On fixed point results for family of multivalued mapping with application

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Abstract. The purpose of this article is to find out some fixed point results for the family of multifunctions on proximinal sets in complete dislocated metric space. We also presented an example which shows he novelty of our results. An application to the system of nonlinear integral equations has been presented. Our results combine, extend and infer several comparable results in the literature.

Keywords: fixed point, proximinal sets, family of multifunctions, dislocated metric space.

1. Introduction

Fixed point theory has a wide range of applications in the different fields of analysis. The most important tool in fixed point theory is Banach contraction principle [5]. Many authors obtained fixed point results in various metric spaces under certain contractive conditions [1]-[22]. A new contraction called $F-$contraction was establish by Wardowski [22] which generalized the several well known results in different aspects. Afterwards, Secelean [17] proved fixed point results which contain $F-$contractions by the systems of iterated function. Piri et al. [12] proved a fixed point result for $F-$Suzuki contractions on the self map in a complete metric space. The notion of generalized set-valued $F-$contraction mappings was introduced by Acar et al. [3]. The extention of setvalued $F-$contraction was given by Acar et al. [2] and in complete metric space they obtained several fixed point results. There are many interesting results on $F-$contractions which can be seen in [13, 14, 15, 16, ?].

In this article, we obtain some fixed point results for the family of multifunctions on proximinal sets in complete dislocated metric space.

Definition 1.1 ([7]). *Let W be a nonempty set. A mapping $d_l : W \times W \to [0, \infty)$ is called a dislocated metric (or simply d_l-metric) if the followings hold, for any $u, v, w \in W$:*

(i) If $d_l(u, v) = 0$, then $u = v$;

(ii) $d_l(u, v) = d_l(v, u)$;

(iii) $d_l(u, v) \leq d_l(u, w) + d_l(w, v)$.

The pair (W, d_l) is called dislocated metric space (D.M.S) or d_l metric space. It can be seen for $d_l(u, v) = 0$, then by above condition (i), $u = v$. But if $u = v$, then $d_l(u, v)$ may not be 0. For example, if $W = \mathbb{R}^+ \cup \{0\}$, then $d_l(u, v) = u + v$ defines a d_l metric on W.

Definition 1.2 ([20]). *Let M be any nonempty subset of D.M.S in W and let $u \in W$. An element $m_0 \in M$ is called a best approximation in M if*

$$d_l(u, M) = d_l(u, m_0), \text{ where } d_l(u, M) = \inf_{m \in M} d_l(u, m).$$

If every $u \in W$ has at least one best approximation in M, then M is called a proximinal set. Here $P(W)$ denotes the collection of all closed proximinal subsets of W.

Definition 1.3 ([20]). *The function* $H_{d_l} : P(W) \times P(W) \to \mathbb{R}^+$, *defined by*

$$H_{d_l}(C, D) = \max\{\sup_{c \in C} d_l(c, D), \ \sup_{d \in D} d_l(C, d)\}$$

is called d_l *Hausdorff metric on* $P(W)$.

Definition 1.4 ([22]). *Suppose* (W, d) *is a metric space and* $H : W \to W$ *is a self mapping. Then* H *is called* F *contraction if there exists* $\tau > 0$ *such that*

$$\forall j, k \in W, \ d(Hj, Hk) > 0 \Rightarrow \tau + F\left(d(Hj, Hk)\right) \leq F\left(d(j, k)\right), \tag{1.1}$$

where F *from* $\mathbb{R}_+$ *to* $\mathbb{R}$ *is a mapping satisfying the followings:*

(C_1) F *is strictly increasing, that is for every* $j, k \in \mathbb{R}_+$ *such that* $j < k$, $F(j) < F(k)$;

(C_2) *For every sequence* $\{\alpha_m\}_{m=1}^{\infty}$ *of positive numbers,* $\lim_{m\to\infty} \alpha_m = 0$ *if and only if* $\lim_{m\to\infty} F(\alpha_m) = -\infty$;

(C_3) *There exists* $r \in (0, 1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^r F(\alpha) = 0$.

We denote by $\triangle_F$, *the set of all functions satisfying the conditions* $(C_1) - (C_3)$.

Definition 1.5 ([22]). *The family of* $\mathcal{F}$ *is not empty.*

(1) $F(u) = \ln(u); u > 0$.
(2) $F(u) = u + \ln(u); u > 0$.
(3) $F(u) = \frac{-1}{\sqrt{u}}; u > 0$.

