# Entire solution of certain type of delay-differential equations

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**Abstract.** In this article, we shall study the conditions regarding the existence of entire solutions of certain type of delay-differential equations. Our results are supplements to some results obtained recently by W. Lu et al., and references therein.
**Keywords:** Nevanlinna theory, entire solutions, delay-differential polynomial.

## 1. Introduction

Let $f$ denote a transcendental function entire or meromorphic function. Assuming the reader is familiar with the basics of Nevanlinna's value distribution theory, we shall adopt the standard notations associated with the theory, such as $T(r, f)$, $N(r, f)$ and $m(r, f)$ (see [1, 2]).

Among many fascinating implementations of the Nevanlinna theory, there are research on the growth and existence of entire and meromorphic solutions

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of discrete types of non-linear differential equations, and one can find prototype for such equations ([3, 4, 5, 6]).

Recently, the Nevanlinna theory has been applied to study types of non-linear difference equations (see, e.g., [7, 8]). Now, we shall utilize Clunie type of theorems for delay -differential polynomials (difference-differential polynomials) to study some non-linear difference equations of more general forms and to obtain some improvements of [7, 9, 10].

## 2. Notation and main results

Given a meromorphic function $f$, recall that $\alpha \not\equiv 0, \infty$ is a small function with respect to $f$, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f) = O(T(r,f))$ as $r \to \infty$, possibly outside a set of $r$ of finite linear measure.

**Definition 2.1.** *Delay-differential equations are differential-difference equations in which the derivative and shifts are taken with respect to same variable.*

For the benefit of the readers, we shall give some related results.

Recently W. Lu, Lin Wu et al., continue discussing several known results for difference or differential equations obtained the following result.

**Theorem I.** *Let $L(z,f)$ denote a difference-differential polynomial of $f$ of degree one with small functions as its coefficients such that $L(z,0) \equiv 0$, and let $\rho_1, \rho_2, \alpha_1, \alpha_2$ be non zero constants such that $\alpha_1 \neq \alpha_2$. If $f$ is an entire solution with $\rho_2 < 1$ to the following equation*

$$f^3 + L(z,f) = \rho_1 e^{\alpha_1 z} + \rho_1 e^{\alpha_2 z} \tag{2.1}$$

*then, one of the following relations holds:*

1. *$f(z) = k_1 e^{\frac{\alpha_1 z}{3}} + k_2 e^{\frac{\alpha_2 z}{3}}$, where $k_1$ and $k_2$ are two non zero constants satisfying $k_1^3 = \rho_1$ $k_2^3 = \rho_2$ and $\alpha_1 + \alpha_2 = 0$*
2. *$f^3(z) = (\rho_1 - k_1)e^{\alpha_1 z}$, and $L(z,f) = k_1 e^{\alpha_1 z} + \rho_1 e^{\alpha_2 z}$, where $k_1$ is a constant.*
3. *$f^3(z) = (\rho_2 - k_2)e^{\alpha_2 z}$, and $L(z,f) = \rho_2 e^{\alpha_1 z} + k_1 e^{\alpha_2 z}$, where $k_2$ is a constant.*

In the present paper we include several known results for difference or differential equations obtained earlier as its special case. In fact, we consider a slightly more general form of (2.1) by replacing $f^3$ by $f^n$ in Theorem I and prove the following results.

**Theorem II.** *Let $L(z,f)$ denote a delay differential of $f$ of degree one with small functions as its coefficients such that $L(z,0) \equiv 0$.*

*Let $n \geq 4$ be an integer, and $\rho_1, \rho_2, \alpha_1, \alpha_2$ be non zero constants such that $\alpha_1 \neq \alpha_2$. If there exists some entire solution $f$ of finite order to (2.2) below.*

$$f^n + L(z, f) = \rho_1 e^{\alpha_1 z} + \rho_2 e^{\alpha_2 z}. \tag{2.2}$$

*Then, one of the following relations holds:*

1 $f(z) = k_1 e^{\frac{\alpha_1 z}{n}}$ *and* $k_1(e^{\frac{\alpha_1}{n}} - 1) = \rho_2, \alpha_1 = n\alpha_2$*;*

2 $f(z) = k_2 e^{\frac{\alpha_2 z}{n}}$ *and* $k_2(e^{\frac{\alpha_2}{n}} - 1) = \rho_1, \alpha_2 = n\alpha_1$*, where* $k_1, k_2$ *are constants satisfying* $k_2^n = \rho_2$*.*

## 3. Preliminaries

In order to prove our conclusions, we need some lemmas.

