On purity and divisibility of S-acts

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Abstract. In this paper we study purity of S-acts where S is a special type of monoid of the form $S = G \dot{\cup} I$, in which G is a group and I is an ideal of S. Here every S-act can be naturally considered as an I^1-act. So the question is that what is the relationship between purity and divisibility of I^1-acts and purity and divisibility of S-acts?

We respond to this question and show that divisibility of an S-act A is extendable from I^1-acts to S-acts. Also, we show that every consistent system of the third type of equations has a solution in A whenever every consistent system of the third type of equations over I^1-act A has a solution in A.
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1. Introduction and preliminaries

The notion of acts of a semigroup or a monoid over a set is one of the very useful notions in many branches of mathematics as well as in computer science which captures the interest of many mathematicians and computer scientists, [2, 5, 6, 7, 8, 9, 11].

Nowadays purity plays a role in different branches of mathematics such as Module Theory, Model Theory, and Category Theory, see [1, 17, 3]. Especially this notion has been studied on G-acts, acts over a monoid or a group G by some authors including Banaschewski, Gould, and Normak [3, 16, 9, 10].

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Divisible act first introduced by Feller and Gantos [6], and later studied by many other authors, see [2, 5, 6, 12, 18].

A. Golchin and J. Renshaw in [7, 8], have studied on acts over a monoid of the form $S = G\dot{\cup}I$, in which G is a group and I is an ideal of S. They show that flatness for these kinds of actions is extendable from I^1-act to S-acts. Meaning that, an S-act A is flat if it is flat as an I^1-act. Thus it is a natural question to ask that: which algebraic properties are extendable from I^1-acts to S-acts, when $S = G\dot{\cup}I$?

Here we answer to this question concerning purity and divisibility. Indeed, we show that divisible property is extendable from I^1-acts to S-acts in general. Also, we show that every consistent system of the third type of equations has a solution in A whenever every consistent system of the third type of equations over I^1-act A has a solution in A. To do so, first we briefly recall some necessary notions.

Given a monoid S, a (right) S-act is a set A together with an action $A \times S \to A$, mapping each (a, s) to as, such that $(i)\,(as)t = a(st)$ and $(ii)\,a1 = a$, for every $a \in A$, $s, t \in S$, in which 1 is the identity element of the monoid S. One can naturally consider the monoid S as an S-act. A subset B of an S-act A is called an *S-subact* of A, denoted by $B \leq A$, whenever $bs \in B$, for every $b \in B$ and $s \in S$. Especially, considering naturally S as an S-act, the S-subacts of S are exactly the right ideals of S. A map $f : A \to B$ between two S-acts A and B is called an *S-homomorphism* or briefly a *homomorphism* if, for each $a \in A$, $s \in S, f(as) = f(a)s$. The usual definitions for *S-monomorphisms*, *S-epimorphisms* and *S-isomorphisms* hold. We denote the *category* of all S-acts and S-homomorphisms between them by **Act-S**.

A non-empty subset $K \subseteq S$ is called an ideal if $KS \subseteq K$ and $SK \subseteq K$ and K is said to be *finitely generated* ideal if $K = \bigcup_{i=1}^{n} s_iS = \bigcup_{i=1}^{n} Ss_i$, for some $n \in \mathbb{N}$ and $s_i \in S$. Also, K is said to be *principal* if $K = sS = Ss$, for some $s \in S$.

An S-act A is called *injective* if for every S-monomorphism $i : B \rightarrowtail C$ and every S-homomorphism $f : B \to A$, there exists an S-homomorphism $\bar{f} : C \to A$ with $\bar{f}i = f$. Also an S-act A is called (*principally, finitely generated) weakly injective* if for every (principal, finitely generated) ideal K of S and any S-homomorphism $f : K \to A$, there exists an S-homomorphism extension $\bar{f}{:}S \to A$ of f, that is, $\bar{f}|_K = f$.

An element θ in an S-act A with $\theta s = \theta$, for all $s \in S$, is called a *zero* or a *fixed element* of A. An element $e \in S$ is called *idempotent* if $e^2 = e$. An element $s \in S$ is called *left cancellable* if $sr = st$, for $r, s \in S$, implies $r = t$. An element $z \in S$ is called *left zero*, if $zs = z$, for all $s \in S$.

