A note on restriction semigroups

Haijun Liu
School of Mathematics and Statistics
Jiangxi Normal University
Nanchang Jiangxi, 330022
People's Republic of China
haijunliu123@163.com

Xiaojiang Guo*
School of Mathematics and Statistics
Jiangxi Normal University
Nanchang Jiangxi, 330022
People's Republic of China
xjguo@jxnu.edu.cn

Abstract. It is proved that if S is a weakly reductive semigroup, then S is a (proper; Clifford; E-reflexive) restriction semigroup if and only if $\Omega(S)$ is a (proper; Clifford; E-reflexive) restriction semigroup and $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$.
Keywords: restriction semigroup, translational hull, weakly reductive semigroup

1. Introduction

Let S be a semigroup. A mapping $\lambda : S \to S$ is called a *left translation* of S if for any $a, b \in S$, $\lambda(ab) = (\lambda a)b$. Similarly, a mapping $\rho : S \to S$ is a *right translation* if for any $a, b \in S$, $(ab)\rho = a(b\rho)$. A left translation λ and a right translation ρ of S are said to be *linked* if for any $a, b \in S$, $a(\lambda b) = (a\rho)b$. In this case, we call the pair (λ, ρ) a *bitranslation* of S. The set $\Lambda(S)$ of all left translations and the set $I(S)$ of all right translations of S form semigroups under the composition of mappings. The *translational hull* of S is the subsemigroup $\Omega(S)$ of $\Lambda(S) \times I(S)$ which consists of all bitranslations (λ, ρ). It is well known that $\Omega(S)$ forms a submonoid of $\Lambda(S) \times I(S)$ under the multiplication:

$$(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho'),$$

where $\lambda\lambda'$ and $\rho\rho'$ respectively represent the composition operation of λ' , λ and of ρ, ρ'. For any $a \in S$, we denote by λ_a [resp. ρ_a] the *inner* left [resp. right] translation,which is defined as $\lambda_a(x) = ax$ [resp. $(x)\rho_a = xa$]. It is easy to check that the pair $\pi_a = (\lambda_a, \rho_a)$ is the *inner bitranslation* induced by a. It is easy to check that the set $\Pi(S) = \{\pi_a : a \in S\}$ is a subsemigroup of $\Omega(S)$. Indeed, $\Pi(S)$ is still an ideal of $\Omega(S)$ (for details, see [10, p.30]).

∗. Corresponding author

The translational hull of a semigroup, introduced by Petrich in 1970, plays an important role in the theory of semigroups. There are many authors having been researching translational hulls of semigroups. It was proved that if S is an inverse semigroup (an adequate semigroup; an ample semigroup, respectively), then so is $\Omega(S)$ (see [1, 2]). After then, there are several classes of semigroups whose translational hulls were researched; for example, strongly right (left) type-A semigroups (see [6]), E-reflexive type-A semigroups (see [7]), inverse wpp semigroups (see [8]), type B semigroups (see [9]), etc.

Restriction semigroups become a larger class of semigroups, including some important class of semigroups; such as, adequate semigroups, especially, inverse semigroups, as its proper subclasses. So, it is a natural topic to study characterizations of translational hulls of restriction semigroups. This is the main aim of this note.

2. Results

To begin with, we recall several concepts.

A *left restriction semigroup* is defined to be an algebra of type $(2,1)$, more precisely, an algebra $S=(S,\cdot,^{+})$ where $(S,\cdot)$ is a semigroup and $^{+}$ is a unary operator such that the following identities are satisfied:

$$x^{+}x=x,\ x^{+}y^{+}=y^{+}x^{+},\ (x^{+}y)^{+}=x^{+}y^{+},\ xy^{+}=(xy)^{+}x. \tag{2.1}$$

A *right restriction semigroup* is dually defined, that is, it is an algebra $(S,\cdot,*)$ satisfying the duals of the identities (2.1). If $S=(S,\cdot,+,*)$ is an algebra of type $(2,1,1)$ where $S=(S,\cdot,+)$ is a left restriction semigroup and $S=(S,\cdot,*)$ is a right restriction semigroup and the identities

$$(x^{+})^{*}=x^{+},\ (x^{*})^{+}=x^{*} \tag{2.2}$$

hold, then it is called a *restriction semigroup.* By definition, the defining properties of a restriction semigroup are left-right dual. Therefore in the sequel dual definitions and statements will not be explicitly formulated. Gould [5] pointed out that in a restriction semigroup, the following identities are satisfied:

$$(xy)^{*}=(x^{*}y)^{*},\ (xy)^{+}=(xy^{+})^{+}. \tag{2.3}$$

Let S be a restriction semigroup. By (2.2), we have $\{x^{+}:x\in S\}=\{x^{*}:x\in S\}$. This set is called the *set of projections of* S and denoted by $P(S)$. Again by (2.1) and its dual, $P(S)$ is a $(2,1,1)$-subsemigroup of S which is indeed a semilattice.

