. The ring R is regular if each of its element is regular. Clearly regular rings and clean rings are $r-$clean. But the converse is not true ([2]).

Remark 2.2. Let R be a $r-$clean ring which is not a field. Then R is a z-clean ring.

Proof. Let $x \in R$, since R is $r-$clean therefore $x = r + e$ such that $r \in Reg(R)$, $e \in Id(R)$. Since $r \in Reg(R)$, there exists $y \in R$ such that $r = ryr$. Hence $r - ryr = 0$, $r(1 - yr) = 0$. Since R is not filed, therefore $1 - yr \neq 0$ and this implies that $r \in Z(R)$. Also $x = r + e$ where $r \in Z(R)$, $e \in Id(R)$. Hence R is z-clean ring. □

In the following, we provide an example for which the converse of Remark 2.2, is not true.

Example 2.5. Suppose that G is a additive Abelian group and we define the multiplicative over G as follow:

$$\forall a, b \in G : a \cdot b = 0.$$

In this case G is z-clean but is not $r-$clean ring.

Remark 2.3. Nill-clean rings are z-clean rings. while the converse may not hold.

Proof. Let R be a nill-clean ring and $x \in R$. Then $x = a+e$ such that $a \in N(R)$ and $e \in Id(R)$.

Since a is nillpotent, therefore there exists $n \in \mathbb{N}$, such that $a^n = 0$. Also $a \cdot a^{n-1} = 0$ and this implies that $a \in Z(R)$. Hence R is z-clean ring.

For the converse, pleas see Example 2.3. □

Proposition 2.1. *Every Abelian z-clean ring is strongly z-clean.*

Proof. For every $x \in R$, we write $x = z + e$, where $e \in Id(R)$ and $z \in Z(R)$. Since R is an Abelian ring, so $ez = ze$ for every $z \in R$. Therefore R is strongly z-clean. □

Proposition 2.2. *Let R be a ring and $a \in R$ be strongly $z-$clean. Then we have the following:*

(1) *$-a$ is strongly $z-$clean.*

(2) *$1 - a$ is strongly $z-$clean.*

Proof. (1). Suppose that $a = z + e$ where such that $z \in Z(R)$, $e \in Id(R)$ and $ez = ze$. Because $z \in Z(R)$, hence there exist $b \in R$, such that $b \neq 0$, $z \cdot b = 0$ and this implies that $-z \cdot b = 0$. Hence $-a = -z + (-e)$ that $-e \in Id(R)$ and $(-e)(-z) = (-z)(-e)$ hence $-a$ is strongly z-clean.
(2). Note $1-a = 1-(z+e) = -z+(1-e)$. Clearly $-z \in Z(R)$, $(1-e) \in Id(R)$, $-z \cdot (1 - e) = (1 - e) \cdot (-z)$. Therefore $1 - a$ is strongly z-clean. □

Proposition 2.3. *Let R be a Abelian ring. If $a \in R$ is a strongly $z-$clean and $e \in Id(R)$, then*

(1) *ae is strongly $z-$clean.*

(2) *If the idempotents are orthogonal, then* $a+e$ *is strongly* $z-$*clean.*

Proof. It is tivial. □

Proposition 2.4. *Let* R *be a strongly* $z-$*clean ring and* e *centeral idempotent. then*

(1) eRe *and* $(1-e)R(1-e)$ *are strongly* $z-$*clean.*

(2) $\forall I \trianglelefteq R$, $\frac{R}{I}$ *is strongly* $z-$*clean.*

Proof. (1) Let $a' \in eRe$. Then there exits $a \in R$ such that $a' = eae$. Since R is strongly z-clean ring, therefore $a = z + e'$ where $z \in Z(R)$, $e' \in Id(R)$ and $ze' = e'z$. It follows that $a' = e(z+e')e = eze + ee'e = eze = ee'$. Now we need to show that eze is zero divisor, ee' is an idempotent and they commute. For this consider

$$(ee')^2 = (ee')(ee') = e(e'e)e' = e(ee')e' = (ee)(e'e') = e^2e'^2 = ee',$$

since R is $z-$clean, then

$$\exists a \in R, \quad a^{'} = eae = e(z+e) = eze + e.$$

Therefore, ee' is an idempotent. Now consider $z \in Z(R)$, therefore there exit $b \in R$ such that $zb = 0$. $(eze)b = ezeb = ezbe = 0$, hence $eze \in Z(eRe)$. Now we show the commutativity of R,

$$(eze)(ee') = ezeee' = ezee' = eze'e = ee'ze = (ee')(ze) = (ee')(zee) = (ee')(eze).$$

Therefore, it is a strongly z-clean ring. (2). Let $\bar{a} = a + I \in \frac{R}{I}$. Since R is strongly z-clean $a = z + e$ and $ze = ez$, where $z \in Z(R)$, $e \in Id(R)$.

Hence, $\bar{a} = z + (e+I)$ that $(e+I) \in Id(\frac{R}{I})$. Consider

$$(z+I)(e+I) = ze + I = ez + I = (e+I)(z+I)$$

since R is strongly z-clean. This implies that $\frac{R}{I}$ is strongly z-clean. □

Corollary 2.1. *Any homomorphic image of a strongly z-clean ring is strongly z-clean.*

Remark 2.4. In general, the converse of Proposition 2.4, may not be hold. For example, $\frac{\mathbb{Z}}{n\mathbb{Z}} \simeq Z_n$, if $n = 2k$ then z_n is strongly z-clean, but $\mathbb{Z}$ is not z-clean ring.