The notions of purity and equational compactness of universal algebras have been studied by Banaschewski and Nelson [4]. Also, Banaschewski deals with these notions in the special case of G-sets for a group G [3]. Indeed, a *free extension* of an algebra A by a set X is an algebra E containing (a copy of) A as a subalgebra, and (a copy of) X as a subset, such that any given homomorphism

$h : A \to C$ and map $u : X \to |C|$, from X to the underlying set of an algebra C, uniquely extend to a homomorphism $f : E \to C$. The free extension $A[X]$ of an algebra A by a set X exists (uniquely up to isomorphism). It is in fact $A \coprod F(X)$, where $F(X)$ is the free algebra over X. In the case of S-acts, $A[X] = A \bigcup (X \times S)$, where the coproduct is merely the disjoint union.

Recall that, any pair $(u, v) \in A[X]^2$ is called an *equation* over A, and is written as $u = v$. The formal definition of a *solution* of an equation $u = v$ over A in an extension B of A is a homomorphism $f : A[X] \to B$ such that $f|_A$ is the inclusion $A \subseteq B$, and $f(u) = f(v)$.

An algebra A is called *pure* in an extension B of A if any system of finitely many equations over A has a solution in A whenever it has a solution in B.

Throughout this paper we take S to be a monoid of the form $S = G\dot{\cup}I$, where G is a group and I is an ideal of S and $I^1 = I\dot{\cup}\{1\}$. Indeed, the group actions, because of the properties of the group, give very special properties. For instance, the group actions with a zero element are injective, Theorem 3.1.9 of [13], or the group actions are absolutely 1-pure, Corollary 3.11 of [9]. Here we are going to extend these properties from the group actions to the acts over a monoid of the form $S = G\dot{\cup}I$, where G is a group and I is an ideal of S and $I^1 = I\dot{\cup}\{1\}$. It is worth noting that, in these kinds of monoids, since I^1 is a submonoid of S, every S-act can be considered as an I^1-act. And also we should note that, for the semigroup $S = G\dot{\cup}I$, the ideal I is a maximal ideal and all the proper ideals of S are included in I. These kinds of monoids are not a few and a large number of monoids can be considered as this form, see the following examples.

Examples:

(i) Every field F is of the form $F = F\backslash\{0\}\cup\{0\}$ with the usual multiplication.

(ii) Every commutative monoid M is of the form $M = G\dot{\cup}I$, in which G is the subset of invertible elements of S and $I = S \setminus G$.

(iii) The non-commutative monoid $S = (M_n(\mathbb{R}), .)$, consisting of the $n \times n$ square matrices with the real number entries, together with the usual multiplication is of the form $S = G\dot{\cup}I$, in which G is the subset of invertible matrices of S and $I = S \setminus G$.

2. When 1-purity is extendable

Concerning the category of S-acts, it is shown that the equations over an S-act A is of one the following three types:

$$xs = yt, \quad xs = xt, \quad xs = a,$$

where $s, t \in S, a \in A$, see [9, 14, 16]. This section is concerned with the equations of the form $xs = a$. Indeed, we show that every consistent system of equations in the form of $xs = a$ over an S-act A has a solution in A whenever every consistent system of equations in the form of $xi = a$ over I^1-act A has a solution in A. But first we give the following definition.

Definition 2.1. (*i*) *A system of equations $\sum$ over an S-act A is said to be consistent if it has a solution in some extension of A.*

(*ii*) *An S-act A is called absolutely pure if every finite consistent system of equations over A has a solution in A.*

(*iii*) *An S-subact A of an S-act B is called* 1*-pure in B if every finite system of equations in one variable over A, that has a solution in B has a solution in A.*

(*iv*) *An act A is called absolutely* 1*-pure if every finite consistent system of equations in one variable over A has a solution in A.*

Lemma 2.1. *Let S be a group and let A be an S-act with a unique zero element θ. Then, A is absolutely pure.*

Proof. By Theorem 3.1.9 of [13], we know that an S-act with a zero element over a group is injective. Now since every injective S-act is absolutely pure, by Proposition 3.6.7 of [13], we get the result. $\square$

Lemma 2.2 ([9])**.** *Let A be an S-act and let $\sum = \{xs_i = a_i \mid s_i \in S, a_i \in A, 1 \leq i \leq n\}$ be a finite system over A. Then the following conditions are equivalent:*