Lemma 1.1 ([20]). *Let* (W, d_l) *be a D.M.S. Let* $(P(W), H_{d_l})$ *be a dislocated Hausdorff metric space on* $P(W)$. *For all* $C, D \in P(W)$ *and for any* $c \in C$ *such that* $d_l(c, D) = d_l(c, d_c)$ *where* $d_c \in D$. *Then* $H_{d_l}(C, D) \geq d_l(c, d_c)$ *holds.*

2. Main result

Let (W, d_l) be a D.M.S, $u_0 \in W$. Consider a family of multivalued mappings $\{S_\beta : \beta \in \Omega\} : W \to P(W)$. Let $u_1 \in S_a u_0$ for some $a \in \Omega$, be an element such that $d_l(u_0, S_a u_0) = d_l(u_0, u_1)$. Let $u_2 \in S_b u_1$ be such that $d_l(u_1, S_b u_1) = d_l(u_1, u_2)$. In the same way, we can construct a sequence u_n of points in W such that $u_{n+1} \in S_\gamma u_n$, $d_l(u_n, S_\gamma u_n) = d_l(u_n, u_{n+1})$, This iterative sequence is denoted by $\{WS_\beta(x_n) : \beta \in \Omega\}$ and called $\{WS_\beta(x_n) : \beta \in \Omega\}$ is a sequence generated by u_0 in W.

We begin with the following definition.

Definition 2.1. *Let* (W, d_l) *be a complete D.M.S. Consider a family of multivalued mappings* $\{S_\beta : \beta \in \Omega\} : W \to P(W)$ *and* $\{WS_\beta(u_n) : \beta \in \Omega\}$ *be a sequence in* W *generated by* u_0. *The family* $\{S_\beta : \beta \in \Omega\}$ *is called to be Ciric type rational multivalued* $F-$*contraction, if for all* $u, v \in \{WS_\beta(u_n)\}$, *we have*

$$\tau + F(H_{d_l}(S_i u, S_j v)) \leq F(D_l(u, v)) \tag{2.1}$$

where $F \in \triangle_F$ *and* $\tau > 0$, *and*

$$D_l(u,v) = \max\left\{ d_l(u,v), \frac{d_l(u,S_iu).d_l(v,S_jv)}{1+d_l(u,v)}, d_l(u,S_iu), d_l(v,S_jv) \right\}. \tag{2.2}$$

Theorem 2.1. *Let* (W,d_l) *be a complete D.M.S and* $\{S_\beta : \beta \in \Omega\}$ *be the family of Ciric type rational multivalued* $F-$*contraction. Then* $\{WS_\beta(u_n)\} \to u \in W$. *Moreover, if (2.1) also holds for* x, *then there is a common fixed point for the family of multivalued mappings* $\{S_\beta : \beta \in \Omega\}$ *in* $\{WS_\beta(u_n) : \beta \in \Omega\}$.

Proof. Suppose $\{WS_\beta(u_n) : \beta \in \Omega\}$ is a complete sequence in W generated by u_0. If $u_0 = u_1$, then u_0 is a common fixed point of S_a for all $a \in \Omega$. Let $D_l(u_0,u_1)) > 0$ for all $u_0, u_1 \in \{WS_\beta(u_n)\}$ with $u_0 \neq u_1$.

Then, from (2.1), and Lemma 1.1 we have

$$F(d_l(u_n,u_{n+1})) \leq F(H_{d_l}(S_iu_{n-1},S_ju_n)) \leq F(D_l(u_{n-1},u_n)) - \tau,$$

for all $i \in \mathbb{N} \cup \{0\}$, and

$$\begin{aligned} D_l(u_{n-1},u_n) &= \max\left\{ \begin{array}{c} d_l(u_{n-1},u_n), \dfrac{d_l(u_{n-1},S_iu_{n-1}).d_l(u_n,S_ju_n)}{1+d_l\left(u_{n-1},u_n\right)}, \\ d_l(u_{n-1},S_iu_{n-1}), d_l(u_n,S_ju_n) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d_l(u_{n-1},u_n), \dfrac{d_l\left(u_{n-1},u_n\right).d_l(u_n,u_{n+1})}{1+d_l\left(u_{n-1},u_n\right)}, \\ d_l\left(u_{n-1},u_n\right), d_l(u_n,u_{n+1}) \end{array} \right\} \\ &= \max\{d_l\left(u_{n-1},u_n\right), d_l(u_n,u_{n+1})\}. \end{aligned}$$

