# Logical entropy of partitions in hyperproduct MV-algebras

**Ronnason Chinram**
*Algebra and Applications Research Unit*
*Department of Mathematics and Statistics*
*Faculty of Science*
*Prince of Songkla University, Hat Yai*
*Songkhla 90110*
*Thailand*
*Centre of Excellence in Mathematics*
*CHE, Si Ayuthaya Road*
*Bangkok 10400*
*Thailand*
*ronnason.c@psu.ac.th*

**Thananya Kaewnoi**
*Algebra and Applications Research Unit*
*Department of Mathematics and Statistics*
*Faculty of Science*
*Prince of Songkla University, Hat Yai*
*Songkhla 90110*
*Thailand*
*thananya.k2538@gmail.com*

**Aiyared Iampan***
*Department of Mathematics*
*School of Science*
*University of Phayao*
*Phayao 56000*
*Thailand*
*aiyared.ia@up.ac.th*

**Abstract.** In this paper, we introduce the algebraic structure hyperproduct MV-algebras which is a generalization of product MV-algebras. In addition, we study the logical entropy and the logical conditional entropy of partitions in a hyperproduct MV-algebra and provide their properties.
**Keywords:** MV-algebras, hyperproduct MV-algebras, partitions, logical entropy.

## 1. Introduction

The Shannon entropy is created by Claude Elwood Shannon, an American mathematician, who is well known as the founder of the information theory. This theory is found in many applications in other areas, including statistical in-

$*$. Corresponding author

ference, cryptography, quantum computing and so on. The basic concept of this theory is a measurable partition of a probability space which is a measurable partition $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ with probabilities $p_i = P(A_i)$ where $i = 1, 2, \dots, n$. The Shannon entropy of a measurable partition $\mathcal{A}$ is the number $h^S(\mathcal{A}) = \sum_{i=1}^{n} S(p_i)$ where $S : [0,1] \to [0,\infty)$ is the Shannon entropy function defined by

$$S(X) = \begin{cases} -x \log x, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The idea of Shannon entropy was generalized in many way, e.g. Ellerman [7] introduced the logical entropy of a measeurable partition $\mathcal{A}$ which is the number $h^l(\mathcal{A}) = \sum_{i=1}^{n} l(p_i)$, where $l : [0,1] \to [0,1]$ is the logical entropy function defined by $l(x) = x(1-x)$. Furthermore, he investigated the relations between logical entropy and the Shannon entropy. The logical entropy plays an important role as a mathematical device in many problem areas, including quantum theory, information theory, computer science, statistics and many other fields. For instant, Ebrahimzadeh introduced the notion of logical entropy of partitions and logical conditional entropy of dynamical systems in quantum logics and proved some of its properties in [4] and [5]. A year later, Ebrahimzadeh, Giski and Markechová [6] defined the logical entropy and logical mutual information of finite partitions and provided its properties. Moreover, they defined the logical entropy of a dynamical system by using the concept of entropy of partitions and proved that logical entropy of dynamical systems is invariant under isomorphisms. Markechová and Riečan defined logical entropy, logical conditional entropy, logical mutual information and logical conditional mutual information of fuzzy partitions of a fuzzy probability space, and studied logical entropy of fuzzy dynamical systems in [8] and [9].

It is well known that the classical two-valued logic, there are only two possible values (TRUE and FALSE), and it can be extended to $n$-valued logic where $n$ is a natural number greater than two. One of those is the Łuksiewicz many-valued logic (cf. [15]). The notion of an MV-algebra (MV := many valued) was originally presented by Chang [3]. The structure of this algebra was explored by using an algebraic counterpart of the Łuksiewicz many-valued logic. MV-algebra is mentioned by Montagna [13] and Riečan [21] that it is a generalization of the fuzzy logic and the probability logic. Consequently, the study of MV-algebras was created, for example, derivations of MV-Algebras was studied in [1] and roughness in MV-algebras was studied in [18].