1. *$\sum$ is consistent.*
2. *For all elements h, k of S if $s_i h = s_j k$ then $a_i h = a_j k$.*

Theorem 2.1. *Let $S = G \dot{\cup} I$ be a monoid and A be an S-act. Then, every finite consistent system $\{xi_k = a_k, i_k \in I^1, a_k \in A, 1 \leq k \leq n\}$ has a solution in A if and only if every finite consistent system $\{xs_k = a_k, s_k \in S, a_k \in A, 1 \leq k \leq n\}$ has a solution in A.*

Proof. The sufficiency part is clear. For the necessity part, first we show that A is finitely generated weakly injective I^1-act. To do so, let $\sum' = \{xi_k = a_k, i_k \in I^1, a_k \in A, 1 \leq k \leq m\}$ be a finite consistent system which has a solution in A. Suppose that J is an ideal of I^1 generated by $X' = \{i_1, ..., i_m\}$, and $f : J \to A$ is an I^1-homomorphism. For every $s, s' \in I^1$ and $i_k, i_{k'} \in X'$, suppose that $i_k s = i_{k'} s'$. Then, we have $f(i_k s) = f(i_{k'} s')$ and so $f(i_k)s = f(i_{k'})s'$. Thus, by Lemma 2.2, the system $\sum_1 = \{xi_k = f(i_k), i_k \in I^1, 1 \leq k \leq m\}$ is consistent. Therefore, by the hypothesis, $\sum_1$ has a solution a in A. That is, $ai_k = f(i_k)$, for every $i_k \in X'$. Since J is generated by X', for every $j \in J$, there exists $t \in I^1$ such that $j = i_k t$. Now, by the hypothesis, we have:

$$f(j) = f(i_k t) = f(i_k)t = (ai_k)t = a(i_k t) = aj.$$

That is A is finitely generated weakly injective as an I^1-act, see Lemma 3.5.2 of [13]. Hence, A is finitely generated weakly injective as an S-act, by Theorem 2.4 of [15].

Now let $\sum = \{xs_i = a_i, s_i \in S, a_i \in A, 1 \leq i \leq n\}$ be a finite consistent system, and K be the generated ideal by $X = \{s_1, ..., s_n\}$ of S. Consider the map $g : K \to A$, defined by $g(s_i t) = a_i t$, for any $s_i \in X, t \in S$. The defined g is well defined because if $s_i t = s_j t'$ then since $\sum$ is consistent system, by Lemma 2.2, for any $t, t' \in S$, we have $a_i t = a_j t'$. Also g is an S-homomorphism, since for any $s, t \in S$ and $s_i \in X$ we have:

$$g((s_i t)s) = g(s_i(ts)) = a_i(ts) = (a_i t)s = g(s_i t)s.$$

So, there exists an S-homomorphism $\bar{g} : S \to A$, shuch that the following diagram is commutative:

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "g"'] & S \arrow[dl, dotted, "\bar{g}"] \\ A & \end{tikzcd}$$

Now, for any $s_i \in X$, we have $\bar{g}(1)s_i = \bar{g}(s_i) = g(s_i) = a_i$. So, $\bar{g}(1) \in A$ is a solution of $\sum$.

□

Lemma 2.3. *Given a soluble system of equations over A in the form $\sum = \{xs = a_s, s \in S, a \in A\}$, has a solution in A if and only if the associated map $f : S \to A$, $f(s) = a_s$, is an S-homomorphism.*

Proof. Let $\sum$ be soluble on A and $a' \in A$ be a solution of $\sum$. Then, for each equation $xs = a_s$ in $\sum$, we have $a's = a_s$. Suppose that $s, t \in S$ and $s = t$ then, we have $a's = a't$ and hence $a_s = a_t$, that is, f is well-defined.

Also, we have:

$$f(st) = a_{st} = a'(st) = (a's)t = a_s t = f(s)t,$$

for every $s, t \in S$. Therefore, f is an S-homomorphism.

Conversely, suppose that $f : S \to A$ is an S-homomorphism. Then, $f(1)$ will be the desired solution. Because $f(1)s = f(s) = a_s$, for every $s \in S$. □

In Theorem 2.2 we use Lemma 2.3 and show that every consistent system of the third type of equations has a solution in A whenever every consistent system of the third type of equations over I^1-act A has a solution in A.

