On some properties of divisible and pure fuzzy multigroups

Paul Augustine Ejegwa
Key Laboratory of Intelligent Information Processing and Control
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
China
and
Department of Mathematics/Statistics/Computer Science
University of Agriculture
P.M.B. 2373, Makurdi
Nigeria
ejegwa.augustine@uam.edu.ng
ocholohi@gmail.com

Yuming Feng*
School of Three Gorges Artificial Intelligence
Chongqing Three Gorges University
Chongqing 404100
China
yumingfeng25928@163.com
ymfeng@sanxiau.edu.cn

Wei Zhang
Chongqing Engineering Research Center of Internet of Things and
Intelligent Control Technology
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
China
weizhang@sanxiau.edu.cn;cqec126@126.com

Jiaxi Zhou
School of Electronics and Information Engineering
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
China

Xin Ming
School of Electronics and Information Engineering
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
China

*. Corresponding author

Abstract. Many group's theoretic notions have been established in fuzzy algebra although with some modifications. The theory of fuzzy multigroups is the study of group theory in the context of fuzzy multisets. In this paper we propose and characterize the ideas of divisible and pure fuzzy multigroups with a number of results which are duly proved. It is shown that the homomorphic image and preimage of divisible and pure fuzzy multigroups are as well divisible and pure fuzzy multigroups. The relationship between divisible fuzzy multigroups and divisible groups as well as between pure fuzzy multigroups and pure groups are instituted using the idea of alpha-cuts. Finally, it is established that a fuzzy multigroup of a divisible cyclic group is constant.

Keywords: fuzzy algebra, fuzzy multigroup, divisible fuzzy multigroup, pure fuzzy multigroup.

1. Introduction

The concept of fuzziness was first studied by Zadeh [29] to curb imprecisions in real-life. After the introduction of fuzzy sets, Rosenfeld [22] orchestrated the study of fuzziness in algebra called fuzzy algebra. Myriad of researchers have extended some group's theoretic notions to fuzzy sets and in so doing, established the theory of fuzzy groups [3, 19, 20]. By a way of generalization, Yager [27] proposed fuzzy multiset as a fuzzy set that allowed the repetitions of the membership functions of elements of a universe of discourse. The idea of fuzzy multisets have been extensively studied and applied to many real-life problems as can be seen in [2, 16, 17, 18, 25].

In a way of application, Shinoj et al. [23] applied group theory to fuzzy multisets to introduce a generalized fuzzy groups called fuzzy multigroups and deduced some related results. As a follow up, Ejegwa[4] studied the analog of subgroups in fuzzy multigroup context and introduced centers and centralizers in fuzzy multigroup context. The idea of commutative fuzzy multigroups have been studied and a number of results were presented [1, 5]. Some group's analog notions like normal subgroups, characteristic subgroups, Frattini subgroups, cosets, quotient groups and homomorphism have been instituted in fuzzy multigroup environment with a number of results [6, 7, 11, 13, 14, 15]. The concept of direct product of fuzzy multigroups and its generalization were discussed in [8, 12]. To enhance a nexus between fuzzy multigroup and group theories, the concept of alpha-cuts of fuzzy multigroups was proposed [9] and the idea of its homomorphism was studied in [10].

Although several concepts of group theory have been extended to fuzzy multigroups via fuzzy multisets, some notions have not been investigated in fuzzy multigroup context. This paper seeks to strengthen fuzzy multigroup theory by characterizing the constructs of divisible and pure fuzzy multigroups which are the applications of divisible and pure groups in fuzzy multisets. The notions of divisible and pure groups have been hitherto studied in fuzzy group and intuitionistic fuzzy group contexts [24, 26, 28]. In this voyage, we first and foremost defined a special fuzzy multisets of a group to enhance the introduction of divisible and pure fuzzy multigroups. We study the homomorphic image and preimage of divisible and pure fuzzy multigroups and show that they are as well divisible and pure fuzzy multigroups. The relationship between divisible and pure fuzzy multigroups with divisible and pure groups are instituted using the idea of alpha-cuts. The rest of the paper is delineated as follows; Section 2 presents the ideas of fuzzy multisets, fuzzy multigroups and some existing results. Section 3 discusses and characterizes divisible fuzzy multigroups and Section 4 explicates pure fuzzy multigroups with related results. Finally, Section 5 summarizes and gives scope of future studies.

2. Preliminaries

Throughout this paper X denotes non-empty set and G denotes an additive group with identity element 0.

Definition 2.1 ([27]). *Assume X is a non-empty set. Then, a fuzzy bag/multiset $\mathtt{A}$ drwan from X is an object of the form*

$$\mathtt{A} = \{\langle \frac{CM_{\mathtt{A}}(x)}{x} \rangle \mid x \in X\}$$

characterized by a count membership function $CM_{\mathtt{A}}$ such that

$$CM_{\mathtt{A}} \colon X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0,1]$ and

$$CM_{\mathtt{A}}(x) = \{\mu^1_{\mathtt{A}}(x), \mu^2_{\mathtt{A}}(x), \ldots, \mu^n_{\mathtt{A}}(x), \ldots\},$$

such that $\mu^1_{\mathtt{A}}(x) \geq \mu^2_{\mathtt{A}}(x) \geq \ldots \geq \mu^n_{\mathtt{A}}(x) \geq \ldots$, whereas in a finite case, we write

$$CM_{\mathtt{A}}(x) = \{\mu^1_{\mathtt{A}}(x), \mu^2_{\mathtt{A}}(x), \ldots, \mu^n_{\mathtt{A}}(x)\},$$

for $\mu^1_{\mathtt{A}}(x) \geq \mu^2_{\mathtt{A}}(x) \geq \ldots \geq \mu^n_{\mathtt{A}}(x)$.

A fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset $\mathtt{A}$ can be characterized by a function

$$CM_{\mathtt{A}} \colon X \to N^I \text{ or } CM_{\mathtt{A}} \colon X \to [0,1] \to N,$$

where $I = [0,1]$ and $N = \mathbb{N} \cup \{0\}$.

Definition 2.2 ([27]). *Let $\mathtt{A}$, $\mathtt{B}$ be fuzzy multisets of X. Then for any $x \in X$, we have*

(i) $\mathtt{A} = \mathtt{B} \Longleftrightarrow CM_{\mathtt{A}}(x) = CM_{\mathtt{B}}(x)$,

(ii) $\mathtt{A} \subseteq \mathtt{B} \Longleftrightarrow CM_{\mathtt{A}}(x) \leq CM_{\mathtt{B}}(x)$,

(iii) $\mathtt{A} \cap \mathtt{B} \Longrightarrow CM_{\mathtt{A} \cap \mathtt{B}}(x) = CM_{\mathtt{A}}(x) \wedge CM_{\mathtt{B}}(x)$, (*iv*) $\mathtt{A} \cup \mathtt{B} \Longrightarrow CM_{\mathtt{A}\cup\mathtt{B}}(x) = CM_{\mathtt{A}}(x) \vee CM_{\mathtt{B}}(x)$,

(*v*) $\mathtt{A} \oplus \mathtt{B} \Longrightarrow CM_{\mathtt{A}\oplus\mathtt{B}}(x) = CM_{\mathtt{A}}(x) \oplus CM_{\mathtt{B}}(x)$,

where $\wedge$ *and* $\vee$ *denote minimum and maximum respectively.*

Suppose $\{\mathtt{A}_i\}_{i\in I}$ is an arbitrary family of fuzzy multisets of X. Then

$$\bigcap_{i\in I} \mathtt{A}_i = \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x),\ \forall x \in X \text{ and } \bigcup_{i\in I} \mathtt{A}_i = \bigvee_{i\in I} CM_{\mathtt{A}_i}(x),\ \forall x \in X.$$

Definition 2.3 ([23])**.** *A fuzzy multiset* $\mathtt{A}$ *of* G *is called a fuzzy multigroup if and only if*

(*i*) $CM_{\mathtt{A}}(x+y) \geq CM_{\mathtt{A}}(x) \wedge CM_{\mathtt{A}}(y)$, $\forall x, y \in G$,

(*ii*) $CM_{\mathtt{A}}(-x) \geq CM_{\mathtt{A}}(x)$, $\forall x \in G$.

Certainly, $CM_{\mathtt{A}}(-x) = CM_{\mathtt{A}}(x)$, $\forall x \in G$*, since*

$$CM_{\mathtt{A}}(x) = CM_{\mathtt{A}}(-(-x)) \geq CM_{\mathtt{A}}(-x).$$

Conditions (*i*) *and* (*ii*) *can be combined as* $CM_{\mathtt{A}}(x-y) \geq CM_{\mathtt{A}}(x) \wedge CM_{\mathtt{A}}(y)$ *for all* $x, y \in G$.

A fuzzy multigroup $\mathtt{A}$ *of* G *is said to be constant or has a constant count membership function if* $CM_{\mathtt{A}}(x) = CM_{\mathtt{A}}(y)$, $\forall x, y \in G$*. In fact, in a fuzzy multigroup* $\mathtt{A}$ *of* G, $CM_{\mathtt{A}}(0) \geq CM_{\mathtt{A}}(x)$, $\forall x \in G$.

Definition 2.4 ([5])**.** *A fuzzy multigroup* $\mathtt{A}$ *of* G *is said to be commutative if* $CM_{\mathtt{A}}(x+y) = CM_{\mathtt{A}}(y+x)$, $\forall x, y \in G$.

Proposition 2.1 ([4])**.** *If* $\mathtt{A}$ *and* $\mathtt{B}$ *are fuzzy multigroups of* G*, then* (*i*) $\mathtt{A}\cap\mathtt{B}$ *and* $\mathtt{A} \oplus \mathtt{B}$ *are fuzzy multigroups of* G*, (ii)* $\mathtt{A} \cup \mathtt{B}$ *is a fuzzy multigroup of* G *provided* $\mathtt{A} \subseteq \mathtt{B}$.

Theorem 2.1 ([10])**.** *Let* $\mathtt{A}$ *be a fuzzy multigroup of* G*. Then, the sets* $\mathtt{A}_\alpha$ *and* $\mathtt{A}^\alpha$ *defined by* $\mathtt{A}_\alpha = \{x \in G \mid CM_{\mathtt{A}}(x) \geq \alpha\}$ *and* $\mathtt{A}^\alpha = \{x \in G \mid CM_{\mathtt{A}}(x) \leq \alpha\}$ *where* $\alpha \in [0,1]$*, are subgroups of* G *for* $\alpha \leq CM_{\mathtt{A}}(0)$ *and* $\alpha \geq CM_{\mathtt{A}}(0)$*, respectively.*

Theorem 2.2 ([10])**.** *Let* $\mathtt{A}$ *be a fuzzy multiset of* G *for* $CM_{\mathtt{A}}(0) = 1$ *and* $\alpha \in [0,1]$.

(*i*) *If every* $\mathtt{A}_\alpha$ *is a subgroup of* G*, then* $\mathtt{A}$ *is a fuzzy multigroup of* G.

(*ii*) *If every* $\mathtt{A}^\alpha$ *is a subgroup of* G*, then* $\mathtt{A}$ *is a fuzzy multigroup of* G.