If, $D_l(u_{n-1},u_n) = d_l(u_n,u_{n+1})$, then

$$F(d_l(u_n,u_{n+1})) \leq F(d_l(u_n,u_{n+1})) - \tau.$$

By (F1) it is a contradiction. So,

$$F(d_l(u_n,u_{n+1})) \leq F(d_l\left(u_{n-1},u_n\right)) - \tau, \tag{2.3}$$

for every $i \in \mathbb{N} \cup \{0\}$. Likewise, we get

$$F(d_l\left(u_{n-1},u_n\right)) \leq F(d_l(u_{n-2},u_{n-1})) - \tau, \tag{2.4}$$

for every $i \in \mathbb{N} \cup \{0\}$. By (2.4) in (2.3), we get

$$F(d_l(u_n,u_{n+1})) \leq F(d_l(u_{n-2},u_{n-1})) - 2\tau.$$

In the same way, we have

$$F(d_l(u_n,u_{n+1})) \leq F(d_l(u_0,u_1)) - n\tau. \tag{2.5}$$

Taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} F(d_l(u_n, u_{n+1})) = -\infty. \tag{2.6}$$

Since $F \in \triangle_F$,

$$\lim_{n\to\infty} d_l(u_n, u_{n+1}) = 0. \tag{2.7}$$

By (2.5), for every $n \in \mathbb{N}$ we get

$$\begin{aligned} &(d_l(u_n, u_{n+1}))^k((F(d_l(u_n, u_{n+1})) - F(d_l(u_0, u_1))) \\ &\leq -(d_l(u_n, u_{n+1}))^k n\tau \leq 0. \end{aligned} \tag{2.8}$$

Taking (2.6), (2.7) and applying limit $n \to \infty$ in (2.8), we get

$$\lim_{n\to\infty} (n(d_l(u_n, u_{n+1}))^k) = 0. \tag{2.9}$$

As (2.9) holds, there exists $n_1 \in \mathbb{N}$ such that $n(d_l(u_n, u_{n+1}))^k \leq 1$, for every $n \geq n_1$ or,

$$d_l(u_n, u_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for every } n \geq n_1. \tag{2.10}$$

By (2.10) and for $m > n > n_1$, we get

$$\begin{aligned} d_l(u_n, u_m) &\leq d_l(u_n, u_{n+1}) + d_l(u_{n+1}, u_{n+2}) + \cdots + d_l(u_{m-1}, u_m) \\ &= \sum_{i=n}^{m-1} d_l(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} d_l(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ entails $\lim_{n,m\to\infty} d_l(u_n, u_m) = 0$. Hence, $\{WS_\beta(u_n)\}$ is a Cauchy sequence in (W, d_l). Since (W, d_l) is a complete D.M.S, there exists $u \in W$ such that $\{WS_\beta(u_n)\} \to u$ that is

$$\lim_{n\to\infty} d_l(u_n, u) = 0. \tag{2.11}$$

Now, by Lemma 1.1, we have

$$\tau + F(d_l(u_{n+1}, S_j u)) \leq \tau + F(Hd_l(S_\gamma u_n, S_j u))). \tag{2.12}$$

As (2.1) holds for u, then we have

$$\tau + F(d_l(u_{n+1}, S_j u)) \leq F(D_l(x_n, u)), \tag{2.13}$$

where,

$$\begin{aligned} D_l(u_n, x) &= \max \left\{ \begin{array}{c} d_l(u_n, x), \frac{d_l(u_n, S_\gamma u_n).d_l(x, S_j x)}{1 + d_l(x_n, u)} , \\ d_l(u_n, S_\gamma u_n), d_l(x, S_j x) \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} d_l(u_n, x), \frac{d_l(u_n, u_{n+1}).d_l(x, S_j x)}{1 + d_l(u_n, x)} , \\ d_l(u_n, u_{n+1}), d_l(x, S_j x) \end{array} \right\}. \end{aligned}$$

Applying limit $n \to \infty$, and by (2.11), we have

$$\lim_{n\to\infty} D_l(u_n, x) = d_l(x, S_j x). \tag{2.14}$$

Since F is strictly increasing, (2.13) implies

$$d_l(u_{n+1}, S_j x) < D_l(u_n, x).$$

Applying limit $n \to \infty$, and by (2.14), we have

$$d_l(x, S_j x) < d_l(x, S_j x).$$

It is a contradiction, hence $d_l(x, S_j x) = 0$ which implies that $x \in S_j x$. Hence $x \in \cap\{S_\beta : \beta \in \Omega\}$.