Theorem 2.2. *Let $S = G\dot{\cup}I$ be a monoid and A be an S-act. Then, every consistent system $\{xi = a_i, i \in I^1, a_i \in A\}$, has a solution in A if and only if every consistent system $\{xs = a_s, s \in S, a_s \in A\}$, has a solution in A.*

Proof. *Necessity.* By Lemma 2.3 for a given system $\sum = \{xs = a_s, s \in S, a_s \in A\}$, it is enough to show that the associated map $f : S \to A$, $f(s) = a_s$, is an S-homomorphism. By the hypothesis, the subsystem $\sum_1 = \{xi = a_i, i \in I^1, a_i \in A\} \subseteq \sum$, has a solution in A. So, the associated map $f|_{I^1} = f_1 : I^1 \to A$,

$f_1(i) = a_i$, is an I^1-homomorphism, by Lemma 2.3. Also, we define the map $f|_G = f_2 : G \to A$ by $f_2(g) = f_1(1)g$. For every $g_1, g_2 \in G$ we have $f_2(g_1.g_2) = f_1(1)g_1.g_2 = (f_1(1)g_1)g_2 = f_2(g_1)g_2$. So f_2 is a G-homomorphism.

Now, patch f_1, f_2 together and we obtain $f : S \to A$ as follows:

$$f(s) = \begin{cases} f_1(s), & s \in I \\ f_2(s), & s \in G. \end{cases}$$

Obviously, f is well-defined. To verify that f is an S-homomorphism it is enough to investigate the following two cases.

1. If $s \in I, t \in G$, then $f(st) = f_1(st) = f_1(1)st = (f_1(1)s)t = f_1(s)t = f(s)t$.

2. If $s \in G, t \in I$, then $f(st) = f_1(st) = f_1(1)st = (f_1(1)s)t = f_2(s)t = f(s)t$.

Sufficiency. is obvious. □

Definition 2.2 ([13]). *An idempotent $e \in S$ is called right special (right fg-special) if for any (finitely generated) right congruence ρ of S there exists an element $k \in eS$ such that*

(1) $(ke)\rho e$,

(2) $u\rho v$, $u, v \in S$, *implies* $(ku)\rho(kv)$.

V. A. R. Gould in 1987 showed that if all S-acts are absolutely 1-pure, then all S-acts are absolutely pure, see Proposition 3.3 of [10]. Here using this Proposition we give a stronger form of this Proposition when S is of the form $S = G\dot{\cup}I$. Indeed, we show that absolutely 1-purity of all I^1-acts implies that all S-acts are absolutely pure when $S = G\dot{\cup}I$, see the following theorem.

Theorem 2.3. *Let $S = G\dot{\cup}I$ be a monoid whose idempotents are central. If all I^1-acts are absolutely* 1*-pure, then all S-acts are absolutely pure.*

Proof. Suppose that all I^1-acts are absolutely 1-pure. Then, S is absolutely 1-pure as an I^1-act. So, S has local left zeros by Lemma 3.6.6 of [13]. Also, every finitely generated ideal of I^1 is generated by right fg-special idempotent by Proposition 3.3 of [10]. Since I^1 contained all ideals and all idempotents of S, and since idempotents of S are central, every finitely generated ideal of S is generated by a central idempotent. Also since every central idempotent is right fg-special, every finitely generated right ideal of S generated by right fg-special. Hence, all S-act are absolutely pure by Proposition 3.3 of [10]. □

The above theorem gives a useful criterion to find the absolutely pure S-acts, where $S = G\dot{\cup}I$. Specially, if $S = G\dot{\cup}\{0\}$, then clearly idempotent elements are central. Now, the above theorem ensures that all S-acts are absolutely pure if all $\{0, 1\}$-acts are absolutely 1-pure. See the following example.