Proposition 2.5. *A direct product* $R = \prod R_i$ *of rings* R_i *is strongly* $z-$*clean if and only if the same is true for each* R_i.

Proof. Suppose that each R_i is strongly $z-$clean ring. For any $a = (a_i) \in R$, we write $a_i = z_i + e_i$ such that $z_i \in Z(R_i)$ and $e_i \in Id(R_i)$ and $z_i e_i = e_i z_i$. Then $a = z + e$, where $e = (e_i)$ is idempotent in $\prod R_i$ and $z = (z_i) \in Z(R)$ with $a = (a_i) = (e_i) + (z_i) \in \prod R_i$, $(e_i)(z_i) = (z_i)(e_i)$. Thus R is strongly z-clean. Conversely let $R = \prod_{i \in I} R_i$ be strongly z-clean. This implies R_i is strongly z-clean from Corollary 2.1, since R_i is a homomorphic image of R. □

Proposition 2.6. *If R is a strongly $z-$clean ring, and $e \in Id(R)$, then eR and $(1-e)R$ are strongly $z-$clean rings.*

Proof. Since

$$\forall e \in Id(R), \quad R = eR \oplus (1-e)R.$$

Therefore it followes by proposition 2.4. □

Proposition 2.7. *If R is a strongly $z-$clean ring, and $R^{'}$ is the diagonal matrixces ring in $M_n(R)$, then $R^{'}$ is a strongly $z-$clean ring.*

Proof. Suppose that $A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \in R^{'}$. Then since R is a strongly $z-$clean, hence we have $\forall 1 \leq i \leq n : \ a_{ii} = z_{ii} + e_{ii}$ such that $z_{ii} \in Z(R), e_{ii} \in Id(R)$ and $z_{ii}e_{ii} = e_{ii}z_{ii}$.

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & z_{nn} \end{bmatrix} + \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{bmatrix}.$$

Since $z_{ii} \in Z(R)$, there exits $b_{ii} \neq 0 \in R$, such that $z_{ii}b_{ii} = 0$, and we have

$$\begin{bmatrix} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & z_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \dots & 0 \\ 0 & b_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & z_{nn} \end{bmatrix} \in Z(R^{'}), \qquad \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{bmatrix} \in Id(R^{'})$$

and

$$\begin{bmatrix} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & z_{nn} \end{bmatrix} \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{bmatrix} \begin{bmatrix} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & z_{nn} \end{bmatrix}.$$

Hence $R^{'}$ is a strongly $z-$clean ring. □

Proposition 2.8. *If R is a strongly $z-$clean ring and $R^{'}$ the matrix ring which has a row or a column with elements of R, then $R^{'}$ is a strongly $z-$clean ring.*

Proof. It is clearly. □

Proposition 2.9. *If R is a commutative ring, then $R[x]$ is not z-clean.*

Proof. See [[2], Theorem 12]. □

Proposition 2.10. *Let R be ring. Then the ring $R[[x]]$ is strongly z-clean if and only if so is R.*

Proof. If $R[[x]]$ is strongly z-clean, then by Remark 2.1, R is strongly z-clean.

Conversely, suppose that R is strongly z-clean. Then by proposition 2.5, $R[[x]]$ is strongly z-clean. □

Proposition 2.11. *Suppose that R is Abelian ring which is not field. If R is a clean ring then $R[[x]]$ is strongly z-clean ring.*

Proof. By [[13], Theorem 3.7], if R is a clean ring then $R[[x]]$ is strongly $r-$clean ring. By Remark 2.2, $R[[x]]$ is strongly z-clean ring. □

Proposition 2.12. *Let a be a strongly clean element of R and $a - a^2 \in Z(R)$. Then a is strongly $z-$clean.*

Proof. Let $a = e + u$ be a strongly cleandecomposition in R and $a - a^2$ be a $z-$clean. Then $a^2 = e + 2eu + u^2$ and so $a - a^2 = (1 - 2e - u)u$.
$a - a^2 \in Z(R)$, then

$$\exists 0 \neq b \in R : (a - a^2)b = 0, \qquad (1 - 2e - u)u \cdot b = 0.$$

It follows that

$$(1 - 2e - u) \in Z(R) \qquad \textit{and} \qquad (-1 + 2e + u) \in Z(R).$$

so

$$a = (1-e) + (-1+2e+u)$$

is a strongly $z-$clean decomposition in R. □

Corollary 2.2. *If $u \in R$ be unit and $1-u \in Z(R)$, then u is strongly $z-$clean.*

For $a \in R$, the commutant if a in R is denoted by $c(a)$, i.e.,

$$c(a) = \{x \in R : ax = xa\}.$$

Proposition 2.13. *Let $a \in R$ and a be a strongly $z-$clean element of R with a strongly $z-$clean decomposition $a = e + b$. then $c(a) \subseteq c(e)$.*

Proof. since

$$\begin{aligned} a &= \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} eb & 0 \\ 0 & (1-e)b \end{bmatrix} \\ &= \begin{bmatrix} e & ex(1-e) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} eb & -ex(1-e) \\ 0 & (1-e)b \end{bmatrix} \\ &= \begin{bmatrix} e & 0 \\ (1-e)xe & 0 \end{bmatrix} + \begin{bmatrix} eb & 0 \\ -(1-e)xe & (1-e)b \end{bmatrix} \end{aligned}$$

are all strongly $z-$clean decomposition of a in R, it follows that $ex(1-e) = (1-e)xe = 0$. so $x \in c(e)$. □

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