**Lemma 3.1** ([7])**.** *Let $f$ be a transcendental meromorphic solution of finite order $\rho$ of a delay equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

*where $H(z, f), P(z, f), Q(z, f)$ are delay polynomials in $f$ such that the total degree of $H(z, f)$ in $f$ and its shifts is $n$, and that the corresponding total degree of Q(z,f) is $\leq n$. If $H(z, f)$ contains just one term of a maximal total degree, then for any $\epsilon > 0$*

$$m(r, P(z, f)) = O(r^{\rho-1+\epsilon}) + S(r, f),$$

*possibly outside of an exponential set of finite logarithmic measure.*

**Remark 3.1.** The following result is a clunie type Lemma 3.1 ([11]) for the delay-differential polynomial of a meromorphic function $f$. It can be proved by applying Lemma 3.1 ([7]) with a suitable reasoning as in [10] and stated as follows.

Let $f(z)$ be a meromorphic function of finite order, and let $P(z, f), Q(z, f)$ be two delay differential polynomials of $f$. If

$$f^n P(z, f) = Q(z, f)$$

holds and if the total degree of $Q(z, f)$ in $f$ and its derivatives and their shifts is $\leq n$, then $m(r, P(z, f)) = S(r, f)$.

**Lemma 3.2** ([12])**.** *Suppose that $m, n$ are positive integer satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there exist no transcendental entire solutions of $f$ and $g$ satisfying the equation $af^n + bg^m = 1$, with $a, b$ being small function of $f$ and $g$ respectively.*

**Lemma 3.3** ([13])**.** *Assume that $c \in \mathbb{C}$ is a non zero constant, $\alpha$ is a non constant meromorphic function. Then the differential equation $f^2 + (c(f^{(n)}))^2 = \alpha$ has no transcendental meromorphic solutions satisfying $T(r, \alpha) = S(r, f)$.*

**Lemma 3.4** ([14, 15])**.** *Let $f$ be a transcendental meromorphic function of finite order $\rho$, then for any complex numbers $d_1, d_2$ and for each*

$$\epsilon > 0, m(r, \frac{f(z+d_1)}{f(z+d_2)}) = O(r^{\rho-1+\epsilon}).$$

References [16] and [17] further pointed out the following.

**Remark 3.2.** If $f$ is non constant finite order meromorphic function and $c \in \mathbb{C}$ then

$$m(r, \frac{f(z+d)}{f(z)}) = S(r, f) \tag{3.1}$$

outside of a possible exceptional set with finite logarithmic measure.

## 4. Proof of Theorem II

Suppose that $f$ is a transcendental entire function of finite order to (2.2)

$$f^n + L(z, f) = \rho_1 e^{\alpha_1 z} + \rho_2 e^{\alpha_2 z}.$$

By differentiating both sides of (2.2), we have

$$nf^{n-1}f' + L'(z, f) = \alpha_1\rho_1 e^{\alpha_1 z} + \alpha_2\rho_2 e^{\alpha_2 z}. \tag{4.1}$$

From (2.2) and (4.1), we obtain

$$\alpha_2 f^n + \alpha_2 L(z, f) - nf^{n-1}f' - L'(z, f) = (\alpha_2 - \alpha_1)\rho_1 e^{\alpha_1 z}, \tag{4.2}$$

$$\alpha_1 f^n + \alpha_1 L(z, f) - nf^{n-1}f' - L'(z, f) = (\alpha_1 - \alpha_2)\rho_2 e^{\alpha_2 z}. \tag{4.3}$$

Differential (4.2) yields

$$\begin{aligned} &n\alpha_2 f^{n-1}f' + \alpha_2 L'(z, f) - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f'' - L'(z, f) \\ &\quad = \alpha_1(\alpha_2 - \alpha_1)\rho_1 e^{\alpha_1 z}. \end{aligned} \tag{4.4}$$