(i) If all projections of S are central, then S is called a *Clifford restriction semigroup.*

(ii) If for all $e\in P(S)$ and $x,y\in S$, $exy\in P(S)$ implies $eyx\in P(S)$, then we call S an *E-reflexive restriction semigroup.*

On S, define: for any $a, b \in S$

$$a\sigma b \Leftrightarrow (\exists e \in P(S))ea = eb;$$
$$\Leftrightarrow (\exists f \in P(S))af = bf.$$

Evidently, σ is an equivalence on S. A restriction semigroup S is *proper* provided for any $a, b \in S$,

(i) if $a^+ = b^+$ and $a\sigma b$, then $a = b$; and

(ii) if $a^* = b^*$ and $a\sigma b$, then $a = b$.

Among restriction semigroups, the notion of subalgebra is understood in type $(2, 1, 1)$, which is emphasised by using the expression $(2, 1, 1)$-subsemigroup.

Recall that a semigroup S is *weakly reductive* if for any $a, b \in S$, $xa = xb$ and $ax = bx$ for all $x \in S$ implies that $a = b$. We have that any restriction semigroup is weakly reductive. Indeed, let $a, b \in S$ and assume that $xa = xb$ and $ax = bx$ for all $x \in S$, we have $a = a^+b$, so that $a^+ = (a^+b)^+ = a^+b^+$; similarly, $b^+ = b^+a^+$, thus $a^+ = b^+$ since $P(S)$ is a semilattice, it follows that $a = a^+b = b^+b = b$, as required.

We now describe our main result:

Theorem 2.1. *If S is a weakly reductive semigroup, then S is a (proper; Clifford; E-reflexive) restriction semigroup if and only if $\Omega(S)$ is a (proper; Clifford; E-reflexive) restriction semigroup and $\Pi(S)$ is a $(2, 1, 1)$-subsemigroup of $\Omega(S)$.*

3. Proofs

To begin with, we prove the following lemma.

Lemma 3.1. *If S is a restriction semigroup, then $S \cong \Pi(S)$.*

Proof. It is obvious that the mapping $\pi : S \to \Pi(S);\ a \mapsto (\lambda_a, \rho_a)$ is a surjective homomorphism. In condition of weakly reductivity, π is still injective. Therefore π is an isomorphism, as required. □

The following lemma is well known; for example, see [2, 7].

Lemma 3.2. *Let S be a restriction semigroup. Then the following statements are equivalent:*

(i) $(\forall e \in P(S))\ \lambda_1 e = \lambda_2 e$ *(respectively, $e\rho_1 = e\rho_2$);*

(ii) $\lambda_1 = \lambda_2$ *(respectively, $\rho_1 = \rho_2$).*

Lemma 3.3. *Let S be a restriction semigroup. If $(\lambda_i, \rho_i) \in \Omega(S)$ with $i = 1, 2$, then the following statements are equivalent:*

(i) $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$*;*

(ii) $\lambda_1 = \lambda_2$;

(iii) $\rho_1 = \rho_2$.

Proof. We need only to verify (ii)⇒(i) because (i)⇒(ii) is obvious, and (i)⇔(ii) is dual to (i)⇔ (iii). Suppose that $\lambda_1 = \lambda_2$, then $\lambda_1 e = \lambda_2 e$ for any $e \in P(S)$, and so $e\rho_1 = (e\rho_1)(e\rho_1)^* = e[\lambda_1(e\rho_1)^*] = e[\lambda_2(e\rho_1)^*] = (e\rho_2)(e\rho_1)^*$; similarly, $e\rho_2 = (e\rho_1)(e\rho_2)^*$. Further,

$$e\rho_1 = (e\rho_2)(e\rho_1)^* = (e\rho_1)(e\rho_2)^*(e\rho_1)^* = (e\rho_1)(e\rho_1)^*(e\rho_2)^* = (e\rho_1)(e\rho_2)^* = e\rho_2.$$

