A modified Newton's method for solving functions of one variable

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Abstract. In this research, we developed the Newton method to diminish the number of iterations to arrive at the optimal solution for the problem, depending on the Taylor series to approximate the derivatives applying the minimum value of the problem. We compare the method presented with the traditional Newton method in requisites of the number of iteration and the time of execution. Shown numerical results effectiveness for the proposed algorithm after applying it to a group of unimodal functions.

Keywords: modified Newton's method, iteration methods, test functions.

1. Introduction

Solving nonlinear problems of one variable is an important part of iteration methods. For details see [10].

The oldest one and the most famed problems in iteration methods, for finding the minimum value of problems of the form :

$$Minimize\left\{\varpi(\upsilon),\upsilon\in R,\varpi:R\rightarrow R\right\}, \tag{1}$$

where $\varpi(\upsilon)$ is a smooth function. For more details see [4,5].

This problems may not be easy to solve. Therefore, we consider briefly a numerical method for its solution. The most widely and the best known used to find the value minimum of optimization methods is the kind Newton's algorithm which converges quadratically. For more details can be found in [12].

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The idea is Newton's formula for approximating the objective function locally by a quadratic model corresponding to the model at a point. The exact approximate function is then improved. For more details see [13]. So, the iteration of Newton's formula is given by:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)}{\varpi''(\upsilon_k)} \tag{2}$$

which is a Newton method.

The secant formula can be a vision as a simplification of Newton's formula that evades calculating the second derivation is obtained as follows:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)(\upsilon_k - \upsilon_{k-1})}{\varpi'(\upsilon_k) - \varpi'(\upsilon_{k-1})}. \tag{3}$$

For more details can be found in [3].

Lately, many studies of Secant-type methods, has become focused on solving nonlinear optimization issues, such that $\varpi''(\upsilon_k)$ this is not required (e.g. [3,5-9 and 14]).

This paper aims to derive a new secant method for improving the iteration of Newton's formula based on avoiding from the evaluation of $\varpi''(\upsilon_k)$.

2. A modified Newton's method

The idea of using Taylor expansion is the key element for derivation the new Newton's methods, we give [11]:

$$\varpi(\upsilon) = \varpi(\upsilon_k) + \varpi'(\upsilon_k)(\upsilon - \upsilon_k) + \frac{1}{2}\varpi''(\upsilon_k)(\upsilon - \upsilon_k)^2. \tag{4}$$

The gradient of $\varpi(\upsilon)$ for υ is given by:

$$\varpi'(\upsilon) = \varpi'(\upsilon_k) + \varpi''(\upsilon_k)(\upsilon - \upsilon_k) = 0. \tag{5}$$

Putting Eq. (5) in Eq. (4) we get:

$$2(\varpi(\upsilon_k) - \varpi(\upsilon)) = \varpi''(\upsilon_k)(\upsilon - \upsilon_k)^2. \tag{6}$$

This yields:

$$\varpi''(\upsilon_k) = \frac{2 * (\varpi(\upsilon_k) - \varpi(\upsilon))}{(\upsilon - \upsilon_k)^2}. \tag{7}$$

Substituting υ_{k-1} in to υ in above equation and can be rewritten as :

$$\varpi''(\upsilon_k) = \frac{(\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}{(\upsilon_{k-1} - \upsilon_k)^2}. \tag{8}$$

Putting Eq. (8) into Eq. (2), we get:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)(\upsilon_{k-1} - \upsilon_k)^2}{2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}, \tag{9}$$

which is a modified Newton's formula.

The modified of Newton's formula above is presented in the following algorithm:

Stage i. Givenε, υ_0 and the function$\varpi(\upsilon_0)$. $\upsilon_1 = \upsilon_0 + 0.1$, set $k = 0$.
Stage ii. Set $k = k + 1$.
Stage iii. Compute $\upsilon_{k+1} = \upsilon_k - [\varpi'(\upsilon_k)(\upsilon_{k-1} - \upsilon_k)^2/2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))]$.
Stage iv. Stop when the absolute value of the derivative of the function of the new iteration is sufficiently small i.e. $|\varpi'(\upsilon_k)| \leq \varepsilon$.

3. Analysis of convergence

In this section, we prove that modified Newton's formula (9) has the likelihood of local convergence.