**Definition 1.1** ([19])**.** *An* MV-algebra *is an algebraic structure* $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ *where* $\oplus$ *is a commutative and associative binary operation on a nonempty set* $A$*,* $\otimes$ *is a binary operation on* $A$*,* $\perp$ *is a unary operation on* $A$*,* $0$ *and* $1$ *belong to* $A$ *and for any* $a, b \in A$ *satisfying the following conditions:*

(1) $a \oplus 0 = a$,

(2) $a \oplus 1 = 1$,

(3) $(a^{\perp})^{\perp} = a$,

(4) $0^{\perp} = 1$,
(5) $a \oplus a^{\perp} = 1$,
(6) $(a^{\perp} \oplus b)^{\perp} \oplus b = (a \oplus b^{\perp})^{\perp} \oplus a$,
(7) $a \otimes b = (a^{\perp} \oplus b^{\perp})^{\perp}$.

**Example 1.1.** Let $A$ be the unit real interval $[0,1]$ and $a, b \in A$. Define $a \oplus b = \min\{a + b, 1\}, a \otimes b = \max\{a + b - 1, 0\}$ and $a^{\perp} = 1 - a$. Then the system $(A, \oplus, \otimes, \perp, 0, 1)$ is an MV-algebra.

Remind that an alelian group $(G, +)$ which is also a partially ordered set such that for all $x, y, z \in G, x + z \leq y + z$ whenever $x \leq y$, is called a commutative lattice order or simply called an abelian $l$-group if the partial order is a lattice, i.e., for all $a, b \in G$, there exist a least upper bound and a greatest lower bound of $a$ and $b$ which are denoted by $a \vee b$ and $a \wedge b$, respectively. An element $u$ in $G$ is a strong unit of $G$ if for all elements $a$ of $G$, there is a natural number $n$ satisfying the condition $a \leq nu$ (cf. [2]).

In [14], Mundici generalized Chang's results [3]. He proved that for every MV-algebra $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, u)$, there exists an abelian $l$-group $L$ with a strong unit $u$ such that $\mathcal{A} \cong A_0(L, u)$ where $\mathcal{A}_0(L, u) := ([0, u], \oplus, \otimes, \perp, 0, u)$ and $[0, u]$ denote the set of elements in $L$ satisfying $0 \leq a \leq u$. We say that $L$ is an abelian $l$-group corresponding to $\mathcal{A}$. Hence, any MV-algebras can be represented by an abelian $l$-group with a strong unit $u$.

**Example 1.2.** Let $(L, +, \leq)$ be an abelian $l$-group, $u \in L$ be a strong unit of $L$ such that $u > 0$ where 0 is a neutral element of $L$, and $[0, u]$ denote the set of elements in $L$ satisfying $0 \leq a \leq u$. Define $a^{\perp} = u - a, a \oplus b = (a + b) \wedge u, a \otimes b = (a + b - u) \vee 0$ and $1 = u$. Then the system $\mathcal{A}_0(L, 1) := ([0, 1], \oplus, \otimes, \perp, 0, 1)$ is an MV-algebra.

Obviously, $a + b \leq u$ implies $a \oplus b = a + b$. In addition, the conditions $a \otimes b = 0$ and $a + b \leq u$ are equivalent. From now on, the operation $\oplus$ in the definition is substituted by the group operation $+$ in the abelian $l$-group $L$ corresponding to an MV-algebra $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ and $u$ is a strong unit of $L$.

**Definition 1.2** ([17])**.** *Let $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ be an MV-algebra. A* partition *in $\mathcal{A}$ is an $n$-tuple $a = (a_1, a_2, \ldots, a_n)$ of elements of $A$ with the property $a_1 + a_2 + \ldots + a_n = u$.*

Consequently, the study of product MV-algebras was created and many researchers have been working on it ever since.

**Definition 1.3** ([20])**.** *A* product MV-algebra *is an algebraic structure $(\mathcal{A}, \cdot) = (A, \oplus, \otimes, \cdot, \perp, 0, 1)$ where $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ is an MV-algebra and $\cdot$ is a commutative and associative binary operation on a nonempty set $A$ satisfying the following conditions:*

(1) *for every $x \in A, u \cdot x = x$,*

(2) *if* $x, y, z \in A$ *such that* $x + y \leq u$, *then (i)* $z \cdot (x + y) = z \cdot x + z \cdot y$ *and (ii)* $z \cdot x + z \cdot y \leq u$.