Definition 2.5 ([7])**.** *Let* $f\colon G \to G'$ *be a homomorphism of groups. Suppose* $\mathtt{A}$ *and* $\mathtt{B}$ *are fuzzy multigroups of* G *and* G'*, respectively. Then,* f *induces a homomorphism from* $\mathtt{A}$ *to* $\mathtt{B}$ *which satisfies* (*i*) $CM_{\mathtt{A}}(f^{-1}(y_1y_2)) \geq CM_{\mathtt{A}}(f^{-1}(y_1)) \wedge CM_{\mathtt{A}}(f^{-1}(y_2))\, \forall y_1, y_2 \in G'$,

(*ii*) $CM_{\mathtt{B}}(f(x_1x_2)) \geq CM_{\mathtt{B}}(f(x_1)) \wedge CM_{\mathtt{B}}(f(x_2))\, \forall x_1, x_2 \in G$,

where

(*i*) *the image of* $\mathtt{A}$ *under* f*, denoted by* $f(\mathtt{A})$*, is a fuzzy multiset of* G' *defined by*

$$CM_{f(\mathtt{A})}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_{\mathtt{A}}(x), & f^{-1}(y) \neq \emptyset \\ 0, & otherwise \end{cases}$$

for each $y \in G'$.

(*ii*) *the preimage of* $\mathtt{B}$ *under* f*, denoted by* $f^{-1}(\mathtt{B})$*, is a fuzzy multiset of* G *defined by* $CM_{f^{-1}(\mathtt{B})}(x) = CM_{\mathtt{B}}(f(x))$, $\forall x \in G$.

Proposition 2.2 ([7])**.** *Let* $f\colon G \to G'$ *be a homomorphism of groups,* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *and* G'*, respectively. Then* $f(\mathtt{A})$ *is fuzzy multigroup of* G' *and if* f *is an isomorphism, then* $f^{-1}(\mathtt{B})$ *is a fuzzy multigroup of* G.

3. Divisible fuzzy multigroups

In a classical sense, a divisible group is an abelian group where every element is an *nth* multiple for each positive integer n. An abelian group G is divisible if for every positive integer n and any $x \in G$, there exists $y \in G$ such that $ny = x$. Equivalently, an abelian group G is divisible if and only if $nG = G$, where n is any positive integer and $nG = \{nx | x \in G\}$.

Definition 3.1. Suppose $\mathtt{A}$ is a fuzzy multiset of G. Then $n\mathtt{A}$, where n is a positive integer is defined by

$$CM_{n\mathtt{A}}(x) = \begin{cases} \bigvee_{x=ny} CM_{\mathtt{A}}(y), & x \in nG \\ 0, & otherwise. \end{cases}$$

Proposition 3.1. *If* $\mathtt{A}$ *and* $\mathtt{B}$ *are fuzzy multisets of* G*, then*

(*i*) $\mathtt{A} \subseteq \mathtt{B} \Longrightarrow n\mathtt{A} \subseteq n\mathtt{B}$,

(*ii*) $n(\mathtt{A} \cap \mathtt{B}) = n\mathtt{A} \cap n\mathtt{B}$,

(*iii*) $n(\mathtt{A} \cup \mathtt{B}) = n\mathtt{A} \cup n\mathtt{B}$,

(*iv*) $n(\mathtt{A} \oplus \mathtt{B}) = n\mathtt{A} \oplus n\mathtt{B}$.

Proof. The proof of (i) is straightforward. Now, we prove (ii) as follows. Suppose $x \notin nG$, then $CM_{n(\mathtt{A}\cap\mathtt{B})}(x) = 0 = CM_{n\mathtt{A}\cap n\mathtt{B}}(x)$. Again, assume $x \in nG$, then we have

$$\begin{aligned} CM_{n\mathtt{A}\cap n\mathtt{B}}(x) &= CM_{n\mathtt{A}}(x) \wedge CM_{n\mathtt{B}}(x) \\ &= \bigvee_{x=ny} CM_{\mathtt{A}}(y) \wedge \bigvee_{x=ny} CM_{\mathtt{B}}(y) \\ &= \bigvee_{x=ny} [CM_{\mathtt{A}}(y) \wedge CM_{\mathtt{B}}(y)] \\ &= \bigvee_{x=ny} CM_{\mathtt{A}\cap\mathtt{B}}(y) \\ &= CM_{n(\mathtt{A}\cap\mathtt{B})}(x). \end{aligned}$$

Hence, $n(\mathtt{A} \cap \mathtt{B}) = n\mathtt{A} \cap n\mathtt{B}$.

The proofs of (iii) and (iv) follow directly from Definitions 2.2, 3.1 and (ii). $\square$

Theorem 3.1. *Let $f : G \to G'$ be a homomorphism such that $\mathtt{A}$ is fuzzy multiset of G. Then for any $n \in \mathbb{N}$, $f(n\mathtt{A}) = nf(\mathtt{A})$.*

Proof of Theorem 3.1. Suppose $x \notin nG$, then $CM_{f(n\mathtt{A})}(x) = 0 = CM_{nf(\mathtt{A})}(x)$. Again, suppose we have $x \in nG$. If $z \in G'$ and $z = f(x)$, then

$$\begin{aligned} CM_{f(n\mathtt{A})}(z) &= \bigvee_{z=f(x)} CM_{n\mathtt{A}}(x) = \bigvee_{z=f(x)} \bigvee_{x=nw} CM_{\mathtt{A}}(w) \\ &= \bigvee_{z=f(x),\, x=nw} CM_{\mathtt{A}}(w) = \bigvee_{z=f(nw)} CM_{\mathtt{A}}(w) \\ &= \bigvee_{z=nf(w)} CM_{\mathtt{A}}(w) = \bigvee_{z=ny} \bigvee_{y=f(w)} CM_{\mathtt{A}}(w) \\ &= \bigvee_{z=f(nw)} CM_{f(\mathtt{A})}(y) = CM_{nf(\mathtt{A})}(z). \end{aligned}$$

Hence, $f(n\mathtt{A}) = nf(\mathtt{A})$.

