On solvability of finite groups with some nonnormal nonabelian subgroups

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Abstract. For a finite group G, the symbol $\mathcal{M}(G)$ denotes the number of conjugacy classes of maximal subgroups of G, and $\mathcal{T}(G)$ denotes the number of conjugacy classes of non-normal non-abelian subgroups of G. In this paper, we prove the finite groups with $\mathcal{T}(G) < \mathcal{M}(G)$ are solvable.

Keywords: maximal subgroup, nonnormal subgroup, nonabelian subgroup.

1. Introduction

All groups considered in this paper are finite. Throughout the following, G always denotes a finite group. For G, we use the symbol $\mathcal{T}(G)$ to denote the number of conjugacy classes of non-normal non-abelian subgroups of G and $\pi(G)$ to denote the set of the prime divisors of $|G|$. Obviously, $\mathcal{T}(G) = 0$ if and only if G is a Meta-Hamilton group (i.e. G is a non-abelian group whose every non-abelian subgroup is normal). Nagrebeckii [5] characterized the structure of the finite non-nilpotent meta-Hamilton group. An [1] gave a complete classification of finite meta-Hamilton p-groups. Recently, Meng et al. [3] proved the groups with $\mathcal{T}(G) \leq |\pi(G)| - 1$ are solvable and the only non-abelian simple group with $\mathcal{T}(G) = |\pi(G)|$ is isomorphic to A_5.

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In the following, we use the symbol $\mathcal{M}(G)$ to denote the number of conjugacy classes of maximal subgroups of G.

Example 1.1. Let $G = A_5$. Then G has three conjugacy classes of maximal subgroups, and three conjugacy classes of non-normal non-abelian subgroups, that is, $\mathcal{T}(G) = \mathcal{M}(G)$.

In the light of Example 1.1, we obtain the main results of this paper as follows:

Theorem 1.1. *If all non-normal non-abelian maximal subgroups of G are conjugate, then G is a solvable group.*

Theorem 1.2. (1) *Let G be a group with $\mathcal{T}(G) < \mathcal{M}(G)$. Then G is a solvable group.*

(2) *Suppose G is a non-abelian simple group. Then $\mathcal{T}(G) = \mathcal{M}(G)$ if and only if $G \cong A_5$.*

Theorem 1.3. (1) *Let G be a group with $\mathcal{T}(G) \leq 2$. Then G is a solvable group.*

(2) *Suppose G is a non-abelian simple group. Then $\mathcal{T}(G) = 3$ if and only if $G \cong A_5$.*

The notation is standard and follows that of Huppert [2].

2. Some lemmas

Lemma 2.1 ([6])**.** *Let $P \in \mathrm{Syl}_p(G)$. If $H \geq N_G(P)$, then $N_G(H) = H$.*

Lemma 2.2 ([6])**.** *If G has an abelian maximal subgroup, then G is solvable.*

Lemma 2.3 ([7])**.** *Let G be an unsolvable group. If every second maximal subgroup of G is nilpotent, then $G \cong A_5$ or $SL(2,5)$.*

Lemma 2.4 ([4])**.** *If G has at most two conjugacy classes of maximal subgroups, then G is solvable.*

3. Proofs

Proof of Theorem 1.1. Let $P \in \mathrm{Syl}_p(G)$ for some $p \in \pi(G)$. Assume that P is normal in G. Consider the group G/P. Let M_1/P and M_2/P are non-normal non-abelian maximal subgroups of G/P. Then M_1 and M_2 are non-normal non-abelian maximal subgroups of G. By the hypothesis, there exists $g \in G$, such that $M_1^g = M_2$. Thus, $(M_1/P)^{\bar{g}} = M_2/P$, that is, all non-normal non-abelian maximal subgroups of G/P are conjugate. By induction, G/P is solvable, and so G is solvable.