Theorem 3.1. *Let $\varpi : I \to R$ bea sufficiently differentiable function and $\upsilon^* \in I$ be a zero of ϖ, where I is an open interval then the proposed method has quadratic convergence. If υ_0 sufficiently close to υ^*.*

Proof. The modified Newton's formula is as:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)(\upsilon_{k-1} - \upsilon_k)^2}{2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}. \tag{10}$$

By subtracting υ^* from both sides of above equation, we get :

$$e_{k+1} = e_k - \frac{\varpi'(\upsilon_k)(e_{k-1} - e_k)^2}{2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}, \tag{11}$$

where $e_k = \upsilon_{k+1} - \upsilon^*$.

By expanding Taylor's series:

$$\begin{aligned}\varpi(\upsilon^*) = \varpi(\upsilon_k) + \varpi'(\upsilon_k)(\upsilon^* - \upsilon_k) + \frac{1}{2!}\varpi''(\upsilon_k)(\upsilon^* - \upsilon_k)^2 \\ + \frac{1}{3!}\varpi'''(c)(\upsilon^* - \upsilon_k)^3,\end{aligned} \tag{12}$$

where c lies between υ^* and υ_k, the derivative of equation (12), we get:

$$\varpi'(\upsilon^*) = \varpi'(\upsilon_k) + \varpi''(\upsilon_k)(\upsilon^* - \upsilon_k) + \frac{1}{2}\varpi'''(c)(\upsilon^* - \upsilon_k)^2 = 0. \tag{13}$$

From (13), we have:

$$-\varpi'(\upsilon_k) = -e_k\varpi''(\upsilon_k) + \frac{e_k^2}{2}\varpi'''(c). \tag{14}$$

Putting equation (14) in to equation (10) we get:

$$e_{k+1} = e_k^2 \frac{\varpi'''(c)}{\varpi''(\upsilon_k)}, \tag{15}$$

which implies that the order of convergence is quadratic. The proving is complete.

4. Application examples

The Newton and modified Newton's methods were applied to single-variable functions using the software Matlab. We use accuracy is $\varepsilon = 10^{-10}$, for computer programs. Comparison between the methods based on the execution time and iteration number for the various functions.

Example 1. Problem $\varpi(\upsilon) = \cos(\upsilon) + (\upsilon - 2)^2$, initial approximation $\upsilon_0 = 2$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**4**	**2.3542**	**0.5790**
New method	**2**	**2.0997**	**0.2040**

Example 2. Problem $\varpi(\upsilon) = e^{\upsilon} - 3\upsilon^2$, initial approximation $\upsilon_0 = 0.25$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**4**	**0.2045**	**0.2660**
New method	**2**	**0.3498**	**0.2030**

Example 3. Problem $\varpi(\upsilon) = e^{-\upsilon} + \upsilon^2$, initial approximation $\upsilon_0 = 1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**5**	**0.3517**	**0.3130**
New method	**3**	**1.0984**	**0.2340**

Example 4. Problem $\varpi(\upsilon) = -\upsilon e^{-\upsilon}$, initial approximation $\upsilon_0 = 0$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**7**	**1**	**0.3900**
New method	**2**	**0.0996**	**0.2040**

Example 5. Problem $\varpi(\upsilon) = 0.65 - 0.75/(1 + \upsilon^2) - 0.65\upsilon \tan^{-1}(1/\upsilon)$, initial approximation $\upsilon_0 = 0.1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**6**	**0.4809**	**0.7970**
New method	**2**	**0.1998**	**0.2960**

Example 6. Problem $\varpi(\upsilon) = \upsilon/\log(\upsilon)$, initial approximation $\upsilon_0 = 0.1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	——	**Fail**	——
New method	**2**	**0.1996**	**0.2190**

Example 7. Problem $\varpi(\upsilon) = \upsilon^4 + 2\upsilon^2 - \upsilon - 3$, initial approximation $\upsilon_0 = 1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**7**	**0.2367**	**0.6870**
New method	**4**	**1.0672**	**0.3130**

5. Conclusion

All methods in some cases may fail to converge to the minimum and it is not necessarily of may converge to global minimum also.

Based on our findings, we now conclude that the new method is the most effective of the Newton methods we studied in this study. We need to evaluate the first derivative at each step and we will likely have difficulty obtaining the global minimum. In the worst case, it is not much worse than its affinity. Can interpret the method performance in multi-dimensional case, as such in [1,2].

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