It follows that for any $a \in S$, $a\rho_1 = a(a^*\rho_1) = a(a^*\rho_2) = a\rho_2$; that is, $\rho_1 = \rho_2$. □

Let S be a restriction semigroup. For $(\lambda, \rho) \in \Omega(S)$, we define $\lambda^+, \lambda^*, \rho^+$ and ρ^* as follows:

$$\lambda^+ a = (a^+\rho)^+ a; \qquad \lambda^* a = (\lambda a^+)^* a;$$
$$a\rho^+ = a(a^*\rho)^+; \qquad a\rho^* = a(\lambda a^*)^*.$$

Lemma 3.4. *Let S be a restriction semigroup with the set of projections $P(S)$. Then for all $e \in P(S)$ and $a \in S$,*

(i) $\lambda^* e = e\rho^* = (\lambda e)^*$;

(ii) $\lambda^+ e = e\rho^+ = (e\rho)^+$;

(iii) $\lambda^* e,\ e\rho^*, \lambda^+ e, e\rho^+ \in P(S)$;

(iv) $(\lambda a)^* = (\lambda^* a)^*,\ (a\rho)^+ = (a\rho^+)^+$.

Proof. (i) By definition, $\lambda^* e = (\lambda e)^* e = e(\lambda e)^* = e\rho^*$, and notice that $(\lambda e)^* e = ((\lambda e)e)^* = (\lambda e)^*$, we have $\lambda^* e = e\rho^* = (\lambda e)^*$.

(ii) Dual to (i).

(iii) Obvious.

(iv) Clearly, $\lambda a = \lambda(a^+ a) = (\lambda a^+)a$, then $(\lambda a)^* = [(\lambda a^+)a]^* = [(\lambda a^+)^* a]^* = (\lambda^* a)^*$. Thus, $(\lambda a)^* = (\lambda^* a)^*$, similarly, $(a\rho)^+ = (a\rho^+)^+$. □

Define $(\lambda, \rho)^+ = (\lambda^+, \rho^+)$ and $(\lambda, \rho)^* = (\lambda^*, \rho^*)$.

Similar to [2], we may verify the following lemma and we omit the detail.

Lemma 3.5. *Let $(\lambda, \rho) \in \Omega(S)$. If S is a restriction semigroup, then $(\lambda, \rho)^+$, $(\lambda, \rho)^* \in \Omega(S)$.*

Lemma 3.6. *Let $(\lambda, \rho), (\lambda_1, \rho_1),\ (\lambda_2, \rho_2) \in \Omega(S)$. If S is a restriction semigroup, then:*

(i) $(\lambda, \rho)^+(\lambda, \rho) = (\lambda, \rho)$;

(ii) $(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)^+ = (\lambda_2,\rho_2)^+(\lambda_1,\rho_1)^+$;

(iii) $[(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)]^+ = (\lambda_1,\rho_1)^+(\lambda_2,\rho_2)^+$;

(iv) $(\lambda_1,\rho_1)(\lambda_2,\rho_2)^+ = [(\lambda_1,\rho_1)(\lambda_2,\rho_2)]^+(\lambda_1,\rho_1)$;

(v) $[(\lambda,\rho)^+]^* = (\lambda,\rho)^+$.

Proof. (i) Compute $(\lambda,\rho)^+(\lambda,\rho) = (\lambda^+,\rho^+)(\lambda,\rho) = (\lambda^+\lambda,\rho^+\rho)$. For any $e \in P(S)$, we have $e\rho^+\rho = (e\cdot e\rho^+)\rho = (e\rho^+)\cdot e\rho = (e\rho)^+e\rho = e\rho$, and so $\rho^+\rho = \rho$. Therefore, $(\lambda,\rho)^+(\lambda,\rho) = (\lambda,\rho)$.

(ii) Compute

$$\lambda_1^+\lambda_2^+e = \lambda_1^+(\lambda_2^+e\cdot e) = (\lambda_1^+e)\cdot(\lambda_2^+e) = (\lambda_2^+e)\cdot(\lambda_1^+e) = \lambda_2^+\lambda_1^+e.$$

Then, $\lambda_1^+\lambda_2^+ = \lambda_2^+\lambda_1^+$. Together with

$$(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)^+ = (\lambda_1^+\lambda_2^+,\rho_1^+\rho_2^+)$$

and

$$(\lambda_2,\rho_2)^+(\lambda_1,\rho_1)^+ = (\lambda_2^+\lambda_1^+,\rho_2^+\rho_1^+),$$

we have $(\lambda_1^+\lambda_2^+,\rho_1^+\rho_2^+) = (\lambda_2^+\lambda_1^+,\rho_2^+\rho_1^+)$ and result (ii).