Theorem 3.2. *Suppose f is a homomorphism from G to G' such that $\mathtt{B}$ is a fuzzy multiset of G'. Then for any $n \in \mathbb{N}$, $nf^{-1}(\mathtt{B}) \subseteq f^{-1}(n\mathtt{B})$. Moreover if f is an isomorphism, $nf^{-1}(\mathtt{B}) = f^{-1}(n\mathtt{B})$.*

Proof of Theorem 3.2. Let $x \in G$. If $x \notin nG$, then $CM_{nf^{-1}(\mathtt{B})}(x) = 0 \leq CM_{f^{-1}(n\mathtt{B})}(x)$. If $x \in nG$, then $f(x) \in nG'$. Thus

$$\begin{aligned} CM_{nf^{-1}(\mathtt{B})}(x) &= \bigvee_{x=nw} CM_{f^{-1}(\mathtt{B})}(w) = \bigvee_{x=nw} CM_{\mathtt{B}}(f(w)) \\ &\leq \bigvee_{f(x)=f(nw)} CM_{\mathtt{B}}(f(w)) \leq \bigvee_{f(x)=ny} CM_{\mathtt{B}}(y) \\ &= CM_{\mathtt{nB}}(ny) = CM_{\mathtt{nB}}(f(x)) \\ &= CM_{f^{-1}\mathtt{B}}(x). \end{aligned}$$

Hence, $nf^{-1}(\mathtt{B}) \subseteq f^{-1}(n\mathtt{B})$.

Suppose f is an isomorphism, then by using the logic in Theorem 3.1, we have

$$CM_{f^{-1}(n\mathtt{B})}(x) = CM_{f^{-1}(n(f(f^{-1}(\mathtt{B}))))}(x) = CM_{f^{-1}(f(n(f^{-1}(\mathtt{B}))))}(x) \leq CM_{n(f^{-1}(\mathtt{B}))}(x),$$

so $f^{-1}(n\mathtt{B}) \subseteq nf^{-1}(\mathtt{B})$. Therefore, $nf^{-1}(\mathtt{B}) = f^{-1}(n\mathtt{B})$.

Definition 3.2. Let G be an abelian group. A fuzzy multigroup $\mathtt{A}$ of G is called divisible if $CM_{n\mathtt{A}}(x) = CM_{\mathtt{A}}(x)$, $\forall x \in G$ or simply, $n\mathtt{A} = \mathtt{A}$ for every positive integer n.

Example 3.1. A fuzzy multigroup of (i) an additive group of rational numbers $\mathbb{Q}$ is a divisible fuzzy multigroup, (ii) a group of complex roots of unity of degrees p^k, $k = 1, 2, \ldots, n$, where p is a prime number is a divisible fuzzy multigroup.

Remark 3.1. (i) Every fuzzy multigroup of a divisible group is a divisible commutative fuzzy multigroup. (ii) If $G = \mathbb{Q}$, then every divisible fuzzy multigroup of G has a constant count membership function over $\mathbb{Q} - \{0\}$.

Proposition 3.2. *Let $f\colon G \to G'$ be a homomorphism of groups, and $\mathtt{A}$ be a divisible fuzzy multigroup of G. Then, the homomorphic image of $\mathtt{A}$ is a divisible fuzzy multigroup of G'.*

Proof. By Theorem 2.2, we see that $f(\mathtt{A})$ is a fuzzy multigroup of G'. Thus, it follows that $nf(\mathtt{A}) = f(n\mathtt{A}) = f(\mathtt{A})$, for every $n \in \mathbb{N}$ (Theorem 3.1). Hence, $f(\mathtt{A})$ is divisible. □

Proposition 3.3. *Suppose $f\colon G \to G'$ is an isomorphism of groups, and $\mathtt{B}$ is a divisible fuzzy multigroup of G'. Then, the homomorphic preimage of $\mathtt{B}$ is a divisible fuzzy multigroup of G.*

Proof. By synthesizing Theorems 2.2 and 3.2, it follows that $f^{-1}(\mathtt{B})$ is a divisible fuzzy multigroup of G. □

Definition 3.3. A fuzzy multigroup $\mathtt{A}$ of G is called p-divisible if and only if $CM_{p^k\mathtt{A}}(x) = CM_{\mathtt{A}}(x)$, $\forall x \in G$ where $k \in \mathbb{N}$ and p is a prime.

Proposition 3.4. *Suppose $\mathtt{A}$ is a fuzzy multigroup of G. Then $\mathtt{A}$ is divisible if and only if it is p-divisible for every prime p.*

Proof. Suppose that $\mathtt{A}$ is divisible fuzzy multigroup of G. Certainly, it is p-divisible since $p^k \in \mathbb{N}$.

Conversely, assume $\mathtt{A}$ is a p-divisible fuzzy multigroup of G for every prime p. Then for every $n \in \mathbb{N}$, we get $n = p_1p_2 \ldots p_m$ where p_i (for $i = 1, 2, \ldots, m$) is prime. Thus, $n\mathtt{A} = (p_1p_2 \ldots p_m)\mathtt{A} = \beta\mathtt{A} = \mathtt{A}$ since $n = \beta$. So, $\mathtt{A}$ is divisible. □

Theorem 3.3. *Let $\mathtt{A}$ be a divisible fuzzy multigroup of G. Then the following are divisible subgroups of G:*

(*i*) $\mathtt{A}_\alpha$, $\alpha \in [0,1]$ *for* $\alpha \leq CM_{\mathtt{A}}(0)$.

(*ii*) $\mathtt{A}^\alpha$, $\alpha \in [0,1]$ *for* $\alpha \geq CM_{\mathtt{A}}(0)$.