A state defined on $(\mathcal{A}, \cdot)$ plays the role of a probability measure on $\mathcal{A}$. It is investigated in [21] by Riečan.

**Definition 1.4** ([23])**.** *A* state *on a product MV-algebra* $(\mathcal{A}, \cdot)$ *is a mapping* $\mu : A \to [0, 1]$ *satisfying the following properties:*
(1) $\mu(u) = 1$,
(2) *if* $x, y \in A$ *such that* $x + y \leq u$, *then* $\mu(x + y) = \mu(x) + \mu(y)$.

In 2018, Markechová, Mosapour and Ebrahimzadeh [11] studied the logical entropy, the logical divergence, and the logical mutual information in a product MV-algebra. In 2019, Markechová and Riečan [10] extended the study of logical entropy of partitions in a product MV-algebra by introducing a general type of entropy of a dynamical system in product MV-algebras. Besides, the entropy of a partition and dynamical system in product MV-algebra was studied in [16] and [22].

The concept of algebraic hyperstructures was first introduced by Marty [12] in 1934 and it has been further studied in various aspects by many authors. Let $H$ be a nonempty set. A hyperoperation $\circ$ on $H$ is a mapping $\circ : H \times H \to P^*(H)$ where $P^*(H)$ is the set of all nonempty subsets of $H$. The order pair $(H, \circ)$ is called a hypergroupoid. If $A$ and $B$ are two nonempty subsets of $H$ and $x \in H$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

In this paper, we introduce the hyperproduct MV-algebras and generalize some results in [11] by using the notion of the hyperproduct MV-algebra. The study of the logical entropy and the logical conditional entropy of partitions in a hyperproduct MV-algebra and their properties are provided in the next section.

## 2. Main results

Firstly, we define hyperproduct MV-algebras as follows:

**Definition 2.1.** *A* hyperproduct MV-algebra *is an algebraic structure* $(A, \oplus, \otimes, \circ, \perp, 0, 1)$ *where* $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ *is an MV-algebra and* $\circ$ *is an associative and commutative binary hyperoperation on* $A$ *satisfying the following properties:*
(1) *for every* $x, y \in A, x \circ y$ *is a finite set,*
(2) *for every* $x \in A, u \circ x = \{x\}$,
(3) *if* $x, y, z \in A$ *such that* $x + y \leq u$, *then*
(*i*) $z \circ (x + y) = z \circ x + z \circ y$ *and* (*ii*) $a \in z \circ x + z \circ y$ *implies* $a \leq u$.

**Example 2.1.** Let $\mathcal{A} = (A, \oplus, \otimes, \cdot, \perp, 0, 1)$ be a product MV-algebra. Define the hyperoperation $\circ$ on $A$ by $x \circ y = \{x \cdot y\}$ for all $x, y \in A$. Then $(A, \oplus, \otimes, \circ, \perp, 0, 1)$ is a hyperproduct MV-algebra.

It can be seen that every product MV-algebra can be considered as a hyperproduct MV-algebra. This implies that hyperproduct MV-algebra is one of generalizations of product MV-algebras. Thoughout the rest of this paper, let $(\mathcal{A}, \circ)$ denote a hyperproduct MV-algebra $(A, \oplus, \otimes, \circ, \perp, 0, 1)$.

### 2.1 Logical entropy of hyperproduct MV-algebras

**Definition 2.2.** *Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a partition of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0, 1]$ be a state on $(\mathcal{A}, \circ)$. The* logical entropy *of $\alpha$ with respect to a state $\mu$ is defined by*

$$h^l_\mu(\alpha) = \sum_{i=1}^{n} \mu(a_i)(1 - \mu(a_i)).$$

**Example 2.2.** Let $\mu : A \to [0, 1]$ be a state on a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. The following statements hold true.

(1) If we put $\alpha_1 = (u)$, then $\alpha_1$ is a partition of $(\mathcal{A}, \circ)$ and $h^l_\mu(\alpha_1) = 0$.

(2) Let $a \in \mathcal{A}$ be such that $\mu(a) \in (0, 1)$. It is obvious that the order pair $\alpha_2 = (a, u - a)$ is a partition of $(\mathcal{A}, \circ)$ and $h^l_\mu(\alpha_2) = 2\mu(a)(1 - \mu(a))$.