(iii) Obviously, $[(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)]^+ = ((\lambda_1^+\lambda_2)^+,(\rho_1^+\rho_2)^+)$. By computing, we have

$$\begin{aligned}(\lambda_1^+\lambda_2)^+e &= (e\rho_1^+\rho_2)^+ = [e(e\rho_1)^+\rho_2]^+\\ &= [(e\rho_1)^+\cdot e\rho_2]^+ = (e\rho_1)^+\cdot(e\rho_2)^+\\ &= (\lambda_1^+e)\cdot(\lambda_2^+e) = \lambda_1^+(e\cdot\lambda_2^+e)\\ &= (\lambda_1^+\lambda_2^+)e.\end{aligned}$$

Thus $(\lambda_1^+\lambda_2)^+ = (\lambda_1^+\lambda_2^+)$. Moreover,$[(\lambda_1^+\lambda_2)^+,(\rho_1^+\rho_2)^+] = (\lambda_1^+\lambda_2^+,\rho_1^+\rho_2^+)$.

(iv) Because $(\lambda_1,\rho_1)(\lambda_2,\rho_2)^+ = (\lambda_1\lambda_2^+,\rho_1\rho_2^+)$ and $[(\lambda_1,\rho_1)(\lambda_2,\rho_2)]^+(\lambda_1,\rho_1) = ((\lambda_1\lambda_2)^+\lambda_1,(\rho_1\rho_2)^+\rho_1)$, we only need to show that $\rho_1\rho_2^+ = (\rho_1\rho_2)^+\rho_1$. For any $e \in P(S)$,

$$\begin{aligned}e\rho_1\rho_2^+ &= (e\rho_1)(e\rho_1)^*\rho_2^+\\ &= (e\rho_1)[(e\rho_1)^*\rho_2^+]\\ &= [(e\rho_1)(e\rho_1)^*\rho_2^+]^+(e\rho_1)\\ &= [(e\rho_1)\rho_2^+]^+(e\rho_1) \quad (\text{by } (a\rho)^+ = (a\rho^+)^+)\\ &= (e\rho_1\rho_2)^+(e\rho_1) \quad (\text{by } (e\rho)^+ = e\rho^+)\\ &= e\cdot e(\rho_1\rho_2)^+\rho_1\\ &= e(\rho_1\rho_2)^+\rho_1.\end{aligned}$$

Thus, $\rho_1\rho_2^+ = (\rho_1\rho_2)^+\rho_1$ and results (iv).

(v) Note that $(\lambda^+)^*e = (\lambda^+e)^* = [(e\rho)^+]^* = (e\rho)^+ = \lambda^+e$. We obtain that $(\lambda^+)^* = \lambda^+$. Therefore $[(\lambda,\rho)^+]^* = (\lambda,\rho)^+$. □

Dual to Lemma 3.6, we have

Lemma 3.7. *Let* $(\lambda,\rho),(\lambda_1,\rho_1),(\lambda_2,\rho_2)\in\Omega(S)$. *If* S *is a restriction semigroup, then:*

(i) $(\lambda,\rho)(\lambda,\rho)^*=(\lambda,\rho)$;

(ii) $(\lambda_1,\rho_1)^*(\lambda_2,\rho_2)^*=(\lambda_2,\rho_2)^*(\lambda_1,\rho_1)^*$;

(iii) $[(\lambda_1,\rho_1)(\lambda_2,\rho_2))^*]^*=(\lambda_1,\rho_1)^*(\lambda_2,\rho_2)^*$;

(iv) $(\lambda_1,\rho_1)^*(\lambda_2,\rho_2)=(\lambda_2,\rho_2)[(\lambda_1,\rho_1)(\lambda_2,\rho_2)]^*$;

(v) $[(\lambda,\rho)^*]^+=(\lambda,\rho)^*$.

Lemmas 3.6 and 3.7 tell us that for a restriction semigroup S, $\Omega(S)$ is still a restriction semigroup. The following lemma gives a structure of the equivalence σ on $\Omega(S)$.

Lemma 3.8. *Let* $(\lambda_1,\rho_1),(\lambda_2,\rho_2)\in\Omega(S)$. *If* S *is a proper restriction semigroup, then the following statements are equivalent:*

(i) $(\lambda_1,\rho_1)\sigma(\lambda_2,\rho_2)$*;*

(ii) *For all* $e\in P(S),(\lambda_1 e)\sigma(\lambda_2 e)$*;*

(iii) *For all* $e\in P(S),(e\rho_1)\sigma(e\rho_1)$.