Now,

$$d_l(x, x) \le d_l(x, S_j x) + d_l(S_j x, x) \le 0.$$

This implies that $d_l(x, x) = 0$. □

Example 2.1. Let $W = [0, 10] \cap Q$ and $d_l(u, v) = u + v$. Then (W, d_l) is a complete D.M.S. Consider a family of multivalued mappings $\{S_\beta : \beta \in \Omega = a, 1, 2, \cdots\} : W \to P(W)$ and $\{WS_\beta(u_n) : \beta \in \Omega\}$ be a sequence in W generated by u_0. Define the mappings $S_n, S_a : W \to P(W)$ as follows:

$$S_n(u) = \left[\frac{1}{5n}u, \frac{2}{5n}u\right] \text{ for } n = 2, 3, \cdots, 14 \text{ and } S_a(u) = \left[\frac{1}{7}u, \frac{2}{7}u\right] \text{ for all } u \in W.$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(u) = \ln(u)$ for all $u \in \mathbb{R}^+, \tau > 0$, $\tau = \ln(1.4)$ and $u, v \in W$. Suppose that, $u_0 = 1$, then $d_l(1, S_a 1) = d_l\left(1, \left[\frac{1}{7}, \frac{2}{7}\right]\right) = 1 + \frac{1}{7} = \frac{8}{7}$. So $u_1 = \frac{1}{7}$. Now, $d_l(u_1, S_1 x_1) = d_l\left(\frac{1}{7}, S_1 \frac{1}{7}\right) = d_l\left(1, \left[\frac{1}{35}, \frac{2}{35}\right]\right) = \frac{6}{35}$.So $u_2 = \frac{1}{35}$. Now, $d_l(u_2, S_2 x_2) = d_l\left(\frac{1}{35}, S_2 \frac{1}{35}\right) = d_l\left(\frac{1}{35}, \left[\frac{1}{175}, \frac{2}{175}\right]\right) = \frac{6}{175}$.So $u_3 = \frac{1}{175}$. We define the sequence $\{WS_\beta(u_n)\} = \{1, \frac{1}{7}, \frac{1}{35}, \frac{1}{175}, \cdots\}$ in W generated by $u_0 = 1$. We have

$$\begin{aligned}
H_{d_l}(S_n u, S_a v) &= \max\left\{\sup_{x\in S_n u} d_l(x, S_a v), \sup_{y\in S_a v} d_l(S_n u, y)\right\} \\
&= \max\left\{\sup_{x\in S_n u} d_l\left(x, \left[\frac{v}{7}, \frac{2v}{7}\right]\right), \sup_{y\in S_a v} d_l\left(\left[\frac{u}{5n}, \frac{2u}{5n}\right], y\right)\right\} \\
&= \max\left\{d_l\left(\frac{2u}{5n}, \frac{v}{7}\right), d_l\left(\frac{u}{5n}, \frac{2v}{7}\right)\right\} \\
&= \max\left\{\frac{2u}{5n} + \frac{v}{7}, \frac{u}{5n} + \frac{2v}{7}\right\}
\end{aligned}$$

where

$$\begin{aligned} D_l(u,v) &= \max\left\{ d_l(u,v), \frac{d_l(u,\left[\frac{u}{5n},\frac{2u}{5n}\right]).d_l(y,\left[\frac{v}{7},\frac{2v}{7}\right]\ )}{1+d_l(u,v)}, \right. \\ &\qquad \left. d_l(u,\left[\frac{u}{5n},\frac{2u}{5n}\right]), d_l(v,\left[\frac{v}{5},\frac{2v}{5}\right]) \right\} \\ &= \max\left\{ d_l(u,v), \frac{d_l(u,\frac{u}{5n}).d_l(v,\frac{v}{7}\ )}{1+d_l(u,v)}, d_l(u,\frac{u}{5n}), d_l(v,\frac{v}{7}) \right\} \\ &= \max\left\{ u+v, \frac{(5nu+u)8v}{35n(1+u+v)}, \frac{5nu+u}{5n}, \frac{8v}{7} \right\} = u+v. \end{aligned}$$