Proof of Theorem 3.3. (i) By Theorem 2.1, $\mathtt{A}_\alpha$ is a subgroup of G. Let $x \in \mathtt{A}_\alpha$, $\alpha \in [0,1]$ and $n \in \mathbb{N}$. Since $\bigvee_{ny=x} CM_{\mathtt{A}}(y) = CM_{\mathtt{A}}(x) \geq \alpha$, then it happens that $y \in \mathtt{A}_\alpha$ with $ny = x$. Hence, $\mathtt{A}_\alpha$ is a divisible subgroup of G.

(ii) Similarly, $\mathtt{A}^\alpha$ is a subgroup of G by Theorem 2.1. If $x \in \mathtt{A}^\alpha$, $\alpha \in [0,1]$ and $n \in \mathbb{N}$. Then since $\bigvee_{ny=x} CM_{\mathtt{A}}(y) = CM_{\mathtt{A}}(x) \leq \alpha$, it follows that $y \in \mathtt{A}^\alpha$ such that $ny = x$, and the result follows.

Theorem 3.4. *If* $\mathtt{A}$ *is a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}_\alpha$, $\alpha \in [0,1]$*, is a divisible subgroup of* G*. Then* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G*.*

Proof of Theorem 3.4. From the given hypotheses , it follows that $\mathtt{A}$ is a fuzzy multigroup of G by Theorem 2.2. Let $x \in G$ and $CM_{\mathtt{A}}(x) = \alpha$. Since $\mathtt{A}_\alpha$ is a divisible subgroup of G for every $n \in \mathbb{N}$, $\exists\ y \in \mathtt{A}_\alpha$ such that $ny = x$, hence $CM_{\mathtt{A}}(y) \geq CM_{\mathtt{A}}(x) = \alpha$. But $CM_{\mathtt{A}}(y) \leq CM_{\mathtt{A}}(x)$ since $\mathtt{A}$ is a fuzzy multigroup of G. Hence, $\bigvee_{ny=x} CM_{\mathtt{A}}(y) = CM_{n\mathtt{A}}(x)$, for $x \in nG$ and so, $n\mathtt{A} = \mathtt{A}$.

Corollary 3.1. *Let* $\mathtt{A}$ *be a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}^\alpha$, $\alpha \in [0,1]$*, is a divisible subgroup of* G*. Then* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G*.*

Proof. Combining Theorems 2.2 and 3.4, the result holds. □

Theorem 3.5. *Let* $\{\mathtt{A}_i\}_{i\in I}$ *be a family of divisible fuzzy multigroups of* G*. Then* $\bigcap_{i\in I} \mathtt{A}_i$ *is a divisible fuzzy multigroup of* G*.*

Proof of Theorem 3.5. By Proposition 2.1, $\bigcap_{i\in I} \mathtt{A}_i$ is a fuzzy multigroup of G. Assume that every $\mathtt{A}_i$ is divisible, then for $x \in nG$ we have

$$\begin{aligned} CM_{n(\bigcap_{i\in I} \mathtt{A}_i)}(x) &= \bigvee_{x=ny} \bigwedge_{i\in I} CM_{\mathtt{A}_i}(y) = \bigwedge_{i\in I} \bigvee_{x=ny} CM_{\mathtt{A}_i}(y) \\ &= \bigwedge_{i\in I} CM_{n\mathtt{A}_i}(x) = CM_{\bigcap_{i\in I}(n\mathtt{A}_i)}(x) \\ &= CM_{\bigcap_{i\in I} \mathtt{A}_i}(x). \end{aligned}$$

If $x \notin nG$, then $CM_{\bigcap_{i\in I} \mathtt{A}_i}(x) = 0 = CM_{n(\bigcap_{i\in I} \mathtt{A}_i)}(x)$. Hence, $\bigcap_{i\in I} \mathtt{A}_i$ is divisible.

Theorem 3.6. *Suppose* $\{\mathtt{A}_i\}_{i\in I}$ *is a family of divisible fuzzy multigroups of* G*. Then*

(*i*) $\bigcup_{i\in I} \mathtt{A}_i$ *is a divisible fuzzy multigroup of* G *if* $\mathtt{A}_1 \subseteq \mathtt{A}_2 \subseteq \ldots \subseteq \mathtt{A}_n$, $i = 1, 2, \ldots, n$.

(*ii*) $\Sigma_{i\in I} \mathtt{A}_i$ *is a divisible fuzzy multigroup of* G*.*

Proof of Theorem 3.6. By using Proposition 2.1 and following the same arguments in Theorem 3.5, the proofs of (i) and (ii) are established.

Theorem 3.7. *If* $\mathtt{A}$ *is a fuzzy multigroup of a divisible cyclic group* G*, then* $\mathtt{A}$ *is constant.*

Proof of Theorem 3.7. Let $G =< a >$ and $x \in G$. Then $x = pa$, $p \in \mathbb{N}$. Suppose that $CM_{\mathtt{A}}(pa) > CM_{\mathtt{A}}((p+1)a)$. Because G is divisible, there exists $y \in G$ such that $py = a$. But also there exists $q \in \mathbb{N}$ such that $y = qa$, and so

$$\begin{aligned} CM_{\mathtt{A}}(pa) &> CM_{\mathtt{A}}((p+1)a) = CM_{\mathtt{A}}((p+1)py) \\ &= CM_{\mathtt{A}}((p+1)pqa) = CM_{\mathtt{A}}((p+1)q(pa)) \\ &\geq CM_{\mathtt{A}}(pa) \wedge CM_{\mathtt{A}}(pa) \wedge \ldots \wedge CM_{\mathtt{A}}(pa) \\ &= CM_{\mathtt{A}}(pa), \end{aligned}$$

which is a contradiction.