Proof. By symmetry, we need only to show (i)⇔(ii). If $(\lambda_1,\rho_1)\sigma(\lambda_2,\rho_2)$, then there exists $(\lambda^*,\rho^*)\in P(\Omega(S))$ such that $(\lambda_1,\rho_1)(\lambda^*,\rho^*)=(\lambda_2,\rho_2)(\lambda^*,\rho^*)$; that is, $(\lambda_1\lambda^*,\rho_1\rho^*)=(\lambda_2\lambda^*,\rho_2\rho^*)$. Then $\lambda_1\lambda^*=\lambda_2\lambda^*$ and for all $e\in P(S)$, $\lambda_1\lambda^*e=\lambda_2\lambda^*e$. Notice that $\lambda^*e\in P(S)$, we have $\lambda_1(\lambda^*e\cdot e)=\lambda_2(\lambda^*e\cdot e)$ and $\lambda_1 e\cdot\lambda^*e=\lambda_2 e\cdot\lambda^*e$. Therefore $(\lambda_1 e)\sigma(\lambda_2 e)$, as required.

Conversely, if for all $e\in P(S),(\lambda_1 e)\sigma(\lambda_2 e)$, then there exists $f\in P(S)$ such that $f\lambda_1 e=f\lambda_2 e$, and so $\lambda_f\lambda_1 e=\lambda_f\lambda_2 e$. By Lemma 3.2, $\lambda_f\lambda_1=\lambda_f\lambda_2$. Thus,

$$(\lambda_f\lambda_1,\rho_f\rho_1)=(\lambda_f\lambda_2,\rho_f\rho_2)$$

That is,

$$(\lambda_f,\rho_f)(\lambda_1,\rho_1)=(\lambda_f,\rho_f)(\lambda_2,\rho_2)$$

By the definition of σ, $(\lambda_1,\rho_1)\sigma(\lambda_2,\rho_2)$, as required. □

Lemma 3.9. *Let* S *be (proper; Clifford; E-reflexive) restriction semigroup. If* T *is a* $(2,1,1)$*-subsemigroup of* S*, then* T *is still a (proper; Clifford; E-reflexive) restriction semigroup.*

Proof. Because T is a $(2,1,1)$-subsemigroup of S, we have that for all $x \in T$, $x^+, x^* \in P(T) \subseteq P(S)$. By definition, the cases for (proper; Clifford) restriction semigroups are obvious. It remains to verify the case for E-reflexive restriction semigroups. To the end, we let $e \in P(T)$, $x, y \in T$ and $exy \in P(T)$. Then $exy \in P(S)$ and as S is E-reflexive, we have $eyx \in P(S)$. So that $eyx = (eyx)^+ = e(yx)^+$ since $yx \in T$ giving $(yx)^+ \in T$. It results that T is an E-reflexive restriction semigroup. □

Lemma 3.10. *Let S be a restriction semigroup. If a is an element in S, then $(\lambda_a, \rho_a)^+ = (\lambda_{a^+}, \rho_{a^+})$ and $(\lambda_a, \rho_a)^* = (\lambda_{a^*}, \rho_{a^*})$. Moreover, $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$.*

Proof. For $a \in S$ and $e \in P(S)$, by Lemma 3.4, we have

$$\lambda_a^+ e = (e\rho_a)^+ = (ea)^+ = ea^+ = a^+e = \lambda_{a^+}e,$$

so that by Lemma 3.2, $\lambda_a^+ = \lambda_{a^+}$. Further, by Lemma 3.3, $(\lambda_a, \rho_a)^+ = (\lambda_{a^+}, \rho_{a^+})$; similarly, $(\lambda_a, \rho_a)^* = (\lambda_{a^*}, \rho_{a^*})$. Now by definition, $\Pi(S)$ is a $(2,1,1)$-subsemigroup since $\Pi(S)$ is a subsemigroup of $\Omega(S)$. The proof is completed. □

Proof of Theorem 2.1: (2.1.1) *The case for restriction semigroups.* By Lemmas 3.6 and 3.10, and Corollary 3.7, we need only to verify the converse part. To the end, we assume that $\Omega(S)$ is a restriction semigroup and $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$. By Lemma 3.1, it suffices to show that $\Pi(S)$ is a restriction semigroup. But $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$, now by Lemma 3.9, $\Pi(S)$ is a restriction semigroup, as required.