**Proposition 2.1.** *Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a partition of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. If $\mu : \mathcal{A} \to [0, 1]$ is a state on $(\mathcal{A}, \circ)$, then*

$$h^l_\mu(\alpha) = 1 - \sum_{i=1}^{n} (\mu(a_i))^2.$$

**Proof.** Since $\sum_{i=1}^{n} \mu(a_i) = \mu(\sum_{i=1}^{n}(a_i)) = \mu(u) = 1$, we have

$$\begin{aligned} h^l_\mu(\alpha) &= \sum_{i=1}^{n} \mu(a_i)(1 - \mu(a_i)) = \sum_{i=1}^{n} \mu(a_i) - (\mu(a_i))^2 \\ &= \sum_{i=1}^{n} \mu(a_i) - \sum_{i=1}^{n} (\mu(a_i))^2 = 1 - \sum_{i=1}^{n} (\mu(a_i))^2. \end{aligned}$$

□

**Proposition 2.2.** *Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a partition of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. If $\mu : A \to [0, 1]$ is a state on $(\mathcal{A}, \circ)$ and $b \in A$, then*

$$\mu(b) = \sum_{i=1}^{n} \sum_{x \in a_i \circ b} \mu(x).$$

**Proof.** Let $b \in A$. Then $\{b\} = u \circ b$. Hence, $\mu(b) = \sum_{y \in \{b\}} \mu(y) = \sum_{y \in u \circ b} \mu(y) = \sum_{y \in (\sum_i^n a_i) \circ b} \mu(y) = \sum_{y \in \sum_i^n (a_i \circ b)} \mu(y) = \sum_{i=1}^n \sum_{x \in a_i \circ b} \mu(x)$. □

Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots b_m)$ be partitions of $(\mathcal{A}, \circ)$. We define the partition $\alpha \vee \beta$ of $(\mathcal{A}, \circ)$ as follows:

$$\alpha \vee \beta = (x \mid x \in a_i \circ b_j \exists i \in \{1, 2, \ldots, n\} \text{ and } \exists j \in \{1, 2, \ldots, m\}).$$

**Proposition 2.3.** *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. Then $\alpha \vee \beta$ is a partition of $(\mathcal{A}, \circ)$.*

**Proof.** Since $\{\sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} x\} = \sum_{i=1}^n \sum_{j=1}^m a_i \circ b_j = (\sum_{i=1}^n a_i) \circ (\sum_{j=1}^m b_j) = u \circ u = \{u\}$, we have $\sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} x = u$ which implies that $\alpha \vee \beta$ is a partition of $(\mathcal{A}, \circ)$. □

In the system of all partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$, we define the *refinement partial order* $\succ$ as follows:

**Definition 2.3.** *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. We say that $\beta$ is a* refinement *of $\alpha$ which is denoted by $\beta \succ \alpha$, if for each $i = 1, 2, \ldots, n$, there exists a subset $I(i)$ of a set $\{1, 2, \ldots, m\}$ such that $a_i = \sum_{j \in I(i)} b_j$.*

**Proposition 2.4.** *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. Then $\alpha \vee \beta \succ \alpha$.*

**Proof.** By the definition of hyperproducts, $\{a_i\} = a_i \circ u = a_i \circ (\sum_{j=1}^m b_j) = \sum_{j=1}^m a_i \circ b_j$, for $i = 1, 2, \ldots, n$. Hence, $a_i = \sum_{j=1}^m \sum_{x \in a_i \circ b_j} x$, so $\alpha \vee \beta \succ \alpha$. □

### 2.2 Logical conditional entropy of partitions in hyperproduct MV-algebras

**Definition 2.4.** *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0, 1]$ be a state on $(\mathcal{A}, \circ)$. The* logical conditional entropy *of $\alpha$ given $\beta$ is defined by*

$$h_\mu^l(\alpha/\beta) = \sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} \mu(x)(\mu(b_j) - \mu(x)).$$

**Proposition 2.5.** *If $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ are partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0, 1]$ is a state on $(\mathcal{A}, \circ)$, then*

$$h_\mu^l(\alpha/\beta) = \sum_{j=1}^m (\mu(b_j))^2 - \sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} (\mu(x))^2.$$