Case i. If, $\max\left\{\frac{2u}{5n}+\frac{v}{7}, \frac{u}{5n}+\frac{2v}{7}\right\} = \frac{u}{5n}+\frac{2v}{7}$, and $\tau = \ln(1.4)$, then we have

$$\begin{aligned} \frac{7u}{n}+10v &\leq 25u+25v, \\ \frac{7}{5}(\frac{u}{5n}+\frac{2v}{7}) &\leq u+v, \\ \ln(1.4)+\ln(\frac{u}{5n}+\frac{2v}{7}) &\leq \ln(u+v), \end{aligned}$$

which implies that

$$\tau + F(H_{d_l}(S_n u, S_a v) \leq F(D_l(u,v)).$$

Case ii. Similarly, if $\max\left\{\frac{2u}{5n}+\frac{v}{7}, \frac{u}{5n}+\frac{2v}{7}\right\} = \frac{2u}{5n}+\frac{v}{7}$, and $\tau = \ln(1.4)$, then we have

$$\begin{aligned} \frac{14u}{n}+5v &\leq 25u+25v, \\ \frac{7}{5}(\frac{2u}{5n}+\frac{v}{7}) &\leq u+v, \\ \ln(1.4)+\ln(\frac{2u}{5n}+\frac{v}{7}) &\leq \ln(u+v). \end{aligned}$$

Hence,

$$\tau + F(H_{d_l}(S_n u, S_a v) \leq F(D_l(u,v).$$

Since all the conditions of Theorem 2.1 are fulfilled, hence $\{S_\beta : \beta \in \Omega\}$ have a common fixed point.

Let (W, d_l) be a D.M.S, $u_0 \in W$ and S be a multivalued mapping from W to $P(W)$. Let $u_1 \in Su_0$ be such that $d_l(u_0, Su_0) = d_l(u_0, u_1)$. Let $u_2 \in Su_1$ be such that $d_l(u_1, Su_1) = d_l(u_1, u_2)$. In the same way, we can construct a sequence u_n of points in W such that $u_{n+1} \in Su_n$, $d_l(u_n, Su_n) = d_l(u_n, u_{n+1})$. This iterative sequence is denoted by $\{WS(u_n)\}$ and called $\{WS(u_n)\}$ is a sequence generated by u_0 in W.

Corollary 2.1. *Let (W, d_l) be a complete D.M.S and $S : W \to P(W)$ be a multivalued mapping such that*

$$\tau + F(H_{d_l}(Su, Sv)) \leq F(D_l(u, v)), \tag{2.17}$$

for all $u, v \in \{SS(u_n)\}$, where $F \in \triangle_F$ and $\tau > 0$, and

$$D_l(u, v) = \max\left\{d_l(u, v), \frac{d_l(u, Su).d_l(v, Sv)}{1 + d_l(u, v)}, d_l(u, Su), d_l(v, Sv)\right\}.$$

Then $\{SS(u_n)\} \to x \in W$. Furthermore, if (2.17) also holds, then S has a fixed point x in W with $d_l(x, x) = 0$.

Theorem 2.2. *Let (W, d_l) be a complete D.M.S and $\{S_\beta : \beta \in \Omega\}$ be a family of Ciric type rational multivalued $F-$contraction. If $F \in \triangle_F$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(H_{d_l}(S_i u, S_j v)) \leq F\left(\begin{array}{c} b_1 d_l(u, v) + b_2 d_l(u, S_i u) + b_3 d_l(v, S_j v) \\ +b_4 \dfrac{d_l^2(u, S_i u).d_l(v, S_j v)}{1 + d_l^2(u, v)} \end{array}\right), \tag{2.18}$$

for all $u, v \in \{WS_\beta(u_n)\}$, with $u \neq v$ where $b_1, b_2, b_3, b_4 > 0$, $b_1+b_2+b_3+b_4 = 1$ and $b_3 + b_4 \neq 1$. Then $\{WS_\beta(u_n)\} \to x \in W$. Furthermore, if (2.18) also holds for x, then there is a common fixed point for the family of multivalued mappings $\{S_\beta : \beta \in \Omega\}$ in $\{WS_\beta(u_n) : \beta \in \Omega\}$.