Now, assume that all Sylow subgroups of G are not normal in G. Let $\pi(G) = \{p_1, \cdots, p_n\}$, and $P_i \in \mathrm{Syl}_{p_i}(G)$ where $1 \leq i \leq n$. Since P_i is not normal in G, we have that the normalizer $N_G(P_i)$ is a proper subgroup of G. Let M_i be a maximal subgroup of G that contains $N_G(P_i)$. By Lemma 2.1, M_i is self-normal in G. If all $M_1, \cdots, M_n$ are not abelian, then $M_1, \cdots, M_n$ are conjugate in G. In particular, $|M_1| = \cdots = |M_n|$. This implies that $|G| = |M_1|$ and $G = M_1$, a contradiction. Thus we must have that some M_i is abelian. By Lemma 2.2, G is solvable. This completes the proof. $\square$

Proof of Theorem 1.2. (1) Let $\{M_1, M_2, \cdots, M_n\}$ be representative system of conjugacy classes of all maximal subgroups of G. If some M_i is abelian, then, by Lemma 2.2, G is solvable and we are done. Thus we may assume that all $M_1, M_2, \cdots, M_n$ are not abelian.

Assume that all $M_1, M_2, \cdots, M_n$ are not normal in G. Thin implies that $\mathcal{T}(G) \geq \mathcal{M}(G)$, which is impossible.

Assume that all $M_1, M_2, \cdots, M_n$ are normal in G. This implies that G is nilpotent, and so G is solvable.

Assume that $M_1, \cdots, M_s$ are normal in G and $M_{s+1}, \cdots, M_n$ are not normal in G, where $1 < s < n$. Then $\mathcal{T}(G) \geq n - s$.

Assume that all $M_1, \cdots, M_s$ are not solvable. By Theorem 1.1, every M_i at least has one non-normal non-abelian maximal subgroup H_i. First, we know that all $H_1, \cdots, H_s$ are not normal in G. Second, $H_i \neq H_j$ when $i \neq j$, otherwise $H_i = H_j = M_i \cap M_j$ is normal in G, which is imposable. Now, we claim that H_i is not conjugate to H_j when $i \neq j$. In fact, if there exists $g \in G$ such that $H_i^g = H_j$, then $H_j = H_i^g \leq M_i^g = M_i$. By maximality of H_i in M_i, we have that $\langle H_i, H_j \rangle = M_i$. Similarly, $\langle H_i, H_j \rangle = M_j$. Thus $M_i = M_j$, a contradiction.

Furthermore, H_i is not conjugate to M_j, where $1 \leq i \leq s$, $s+1 \leq j \leq n$. In fact, if there exists $g \in G$ such that $H_i^g = M_j$, then $M_j = H_i^g \leq M_i^g = M_i$, a contradiction.

Now we know that $H_1, \cdots, H_s$, $M_{s+1}, \cdots, M_n$ are some representative elements of conjugacy classes of non-normal non-abelian subgroups of G. Then $\mathcal{T}(G) \geq n$, which contradicts the hypothesis.

Thus we must have that some M_i is solvable. By induction, G/M_i is solvable, so G is solvable.

(2) By Example 1.1, it is clear that if G is isomorphic to A_5, then $\mathcal{T}(G) = \mathcal{M}(G)$.

Conversely, assume that G is a non-abelian simple group with $\mathcal{T}(G) = \mathcal{M}(G)$. Let $\{M_1, M_2, \cdots, M_n\}$ be representative system of conjugacy classes of all maximal subgroups of G. Since G is a non-abelian simple group, by Lemma 2.2, we have that all $M_1, M_2, \cdots, M_n$ are non-abelian non-normal. Thus, by the hypothesis, every proper subgroup of M_i is abelian or normal for all $1 \leq i \leq n$. Since G is a non-abelian simple group, it follows that every subgroup of M_i is abelian. Thus G is isomorphic to A_5 by Lemma 2.3, as desired. $\square$

Proof of Theorem 1.3. (1) If G is unsolvable, then $\mathcal{T}(G) \geq \mathcal{M}(G)$ by Theorem 1.2(1). Then $\mathcal{M}(G) \leq 2$. It follows from the Lemma 2.4 that G is solvable, a contradiction. Thus, G is solvable, as desired.

(2) By Example 1.1, it is clear that if G is isomorphic to A_5, then $\mathcal{T}(G) = 3$.

Conversely, assume that G is a non-abelian simple group with $\mathcal{T}(G) = 3$. By Theorem 1.2(1) and Lemma 2.4, we have that $\mathcal{T}(G) = \mathcal{M}(G)$. By Theorem 1.2(2), G is isomorphic to A_5, as desired. $\square$

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