Also, if $CM_{\mathtt{A}}((p+1)a) > CM_{\mathtt{A}}(pa)$. Then there exists $z \in G$ such that $(p+1)z = a$, $z = ra$, $r \in \mathbb{N}$ since G is divisible. Thus

$$\begin{aligned} CM_{\mathtt{A}}((p+1)a) &> CM_{\mathtt{A}}(pa) = CM_{\mathtt{A}}(p(p+1)ra) = CM_{\mathtt{A}}(pr(p+1)a) \\ &\geq CM_{\mathtt{A}}((p+1)a) \wedge CM_{\mathtt{A}}((p+1)a) \wedge \ldots \wedge CM_{\mathtt{A}}((p+1)a) \\ &= CM_{\mathtt{A}}((p+1)a), \end{aligned}$$

which is also a contradiction. Hence, $CM_{\mathtt{A}}((p+1)a) = CM_{\mathtt{A}}(pa)$. Since x is arbitrary, if $x = a$, then $CM_{\mathtt{A}}(a) = CM_{\mathtt{A}}(2a) = CM_{\mathtt{A}}(3a) = \ldots$, and so $\mathtt{A}$ is a constant fuzzy multigroup of a divisible cyclic group G.

4. Pure fuzzy multigroups

Recall that a subgroup H of a group G is called pure if $nH = H \cap nG$, for every positive integer n. Now, we extend the concept to fuzzy multigroup of G as follows.

Definition 4.1. Let $\mathtt{A}$ and $\mathtt{B}$ be fuzzy multigroups of G such that $\mathtt{A} \subseteq \mathtt{B}$. Then $\mathtt{A}$ is a pure fuzzy multigroup of G if $n\mathtt{A} = \mathtt{A} \cap n\mathtt{B}$ for every $n \in \mathbb{N}$, that is $CM_{n\mathtt{A}}(x) = CM_{\mathtt{A}}(x) \wedge CM_{n\mathtt{B}}(x)\ \forall x \in G$.

Remark 4.1. Let $\mathtt{A}$, $\mathtt{B}$ and $\mathtt{C}$ be fuzzy multigroups of G such that $\mathtt{A}$ and $\mathtt{B}$ are contained in $\mathtt{C}$. If $\mathtt{A}$ and $\mathtt{B}$ are pure, then $\mathtt{A} \cap \mathtt{B}$ and $\mathtt{A} \cup \mathtt{B}$ are pure fuzzy multigroups of G since $n(\mathtt{A} \cap \mathtt{B}) = (\mathtt{A} \cap \mathtt{B}) \cap n\mathtt{C}$ and $n(\mathtt{A} \cup \mathtt{B}) = (\mathtt{A} \cup \mathtt{B}) \cap n\mathtt{C}$.

Definition 4.2. Let $\mathtt{A}$ and $\mathtt{B}$ be fuzzy multigroups of G such that $\mathtt{A} \subseteq \mathtt{B}$. Then $\mathtt{A}$ is p-pure if $p^r\mathtt{A} = \mathtt{A} \cap p^r\mathtt{B}$ for every $r \in \mathbb{N}$ and p is prime, that is $CM_{p^r\mathtt{A}}(x) = CM_{\mathtt{A}}(x) \wedge CM_{p^r\mathtt{B}}(x)\ \forall x \in G$.

Remark 4.2. Every p-pure fuzzy multigroup of a p-divisible group is p-divisible.

Proposition 4.1. *Let $f\colon G \to G'$ be an isomorphism of divisible groups, $\mathtt{A}$, $\mathtt{C}$ be fuzzy multigroups of G and $\mathtt{B}$, $\mathtt{D}$ be fuzzy multigroups of G' such that $\mathtt{A} \subseteq \mathtt{C}$ and $\mathtt{B} \subseteq \mathtt{D}$. If $\mathtt{A}$ and $\mathtt{B}$ are pure fuzzy multigroups of G and G', respectively then*

(*i*) *$f(\mathtt{A})$ is a pure fuzzy multigroup of G'.*

(*ii*) *$f^{-1}(\mathtt{B})$ is a pure fuzzy multigroup of G.*

Proof. Let $x, y \in G$ and $w, z \in G'$ such that $f(x, y) = w, z$. Since G, G' are divisible groups for every $n \in \mathbb{N}$, $\exists\, y \in G$ and $z \in G'$ such that $ny = x$, $nz = w$. From Proposition 2.2, it follows that $f(\mathtt{A})$ and $f^{-1}(\mathtt{B})$ are fuzzy multigroups of G and G', respectively. Assume that $\mathtt{C}$ and $\mathtt{D}$ are constant, then by Theorems 3.1 and 3.2, we deduce that, for any $w \in G'$

$$\begin{aligned} CM_{f(n\mathtt{A})}(w) &= CM_{nf(\mathtt{A})}(w) = CM_{f(\mathtt{A})}(w) \wedge CM_{nf(\mathtt{C})}(w) \\ &= CM_{f(\mathtt{A})}(w) \wedge \bigvee_{w=nz} CM_{f(\mathtt{C})}(z) \\ &= CM_{f(\mathtt{A})}(w) \end{aligned}$$

and

$$\begin{aligned} CM_{f^{-1}(n\mathtt{B})}(x) &= CM_{nf^{-1}(\mathtt{B})}(x) = CM_{f^{-1}(\mathtt{B})}(x) \wedge CM_{nf^{-1}(\mathtt{D})}(x) \\ &= CM_{f^{-1}(\mathtt{B})}(x) \wedge \bigvee_{x=ny} CM_{f^{-1}(\mathtt{D})}(y) \\ &= CM_{f^{-1}(\mathtt{B})}(x). \end{aligned}$$

Hence, the results. □

Theorem 4.1. *Suppose $\{\mathtt{A}_i\}_{i\in I}$ and $\{\mathtt{B}_j\}_{j\in J}$ are families of fuzzy multigroups of G such that $\{\mathtt{A}_i\}_{i\in I} \subseteq \{\mathtt{B}_j\}_{j\in J}$ and $\{\mathtt{B}_j\}_{j\in J}$ is constant. If $\{\mathtt{A}_i\}_{i\in I}$ is pure, then $\bigcap_{i\in I} \mathtt{A}_i$ is a pure fuzzy multigroup of G.*

Proof of Theorem 4.1. Certainly, $\bigcap_{i\in I} \mathtt{A}_i$ is a fuzzy multigroup of G by Proposition 2.1. If every $\mathtt{A}_i$ is pure, then we consider the following cases.