Proof. As $u_1 \in S_i u_0$ and $u_2 \in S_j u_1$, by using Lemma 1.1

$$\begin{aligned}
\tau + F(d_l(u_1, u_2)) &= \tau + F(d_l(S_i u_0, S_j u_1)) \leq \tau + F(H_{d_l}(S_i u_0, S_j u_1)) \\
&\leq F\left(\begin{array}{c} b_1 d_l(u_0, u_1) + b_2 d_l(u_0, u_1) + b_3 d_l(u_1, Tu_1) \\ +b_4 \dfrac{d_l^2(u_0, S_i u_0).d_l(u_1, S_j u_1)}{1 + d_l^2(u_0, u_1)} \end{array}\right) \\
&\leq F\left(\begin{array}{c} b_1 d_l(u_0, u_1) + b_2 d_l(u_0, u_1) + b_3 d_l(u_1, u_2) \\ +b_4 d_l(u_1, S_j u_1)\left(\dfrac{d_l^2(u_0, u_1)}{1 + d_l^2(u_0, u_1)}\right) \end{array}\right) \\
&\leq F((b_1 + b_2)d_l(u_0, u_1) + (b_3 + b_4)d_l(u_1, u_2)).
\end{aligned}$$

Since F is strictly increasing, we get

$$\begin{aligned}
d_l(u_1, u_2) &< (b_1 + b_2)d_l(u_0, u_1) + (b_3 + b_4)d_l(u_1, u_2) \\
&< \left(\frac{b_1 + b_2}{1 - b_3 - b_4}\right) d_l(u_0, u_1).
\end{aligned}$$

From $b_1 + b_2 + b_3 + b_4 = 1$ and $b_3 + b_4 \neq 1$, we deduce $1 - b_3 - b_4 > 0$ and so

$$d_l(u_1, u_2) < d_l(u_0, u_1).$$

Consequently

$$F(d_l(u_1,u_2)) \leq F(d_l(u_0,u_1)) - \tau.$$

As we have $u_{2i+1} \in S_i u_{2i}$ and $u_{2i+2} \in S_j u_{2i+1}$ then from (2.18), and Lemma 1.1 we have

$$\begin{aligned}
&\tau + F(d_l(u_{2i+1},u_{2i+2})) \\
&= \tau + F(d_l(u_{2i+1},S_j u_{2i+1})) \leq \tau + F(H_{d_l}(S_i u_{2i},S_j u_{2i+1})) \\
&\leq F(b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},S_i u_{2i}) \\
&+ b_3 d_l(u_{2i+1},S_j u_{2i+1}) + b_4 \frac{d_l^2(u_{2i},S_i u_{2i}).d_l(u_{2i+1},S_j u_{2i+1})}{1+d_l^2(u_{2i},u_{2i+1})} \\
&\leq F(b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},u_{2i+1}) + b_3 d_l(u_{2i+1},u_{2i+2}) \\
&+ \frac{b_4 d_l(u_{2i+1},u_{2i+2}).d_l^2(u_{2i},u_{2i+1})}{1+d_l^2(u_{2i},u_{2i+1})}) \\
&\leq F(b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},u_{2i+1}) + b_3 d_l(u_{2i+1},u_{2i+2}) + b_4 d_l(u_{2i+1},u_{2i+2})).
\end{aligned}$$

Since F is strictly increasing, and $b_1+b_2+b_3+b_4 = 1$ where $b_3+b_4 \neq 1$, we deduce $1-b_3-b_4 > 0$ so we obtain

$$\begin{aligned}
d_l(u_{2i+1},u_{2i+2}) &< b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},u_{2i+1}) + b_3 d_l(u_{2i+1},u_{2i+2}) \\
&\quad + b_4 d_l(u_{2i+1},u_{2i+2})) \\
&< (b_1+b_2) d_l(u_{2i},u_{2i+1}) + (b_3+b_4) d_l(u_{2i+1},u_{2i+2}) \\
d_l(u_{2i+1},u_{2i+2}) &< \left(\frac{b_1+b_2}{1-b_3-b_4}\right) d_l(u_{2i},u_{2i+1}) = d_l(u_{2i},u_{2i+1}).
\end{aligned}$$