Example 2.1. (1) All the $\mathbb{Q}$-acts are absolutely pure, in which $\mathbb{Q}$ is the rational numbers with the usual multiplication. Indeed, one can consider $\mathbb{Q} = (\mathbb{Q} - \{0\}\dot{\cup}\{0\}$, , where $G = \mathbb{Q} - \{0\}$ is a group, $I = \{0\}$ is an ideal of S, and $I^1 = \{0, 1\}$. It worths noting that for any $a \in A$, every one variable system of equations over A, is in the form of $\sum = \{x.0 = \theta, x.1 = a, 0, 1 \in I^1\}$, which $a \in A$ is a solution of $\sum$. That is, A is absolutely 1-pure as an I^1-act and hence all I^1-acts are absolutely 1-pure.

(2) Analogously, one can see that all the $\mathbb{R}$-acts are absolutely pure, in which $\mathbb{R}$ is the monoid of the real numbers with the usual multiplication.

3. Divisibility is extendable

Recall that an element $a \in A$ is called *divisible by* $s \in S$ if there exists $b \in B$ such that $bs = a$. An S-act A is said to be divisible if $Ac = A$, for any left cancellable element $c \in S$. Also, it is known that every absolutely 1-pure S-act is divisible, Proposition 3.6.15 and Corollary 3.3.3 of [13]. In this short section, we focus on the divisible S-acts and we show that, for the monoid $S = G\dot{\cup}I$, an S-act A is divisible if it is divisible as an I^1-act. We then give a criterion to recognize divisible S-acts.

Theorem 3.1. *Let $S = G\dot{\cup}I$ be a monoid. Then A is divisible as an S-act whenever it is divisible as an I^1-act.*

Proof. Given a left cancellable element $c \in S$ and $a \in A$, we have to find $b \in A$ with $bc = a$. But since S is of the form $S = G\dot{\cup}I$, two cases may occur for c. (1) If $c \in I$ then, by the hypothesis, we get the result. (2) If $c \in G$ then, for every $a \in A$, we have $a = (ac^{-1})c$. That is, A is divisible as an S-act. $\square$

Corollary 3.1. *Let $S = G\dot{\cup}I$ be a monoid. Then divisibility of all I^1-acts implies the divisibility of all S-acts.*

By the above theorem we get a useful criterion to check divisibility of S-acts. See the following examples.

Example 3.1. (1) Let $S = G\dot{\cup}\{0\}$ be a monoid. Then, an S-act is divisible, if it is divisible as an $\{0, 1\}$-act.

(2) Let $S = (\mathbb{Q}, \cdot)$ be a monoid of all rational numbers with the usual multiplication and A be an I^1-act. Consider $S = (\mathbb{Q} - \{0\})\dot{\cup}\{0\}$, where $G = \mathbb{Q} - \{0\}$ is a group and $I = \{0\}$ is an ideal of S. Since the only left cancellable element in the monoid $I^1 = \{0, 1\}$ is 1, implies that A is divisible as an I^1-act, and so all I^1-acts are divisible. Hence, by the above theorem all $\mathbb{Q}$-acts are divisible.

(3) Analogously, one can see that all $\mathbb{R}$-acts are divisible, in which $\mathbb{R}$ is the monoid of all real numbers with usual multiplication, $(\mathbb{R}, .)$.

Theorem 3.2. *Let $S = G\dot{\cup}I$ be a monoid and I does not contain any left cancellable element of S. Then every S-act is divisible.*

Proof. Given an S-act A, consider a left cancellable element $c \in S$ and $a \in A$. By the hypothesis, since I does not contain any left cancellable element of S, $c \in G$. So, we have $a = a(c^{-1}c) = (ac^{-1})c \in Ac$. Therefore, $A \subseteq Ac$. Also clearly $Ac \subseteq A$, and hence $A = Ac$. That is, A is divisible as an S-act. □

Definition 3.1 ([13]). *An element $s \in S$ is called right almost regular if there exist elements $r, r_1, ..., r_n, s_1, ..., s_n \in S$ and left cancellable elements $c_1, ..., c_n \in S$ such that $r_1 s = c_1 s_1$, $r_2 s_1 = c_2 s_2$, ..., $r_n s_{n-1} = c_n s_n$, $s = srs_n$. If all elements of S are almost regular then S is called a (right) almost regular monoid.*

Corollary 3.2. *If I^1 is an almost regular monoid and I does not contain any left cancellable element of S. Then, every S-act is principally weakly injective.*

Proof. The result follows directly by Theorem 3.2, and Theorem 4.1.5 of [13]. □

Example 3.2. For all the monoids in the form of $S = G\dot{\cup}\{0\}$, (for example, all fields with the usual multiplication, $(F, \cdot)$), clearly $I = \{0\}$, does not contain any left cancellable element of S and $I^1 = \{0, 1\}$ is an almost regular monoid. So, by the above theorem every F-act is principally weakly injective and divisible, in which F is the field with usual multiplication, $(F, \cdot)$.