Case I: Suppose $x \in nG$, we have

$$\begin{aligned} CM_{n(\bigcap_{i\in I} \mathtt{A}_i)}(x) &= CM_{(\bigcap_{i\in I} \mathtt{A}_i)\cap n(\bigcap_{j\in J} \mathtt{B}_j)}(x) \\ &= \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x) \wedge \bigvee_{x=ny} \bigwedge_{j\in J} CM_{\mathtt{B}_j}(y) \\ &= \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x) \wedge \bigwedge_{j\in J} \bigvee_{x=ny} CM_{\mathtt{B}_j}(y) \\ &= \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x) = CM_{\bigcap_{i\in I} \mathtt{A}_i}(x). \end{aligned}$$

Case II: Suppose $x \notin nG$, then we have

$$CM_{n(\bigcap_{i\in I} \mathtt{A}_i)}(x) = CM_{(\bigcap_{i\in I} \mathtt{A}_i)\cap n(\bigcap_{j\in J} \mathtt{B}_j)}(x) = 0 = CM_{\bigcap_{i\in I} \mathtt{A}_i}(x).$$

Hence, $\bigcap_{i\in I} \mathtt{A}_i$ is a pure fuzzy multigroup of G.

Theorem 4.2. *Let* $\{\mathtt{A}_i\}_{i\in I}$ *and* $\{\mathtt{B}_j\}_{j\in J}$ *be families of fuzzy multigroups of* G *such that* $\{\mathtt{B}_j\}_{j\in J}$ *is constant and contains* $\{\mathtt{A}_i\}_{i\in I}$*. If* $\{\mathtt{A}_i\}_{i\in I}$ *is pure, then*

(i) $\bigcup_{i\in I}\mathtt{A}_i$ *is a pure fuzzy multigroup of* G *for* $\mathtt{A}_1 \subseteq \mathtt{A}_2 \subseteq \ldots \subseteq \mathtt{A}_n$, $i = 1, 2, \ldots, n$.

(ii) $\Sigma_{i\in I}\mathtt{A}_i$ *is a pure fuzzy multigroup of* G.

Proof of Theorem 4.2. By synthesizing Proposition 2.1 and following the same logic in Theorem 4.1, the proofs of (i) and (ii) are established.

Theorem 4.3. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. Then* $\mathtt{A}$ *is pure if and only if* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G.

Proof of Theorem 4.3. *Suppose* $\mathtt{A}$ *is a pure fuzzy multigroup of* G*. Then* $n\mathtt{A} = \mathtt{A} \cap n\mathtt{B}$ *for* $n \in \mathbb{N}$*. Certainly,* $\mathtt{A} \cap n\mathtt{B} = \mathtt{A}$ *because*

$$\begin{aligned} CM_{\mathtt{A}}(x) \wedge CM_{n\mathtt{B}}(x) &= CM_{\mathtt{A}}(x) \wedge \bigvee_{ny=x} CM_{\mathtt{B}}(y) \\ &= CM_{\mathtt{A}}(x)\ \forall x \in X. \end{aligned}$$

Thus, $n\mathtt{A} = \mathtt{A}$ *for* $n \in \mathbb{N}$*, and hence* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G.

Conversely, assume $\mathtt{A}$ *is a divisible fuzzy multigroup of* G*. Then* $n\mathtt{A} = \mathtt{A}$ *for* $n \in \mathbb{N}$*. Since* $\mathtt{A} \cap n\mathtt{B} = \mathtt{A}$*, we have* $n\mathtt{A} = \mathtt{A} \cap n\mathtt{B}$ $n \in \mathbb{N}$*, and so* $\mathtt{A}$ *is a pure fuzzy multigroup of* G.

On the strength of Theorem 4.3, we state the following results without proofs because their proofs are similar to their equivalent proofs in Section 3.

Proposition 4.2. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. Then* $\mathtt{A}$ *is pure if and only if it is* p*-pure for every prime* p.

Proof. Similar to Proposition 3.4. □

Proposition 4.3. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. If* $\mathtt{A}$ *is a pure fuzzy multigroup of* G*, then* $\mathtt{A}_\alpha$, $\alpha \in [0, 1]$ *is a pure subgroup of* G *for* $\alpha \leq CM_{\mathtt{A}}(0)$ *and* $\mathtt{A}^\alpha$, $\alpha \in [0, 1]$ *is a pure subgroup of* G *for* $\alpha \geq CM_{\mathtt{A}}(0)$.

Proof. Similar to Theorem 3.3. □

Proposition 4.4. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multisets of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. If* $\mathtt{A}$ *is a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}_\alpha$, $\alpha \in [0, 1]$*, is a pure subgroup of* G*. Then* $\mathtt{A}$ *is a pure fuzzy multigroup of* G.

Proof. Similar to Theorem 3.4. □

Corollary 4.1. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multisets of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. Let* $\mathtt{A}$ *be a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}^\alpha$, $\alpha \in [0, 1]$*, is a pure subgroup of* G*. Then* $\mathtt{A}$ *is a pure fuzzy multigroup of* G.