(2.1.2) *The case for proper restriction semigroups.* Assume that S is a proper restriction semigroup. By Lemma 3.10, it suffices to verify that $\Omega(S)$ is a proper restriction semigroup. Let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ be such that $(\lambda_1, \rho_1)^* = (\lambda_2, \rho_2)^*$ and $(\lambda_1, \rho_1)\sigma(\lambda_2, \rho_2)$. Then by Lemmas 3.4 and 3.8, $(\lambda_1 e)^* = (\lambda_2 e)^*$ and $(\lambda_1 e)\sigma(\lambda_2 e)$ for all $e \in P(S)$, thus $\lambda_1 e = \lambda_2 e$ since S is a proper restriction semigroup, so that by Lemma 3.2, $\lambda_1 = \lambda_2$. From Lemma 3.3, it follows that $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. Obviously, $(\lambda_1, \rho_1)^+ = (\lambda_2, \rho_2)^+$. Together with $(\lambda_1, \rho_1)\sigma(\lambda_2, \rho_2)$, this shows that $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. Therefore $\Omega(S)$ is a proper restriction semigroup.

For the converse, by Lemma 3.9, $\Pi(S)$ is a proper restriction semigroup, and by Lemma 3.1, S is a proper restriction semigroup. The proof is completed.

(2.1.3) *The case for Clifford restriction semigroups.* If $(\lambda_1, \rho_1) \in \Omega(S)$ and $(\lambda, \rho)^+ \in P(\Omega(S))$, then

$$\lambda_1 \lambda^+ e = \lambda_1 e \cdot \lambda^+ e = \lambda^+(e \cdot \lambda_1 e) = \lambda^+(\lambda_1 e \cdot e) = \lambda^+ \lambda_1 e$$

and so $\lambda_1 \lambda^+ = \lambda^+ \lambda_1$, it follows that $(\lambda, \rho)^+(\lambda_1, \rho_1) = (\lambda_1, \rho_1)(\lambda, \rho)^+$. Therefore $\Omega(S)$ is a Clifford restriction semigroup. Together with Lemma 3.10, we have proved the direct part.

The converse part is immediate from Lemmas 3.1 and 3.9.

(2.1.4) *The case for E-reflexive restriction semigroups.* By Lemmas 3.1 and 3.9, it suffices to show the direct part. Now let S be an E-reflexive restriction semigroup. We notice that in [9, Theorem 4.6], the proof that $\Omega(S)$ is an E-reflexive type B semigroup used only the identities (2.1) and their duals. So, these arguments in [9, Theorem 4.6] have proved that $\Omega(S)$ is of the property that for all $e \in P(\Omega(S))$ and $x, y \in \Omega(S)$, $exy \in P(\Omega(S))$ implies that $eyx \in P(\Omega(S))$. So, $\Omega(S)$ is indeed an E-reflexive restriction semigroup. □

Acknowledgements

The authors are thankful to the referees for their valuable comments. And this research is jointly supported by the National Natural Science Foundation of China (grant: 11761034; 11661042).

References

[1] J.E. Ault, *The translational hull of an inverse semigroup,* Glasgow Math. J., 14 (1973), 56-64.

[2] J.B. Fountain and M.V. Lawson, *The translational hull of an adequate semigroup*, Semigroup Forum, 32 (1985), 79-86.

[3] J.M. Howie, *Fundamentals of semigroup theory*, Oxford University Press, 1995.

[4] M. Gould, *An easy of Ponizovski's theorem*, Semigroup Forum, 15 (1977), 181-182.

[5] V. Gould, *Notes on restriction semigroups and related structures*, notes available at http: //www-users.york.ac.uk/ varg1.

[6] X.J. Guo,Y.Q. Guo, *The translational hull of a strongly right type-A semigroup*, Sci. China (Ser. A), 43 (2000), 6-12.

[7] X.J. Guo,K.P. Shum, *On translational hulls of type-A semigroups*, J. Algebra, 269 (2003), 240-249.

[8] Z.B. Hu, X.J. Guo, *The translational hulls of inverse wpp semigroups*, Int. Math. Forum, 4 (2009), 1397-1404.

[9] C.H. Li,L.M. Wang, *On the translational hull of type B semigroup*, Semigroup Forum, 82 (2011), 516-529.

[10] M. Petrich,N.R. Reilly, *Completely regular semigroups*, John Wiley & Sons, INC., 1999.

Accepted: June 11, 2020