This implies that,

$$F(d_l(u_{2i+1},u_{2i+2})) \leq F(d_l(u_{2i},u_{2i+1})) - \tau.$$

Following similar arguments as given in Theorem 2.2, we have $\{WS_\beta(u_n)\} \to x$ that is

$$\lim_{n\to\infty} d_l(u_n,x) = 0. \tag{2.19}$$

Now, by Lemma 1.1, we have

$$\tau + F(d_l(u_{2n+1},S_j x)) \leq \tau + F(H d_l(S_i u_{2n},S_j x)).$$

By using inequality (2.18), we have

$$\begin{aligned}
\tau + F(d_l(u_{2n+1},S_j x)) &\leq F(b_1 d_l(u_{2n},x) + b_2 d_l(u_{2n},S_i u_{2n}) + b_3 d_l(x,S_j x) \\
&\quad + b_4 \frac{d_l^2(u_{2n},S_i u_{2n}).d_l(x,S_j x)}{1+d_l^2(u_{2n},x)}) \\
&\leq F(b_1 d_l(u_{2n},x) + b_2 d_l(u_{2n},u_{2n+1}) + b_3 d_l(x,S_j x) \\
&\quad + b_4 \frac{d_l^2(u_{2n},u_{2n+1}).d_l(x,S_j x)}{1+d_l^2(u_{2n},x)}).
\end{aligned}$$

As F is strictly increasing, we get

$$\begin{aligned} d_l(u_{2n+1}, S_j x) &< b_1 d_l(u_{2n}, x) + b_2 d_l(u_{2n}, u_{2n+1}) + b_3 d_l(x, S_j x) \\ &\quad + b_4 \frac{d_l^2(u_{2n}, u_{2n+1}).d_l(x, S_j x)}{1 + d_l^2(u_{2n}, x)}. \end{aligned}$$

Applying limit $n \to \infty$, and using (2.19), we have

$$d_l(x, S_j x) < b_3 d_l(x, S_j x).$$

It is a contradiction, hence $d_l(x, S_j x) = 0$ or $x \in S_j x$. Similarly by Lemma 1.1, using (2.18),(2.19), and the inequality

$$\tau + F(d_l(u_{2n+2}, S_i x)) \leq \tau + F(H_{d_l}(S_j u_{2n+1}, S_i x))$$

we can show that $d_l(x, S_i x) = 0$ or $x \in S_i x$. Hence the S_i and S_j have a common fixed point x in (W, d_l). Now,

$$d_l(x, x) \leq d_l(x, S_j x) + d_l(S_j x, x) \leq 0.$$

This implies that $d_l(x, x) = 0$. □

Corollary 2.2. *Let (W, d_l) be a complete D.M.S and $S : W \to P(W)$ be the multivalued mappings. If $F \in \triangle_F$ and $\tau \in \mathbb{R}_+$ such that*

$$\begin{aligned} &\tau + F(H_{d_l}(Su, Sv)) \\ &\leq F\left(b_1 d_l(u, v) + b_2 d_l(u, Su) + b_3 d_l(v, Sv) + b_4 \frac{d_l^2(u, Su).d_l(v, Sv)}{1 + d_l^2(u, v)}\right), \end{aligned} \tag{2.20}$$

for all $u, v \in \{SS(u_n)\}$, with $u \neq v$ where $b_1, b_2, b_3, b_4 > 0$, $b_1 + b_2 + b_3 + b_4 = 1$ and $b_3 + b_4 \neq 1$, then $\{SS(u_n)\} \to u \in W$. Furthermore, if for $x \in W$ (2.20) holds, then $x \in W$ is a fixed point for S and $d_l(x, x) = 0$.

3. Single valued result with application to system of integral equations

Definition 3.1. *Let (W, d_l) be a complete D.M.S and $\{S_\beta : \beta \in \Omega\}$ be a family of mapping from W to W and $\{WS_\beta(x_n) : \beta \in \Omega\}$ be a sequence in W generated by x_0. The family $\{S_\beta : \beta \in \Omega\}$ is called to be Ciric type rational $F-$contraction, if for all $x, y \in \{WS_\beta(x_n)\}$, we have*

$$\tau + F(d_l(S_i x, S_j v)) \leq F(D_l(x, v)) \tag{3.1}$$

where $F \in \triangle_F$ and $\tau > 0$, and

$$D_l(x, v) = \max\left\{d_l(x, v), \frac{d_l\,(x, S_i x)\,.d_l\,(y, S_j y)}{1 + d_l\,(x, y)}, d_l(x, S_i x), d_l(y, S_j y)\right\}. \tag{3.2}$$

Theorem 3.1. *Let (W, d_l) be a complete D.M.S and $\{S_\beta : \beta \in \Omega\}$ be a family of self mapping of Ciric type rational $F-$contraction. Then $\{S_\beta : \beta \in \Omega\}$ have a common fixed point x in W and $d_l(x, x) = 0$.*