Proof. Similar to Corollary 3.1. □

5. Conclusion

The notion of fuzzy multigroups is the application of group theory to fuzzy multisets. Several analog concepts of group theory have been established in fuzzy multigroup environment. In this paper the concepts of divisible and pure fuzzy multigroups were established and characterized with some duly proved related results. The nexus between divisible fuzzy multigroups and divisible groups as well as between pure fuzzy multigroups and pure groups were instituted with the aid of alpha-cuts. More properties of divisible and pure fuzzy multigroups could be constructed in future research.

Acknowledgements

This work is supported by Foundation of Chongqing Municipal Key Laboratory of Institutions of Higher Education ([2017]3), Foundation of Chongqing Development and Reform Commission (2017[1007]), and Foundation of Chongqing Three Gorges University.

References

[1] A. Baby, T. K. Shinoj, J. J. Sunil, *On abelian fuzzy multigroups and order of fuzzy multigroups*, Journal of New Theory, 5 (2015), 80-93.

[2] R. Biswas, *An application of Yager's bag theory in multicriteria based decision making problems*, International Journal of Intelligent Systems, 14 (1999), 1231-1238.

[3] P. Bhattacharya, N. P. Mukherjee, *Fuzzy groups: some group theoretic analogs II*, Information Science, 41 (1987), 77-91.

[4] P. A. Ejegwa, *On fuzzy multigroups and fuzzy submultigroups*, Journal of Fuzzy Mathematics, 26 (2018), 641-654.

[5] P. A. Ejegwa, *On abelian fuzzy multigroups*, Journal of Fuzzy Mathematics, 26 (2018), 655-668.

[6] P. A. Ejegwa, *On normal fuzzy submultigroups of a fuzzy multigroup*, Theory and Applications of Mathematics and Computer Science, 8 (2018), 64-80.

[7] P. A. Ejegwa, *Homomorphism of fuzzy multigroups and some of its properties*, Applied and Application of Mathematics, 13 (2018), 114-129.

[8] P. A. Ejegwa, *Direct product of fuzzy multigroups*, Journal of New Theory, 28 (2019), 62-73.

[9] P. A. Ejegwa, *On alpha-cuts homomorphism of fuzzy multigroups*, Annals of Fuzzy Mathematics and Informatics, 19 (2020), 73-87.

[10] P. A. Ejegwa, *Some properties of alpha-cuts of fuzzy multigroups*, Journal of Fuzzy Mathematics, 28 (2020), 201-222.

[11] P. A. Ejegwa, *Some group's theoretic notions in fuzzy multigroup context*, In: Handbook of Research on Emerging Applications of Fuzzy Algebraic Structures, pp. 34-62, IGI Global Publisher, Hershey, Pennsylvania 17033-1240, USA, 2020.

[12] P. A. Ejegwa, *On generalized direct product of fuzzy multigroups*, Annals of Communications in Mathematics, 3 (2020), 35-45.

[13] P. A. Ejegwa, *Some characterizations of fuzzy comultisets and quotient fuzzy multigroups*, Ratio Mathematica, 38 (2020), 237-259.

[14] P. A. Ejegwa, J. M. Agbetayo, *On commutators of fuzzy multigroups*, Earthline Journal of Mathematics, 4 (2020), 189-210.

[15] P. A. Ejegwa, J. M. Agbetayo, J. A. Otuwe, *Characteristic and Frattini fuzzy submultigroups of fuzzy multigroups*, Annals of Fuzzy Mathematics and Informatics, 19 (2020), 139-155.

[16] S. Miyamoto, *Basic operations of fuzzy multisets*, Journal of Japan Society of Fuzzy Theory and Systems, 8 (1996), 639-645.

[17] S. Miyamoto, K. Mizutani, *Fuzzy multiset model and methods for nonlinear document clustering for information retrieval*, Springer-Verlag Berlin Heidelberg, 2004.

[18] K. Mizutani, R. Inokuchi, S. Miyamoto, *Algorithms of nonlinear document clustering based on fuzzy multiset model*, International Journal of Intelligent Systems, 23 (2008), 176-198.

[19] J. M. Mordeson, K. R. Bhutani, A. Rosenfeld, *Fuzzy group theory*, Springer-Verlag Berlin Heidelberg, 2005.

[20] N. P. Mukherjee, P. Bhattacharya, *Fuzzy groups: some group theoretic analogs*, Information Science, 39 (1986), 247-268.

[21] R. Rasuli, *$t-$norms over fuzzy multigroups*, Earthline Journal of Mathematics, 3 (2020), 207-228.

[22] A. Rosenfeld, *Fuzzy subgroups*, Journal of Mathematical Analysis and Applications, 35 (1971), 512-517.

[23] T. K. Shinoj, A. Baby, J. J. Sunil, *On some algebraic structures of fuzzy multisets*, Annals of Fuzzy Mathematics and Informatics, 9 (2015), 77-90.

[24] F. I. Sidky, M. A. Mishref, *Divisible and pure fuzzy subgroups*, Fuzzy Sets and Systems, 34 (1990), 377-382.

[25] A. Syropoulos, *On generalized fuzzy multisets and their use in computation*, Iranian Journal of Fuzzy Systems, 9(2) (2012), 113-125.

[26] G. Wenxiang, L. Tu, *The properties of fuzzy divisible groups*, Fuzzy Sets and Systems, 56 (1993), 195-198.

[27] R. R. Yager, *On the theory of bags*, International Journal of General Systems, 13 (1986), 23-37.

[28] S. Yamak, O. Kazanc, B. Davvaz, *Divisible and pure intuitionistic fuzzy subgroups and their properties*, International Journal of Fuzzy Systems, 10 (2008), 298-307.

[29] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

Accepted: June 01, 2020