It follows from (4.2) and (4.4) that

$$f^{n-2}\varphi = U(z, f), \tag{4.5}$$

where

$$\begin{aligned} &\varphi = \alpha_1\alpha_2 f^2 - n(\alpha_1 + _2)f'f + n(n-1)(f')^2 + nf''f, \\ &U(z, f) = -\alpha_1\alpha_2 L(z, f) + (\alpha_1 + \alpha_2)L'(z, f) - L''(z, f). \end{aligned}$$

Next, we shall prove $\varphi \equiv 0$. In fact, since $U$ is a delay-differential polynomial in $f$, and its degree at most 1. By (4.5) and Remark 3.1 after Lemma 3.1, we have $m(r, \varphi) = S(r, f)$, and $T(r, \varphi) = S(r, f)$, on the other hand, we can rewrite (4.5) as

$$f^{n-3} f\varphi = U(z, f),$$

which implies that

$$m(r, f\varphi) = S(r, f)$$

and $T(r, f\varphi) = S(r, f)$.

If $\varphi \not\equiv 0$, then $T(r, f) = T(r, \frac{f\varphi}{\varphi}) = S(r, f)$ and this is impossible. Hence, $\varphi \equiv 0$, and $T \equiv 0$ i.e,

$$-\alpha_1\alpha_2 L(z, f) + (\alpha_1 + \alpha_2)L^{'}(z, f) - L^{''}(z, f) \equiv 0. \tag{4.6}$$

If $L(z, f) \equiv 0$, then (2.2) can be rewritten as $\frac{1}{\rho_1}(fe^{\frac{-\alpha_1 z}{n}})^n - \frac{\rho_2}{\rho_1}(e^{\frac{1}{3}(\alpha_2 - \alpha_1)z})^3 = 1$, which is impossible by Lemma 3.2. Thus, $L(z, f) \not\equiv 0$. It is easily seen from $\alpha_1 \neq \alpha_2$ that

$$\alpha_1 L(z, f) - L^{'}(z, f) \equiv 0$$

and

$$\alpha_2 L(z, f) - L^{'}(z, f) \equiv 0,$$

cannot hold simultaneously.

First of all, we assume that $\alpha_2 L(z, f) - L^{'}(z, f) \not\equiv 0$ then (4.6) given $\alpha_2 L(z, f) - L^{'}(z, f) = Ae^{\alpha_1 z}$, where $A$ is a non zero constant, substituting the above expression into (4.2), we obtain

$$f^{n-1}(\alpha_2 f - nf^{'}) = \frac{(\alpha_2 - \alpha_1)P_1 - A}{A}[\alpha_2 L(z, f) - L^{'}(z, f)].$$

Again, Lemma 3.1 shows that $\alpha_2 f - nf^{'} \equiv 0$, $(\alpha_2 - \alpha_1)\rho_1 = A$. So

$$f^n = Be^{\alpha_2 z}, \tag{4.7}$$

where $B$ is constant.

Substituting (4.7) in (2.2), we find

$$(1 - \frac{P_2}{B})f^n = \frac{\rho_1(\alpha_2 L(z, f) - L^{'}(z, f))}{A} - L(z, f).$$

If $B \neq P_2$, by Lemma 3.1, we get $T(r, f) = m(r, f) = S(r, f)$, which is absurd, so $B = P_2$, and $f = k_2 e^{\frac{\alpha_2 z}{n}} = k_2 e^{\alpha_1 z}$, $k_2^n = B = \rho_2$, $k_2(e^{\frac{\alpha_2}{n}} - 1) = \rho_1$ If $\alpha_1 L(z, f) - L^{'}(z, f) \not\equiv 0$, by (4.3) and using similar arguments as above, we can derive $f = k_1 e^{\frac{\alpha_1 z}{n}} = k_1 e^{\alpha_2 z}$, $k_1^n = \rho_1$, $k_1(e^{\frac{\alpha_1}{n}} - 1) = \rho_2$.

This conclude the verification of Theorem II.

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