**Proof.** By Proposition 2.2, $\mu(b_j) = \sum_{i=1}^{n} \sum_{x \in a_i \circ b_j} \mu(x)$. Then

$$\begin{aligned}
h_\mu^l(\alpha/\beta) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} \mu(x)(\mu(b_j) - \mu(x)) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} \mu(x)\mu(b_j) - (\mu(x))^2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} \mu(x)\mu(b_j) - \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} (\mu(x))^2 \\
&= \sum_{j=1}^{m} (\mu(b_j))^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} (\mu(x))^2.
\end{aligned}$$

□

Note that the logical conditional entropy $h_\mu^l(\alpha/\beta)$ is always nonnegative. It can be easily verified that $h_\mu^l(\alpha/\beta) = h_\mu^l(\alpha)$ if the partition $\beta = (u)$.

**Theorem 2.1.** *Let $\alpha, \beta$ be two arbitrary partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0,1]$ be a state on $(\mathcal{A}, \circ)$. Then*

$$h_\mu^l(\alpha \vee \beta) = h_\mu^l(\alpha) + h_\mu^l(\beta/\alpha).$$

**Proof.** Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ be partitions of $(\mathcal{A}, \circ)$. By Proposition 2.1 and Proposition 2.5,

$$\begin{aligned}
h_\mu^l(\alpha) + h_\mu^l(\beta/\alpha) &= [1 - \sum_{i=1}^{n} (\mu(a_i))^2] + [\sum_{i=1}^{n} (\mu(a_i))^2 - \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{x \in b_j \circ a_i} (\mu(x))^2] \\
&= 1 - \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} (\mu(x))^2 = h_\mu^l(\alpha \vee \beta).
\end{aligned}$$

□

**Corollary 2.1.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. Then*

$$h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) = h_\mu^l(\alpha_1) + \sum_{i=2}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1})).$$

**Proof.** By Theorem 2.1,

$$
\begin{aligned}
& h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) \\
& = h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1}) + h_\mu^l(\alpha_n/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1})) \\
& = [h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}) + h_\mu^l(\alpha_{n-1}/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}))] \\
& \quad + h_\mu^l(\alpha_n/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1})) \\
& = h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}) + \sum_{i=n-1}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1})) \\
& \qquad \vdots \\
& = h_\mu^l(\alpha_1) + \sum_{i=2}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1})).
\end{aligned}
$$

□

**Theorem 2.2.** *For arbitrary partitions $\alpha, \beta, \gamma$ of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and a state $\mu : A \to [0,1]$ on $(\mathcal{A}, \circ)$,*

$$h_\mu^l((\alpha \vee \beta)/\gamma) = h_\mu^l(\alpha/\gamma) + h_\mu^l(\beta/(\alpha \vee \gamma)).$$

**Proof.** Let $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_m)$ and $\gamma = (c_1, c_2, \ldots, c_r)$ be partitions of $(\mathcal{A}, \circ)$. By Proposition 2.5,

$$
\begin{aligned}
h_\mu^l(\alpha/\gamma) + h_\mu^l(\beta/\alpha \vee \gamma) &= [\sum_{k=1}^{r} (\mu(c_k))^2 - \sum_{i=1}^{n} \sum_{k=1}^{r} \sum_{x \in a_i \circ c_k} (\mu(x))^2] \\
& + [\sum_{i=1}^{n} \sum_{k=1}^{r} \sum_{x \in a_i \circ c_k} (\mu(x))^2 - \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{r} \sum_{x \in b_j \circ a_i \circ c_k} (\mu(x))^2] \\
& = \sum_{k=1}^{r} (\mu(c_k))^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} \sum_{x \in a_i \circ b_j \circ c_k} (\mu(x))^2 = h_\mu^l((\alpha \vee \beta)/\gamma).
\end{aligned}
$$

□

**Corollary 2.2.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0,1]$ be a state on $(\mathcal{A}, \circ)$. Then*

$$h_\mu^l((\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)/\gamma) = h_\mu^l(\alpha_1/\gamma) + \sum_{i=2}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1} \vee \gamma)).$$

**Theorem 2.3.** *For arbitrary partitions $\alpha, \beta$ of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0,1]$ is a state on $(\mathcal{A}, \circ)$, each of the following statements hold true:*

(*i*) $h_\mu^l(\alpha/\beta) \leq h_\mu^l(\alpha)$, *(ii)* $h^l_\mu(\alpha \vee \beta) \leq h^l_\mu(\alpha) + h^l_\mu(\beta)$.