Proof. The proof of this theorem is same as given for Theorem 2.1 We have to prove the uniquness only. Let v be another common fixed point of S. Suppose $d_l(S_i x, S_j y) > 0$ then we have

$$\begin{aligned}
\tau + F(d_l(S_i x, S_j y)) &\leq F(D_l(x, y)) \\
&= F(\max\left\{ d_l(x, y), \frac{d_l\,(x, S_i x)\,.d_l\,(y, S_j y)}{1 + d_l\,(x, y)}, \right. \\
&\quad \left. d_l(x, S_i x), d_l(y, S_j y)\right\} \\
&= F(\max\{d_l(x, y)\}), \\
F(d_l(S_i x, S_j y)) &< d_l(x, y), \\
d_l(x, y) &< d_l(x, y)
\end{aligned}$$

which is a contradiction so $d_l(S_i x, S_j y) = 0$. Hence $x = y$.

In this section, we discuss the application of fixed point Theorem 3.1 in form of unique solution of a family of Volterra type integral equations given as:

$$p(a) = \int_0^a T_i(a, z, u(z))dz + f(a), \tag{3.3}$$

for all $a \in [0, 1]$, $i \in \Omega$ and T_i be the mapping on $[0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+) \to \mathbb{R}$. We find the solution of (3.3). Let $W = C([0, 1])$ be the set of all real continuous functions on $[0, 1]$. For $u \in C([0, 1], \mathbb{R}_+)$, we define:

$$\|p\|_\tau = \max\left\{ \sup_{a\in[0,1]} |p(a)|\, e^{-\tau a} \right\},$$

where $\tau > 0$. Then,

$$d_\tau(p, q) = \max \sum \left\{ \sup_{a\in[0,1]} \| \, |p(a) + q(a)|\, e^{-\tau a} \|_\tau \right\},$$

for all $p, q \in C([0, 1], \mathbb{R}_+)$. With these setting, $C([0, 1], \mathbb{R}, \|\cdot\|_\tau)$ becomes complete D.M.S. □

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 3.2. *Assume the following conditions are satisfied:*

(i) $T_i : [0, 1] \times [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ *and* $f, g : [0, 1] \to \mathbb{R}$ *are continuous;*

(ii) Define

$$S_\alpha u(a) = \int_0^a T_i(a, z, u(z))dz + f(a).$$

Suppose there exists $\tau > 1$, *such that*

$$|T_i(a,z,u) + T_j(a,z,v)| \leq \tau e^{-\tau}[N(u,v)]$$

for all $a, z \in [0,1]$ *and* $u, v \in C([0,1],\mathbb{R})$, *where*

$$N(u,v) = \max \left\{ \begin{array}{c} |u(a)+v(a)|, \dfrac{|u(a)+S_\alpha u(a)|\,.\,|v(a)+S_\beta v(a)|}{1+|u(a)+v(a)|}, \\ |u(a)+S_\alpha u(a)|, |v(a)+S_\beta v(a)| \end{array} \right\}.$$

Then integral equation (3.3) *has a solution.*

Proof. By assumption (ii)

$$\begin{aligned} |S_\alpha u(a) + S_\beta v(a)| &= \int_0^a |T_i(a,z,u(z)) + T_j(a,z,v(z))|\, dz \\ &\leq \int_0^a \tau e^{-\tau}([N(u,v)]e^{-\tau z})e^{\tau z} dz \\ &\leq \int_0^a \tau e^{-\tau}\|N(u,v)\|_\tau e^{\tau z} dz \\ &\leq \tau e^{-\tau}\|N(u,v)\|_\tau \int_0^a e^{\tau z} dz \\ &\leq \tau e^{-\tau}\|N(u,v)\|_\tau \frac{1}{\tau} e^{\tau a} \\ &\leq e^{-\tau}\|N(u,v)\|_\tau e^{\tau a}. \end{aligned}$$

This implies

$$|S_\alpha u(a) + S_\beta v(a)|\, e^{-\tau a} \leq e^{-\tau}\|N(u,v)\|_\tau.$$

That is

$$\|S_\alpha u(a) + S_\beta v(a)\|_\tau \leq e^{-\tau}\|N(u,v)\|_\tau,$$

which further implies

$$\tau + \ln \|S_\alpha u(a) + S_\beta v(a)\|_\tau \leq \ln \|N(u,v)\|_\tau.$$

Since all the conditions of Theorem (3.1) are fullfilled. So, integral equation given in (3.3) has a unique common solution □

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