By Proposition 3.3.2 of [13], we know that every principally weakly injective act is divisible. But there exist examples (see, example 3.3.11 of [13]), which show that the converse is not true. In the following theorem we show that if I^1 is a left cancellable principal right ideal monoid then divisibility as an I^1-act leads to weakly injectivity as an S-act.

Theorem 3.3. *Let $S = G\dot{\cup}I$, A be an S-act and I^1 be a left cancellable principal ideal monoid. Then, divisibility of A as an I^1-act implies weakly injectivity of A as an S-act.*

Proof. First we show that A is weakly injective as an I^1-act. To do so, suppose that K is an ideal of I^1 and $f : K \to A$ is an I^1-homomorphism. By the hypothesis, there exists $i \in I^1$ such that $K = iI^1$. So, for every $x \in K$, there exists $j \in I^1$ such that $x = ij$. Since $f(i) \in A$ and A is divisible as an I^1-act, and I^1 is a left cancellable, there exists $a \in A$ such that $f(i) = ai$. Now, we have:

$$f(x) = f(ij) = f(i)j = (ai)j = a(ij) = ax,$$

for every $x \in K$. This means that A is weakly injective as an I^1-act, by Lemma 3.5.2 of [13]. Hence, A is weakly injective as an S-act, by Theorem 2.4 of [15]. □

References

[1] J. Adamek, J. Rosicky, *Locally presentable and accessible categories*, Cambridge University Press, Cambridge, 1994.

[2] J. Ahsan, M. F. Khan, M. Shabir, M. Takahashi, *Characterizations of monoids by p-injective and normal S-systems*, Kobe J. Math., 8 (1991), 173-190.

[3] B. Banaschewski, *Equational compactness of G-sets*, Canad. Math. Bull., 17 (1974), 11-18.

[4] B. Banaschewski, E. Nelson, *Equational compactness in equational classes of algebras*, Algebra Universalis, 2 (1972), 152-165.

[5] M. P. Dorofeeva, *Injective and flat acts over hereditary monoids*, Vestnik Moskov. Univ., 1 (1973), 47-51.

[6] E. H. Feller, R. L. Gantos, *Indecomposable and injective S-systems with zero*, Math. Nachr., 41 (1969), 37-48.

[7] A. Golchin, *On flatness of acts*, Semigroup Forum, 67 (2003), 262-270.

[8] A. Golchin, *Periodic monoids over which all flat cyclic right acts satisfy condition* (P), Semigroup Forum, 54 (1997), 261-263.

[9] V. Gould, *The characterization of monoids by properties of their S-systems*, Semigroup Forum, 32 (1985), 251-265.

[10] V. Gould, *Completely right pure monoids*, Proc. Roy. Irish Acad. Sect A., 87 (1987), 73-82.

[11] V. Gould, *Axiomatisability problems for S-systems*, Comment. Math. Univ. Carolinae., 19 (1978), 27-35.

[12] V. Gould, *Divisible S-systems and R-modules*, Edinburgh Math. Soc., II., 30 (1987), 187-200.

[13] M. Kilp, U. Knauer, A. Mikhalev, *Monoids, acts and categories*, Walter de Gruyter, Berlin, New York, 2000.

[14] M. Mahmoudi, M. M. Ebrahimi, *Purity and equational compactness of projection algebras*, Appl. Categ. Struc., 9 (2001), 381-394.

[15] Gh. Moghaddasi, M. Haddadi, S. Delavary, *On injectivity of S-acts*, Scientific Bulletin, Series A: Applied Mathematics and Physics, 79 (2017), 189-198.

[16] P. Normak, *Purity in the category of M-sets*, Semigroup Forum, 20 (1980), 157-170.

[17] P. Rothmaler, *Advances in algebra and model theory*, CRC Press, London, 1997.

[18] M. Satyanarayana, *Quasi- and weakly-injective S-systems*, Math. Nachr., 71 (1976), 183-190.

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