**Proof.** Let $\alpha = (a_1, a_2, \dots a_n)$ and $\beta = (b_1, b_2, \dots, b_m)$ be partitions of $(\mathcal{A}, \circ)$.

(i) By Proposition 2.2, $\mu(a_i) = \sum_{j=1}^m \sum_{x\in a_i\circ b_j} \mu(x)$ for $i = 1, 2, \dots, n$. Then we obtain that

$$\sum_{j=1}^m \sum_{x\in a_i\circ b_j} \mu(x)(\mu(b_j) - \mu(x)) \leq [\sum_{j=1}^m \sum_{x\in a_i\circ b_j} \mu(x)][\sum_{j=1}^m \sum_{x\in a_i\circ b_j} (\mu(b_j) - \mu(x))]$$
$$= \mu(a_i)[\sum_{j=1}^m \mu(b_j) - \sum_{j=1}^m \sum_{x\in a_i\circ b_j} \mu(x)] = \mu(a_i)(1 - \mu(a_i)).$$

Hence, $h^l_\mu(\alpha/\beta) = \sum_{i=1}^n \sum_{j=1}^m \sum_{x\in a_i\circ b_j} \mu(x)(\mu(b_j) - \mu(x)) \leq \sum_{i=1}^n \mu(a_i)(1 - \mu(a_i)) = h^l_\mu(\alpha)$.

(ii) By using Theorem 2.1 and (i). □

**Theorem 2.4.** *For arbitrary partitions $\alpha, \beta$ of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and a state $\mu : A \to [0, 1]$ on $(\mathcal{A}, \circ)$, each of the following statements hold true:*

*(i) $\beta \succ \alpha$ implies $h^l_\mu(\beta) \geq h^l_\mu(\alpha)$,*

*(ii)* $h^l_\mu(\alpha \vee \beta) \geq \max\{h^l_\mu(\alpha), h^l_\mu(\beta)\}$.

**Proof.** (i) Let $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_m)$ be partitions of $(\mathcal{A}, \circ)$. By assumption, for each $i = 1, 2, \dots, n$, there exists a subset $I(i)$ of a set $\{1, 2, \dots, m\}$ such that $a_i = \sum_{j\in I(i)} b_j$ for each $i = 1, 2, \dots, n$. Therefore, $h^l_\mu(\alpha) = 1 - \sum_{i=1}^n (\mu(a_i))^2 = 1 - \sum_{i=1}^n (\mu(\sum_{j\in I(i)} b_j))^2 = 1 - \sum_{i=1}^n (\sum_{j\in I(i)} \mu(b_j))^2 \leq 1 - \sum_{i=1}^n \sum_{j\in I(i)} (\mu(b_j))^2 = 1 - \sum_{j=1}^m (\mu(b_j))^2 = h^l_\mu(\beta)$.

(ii) According to Proposition 2.4, it holds $\alpha \vee \beta \succ \alpha$, and $\alpha \vee \beta \succ \beta$. Then the property (ii) is a direct consequence of the property (i). □

## Acknowledgment

This paper was supported by Algebra and Applications Research Unit, Prince of Songkla University.

## References

[1] N. O. Alshehri, *Derivations of MV-algebras*, International Journal of Mathematics and Mathematical Sciences, 2010 (2010), Article number 312027.

[2] M. Anderson, T. Feil, *Lattice ordered groups*, Kluwer, Dordrecht, 1988.

[3] C. C. Chang, *Algebraic analysis of many valued logics*, Transactions of the American Mathematical Society, 88 (1958), 467-490.

[4] A. Ebrahimzadeh, *Logical entropy of quantum dynamical systems*, Open Physics, 14 (2016), 1-5.

[5] A. Ebrahimzadeh, *Quantum conditional logical entropy of dynamical systems*, Italian Journal of Pure and Applied Mathematics, 36 (2016), 879-886.

[6] A. Ebrahimzadeh, Z.E. Giski, D. Markechová, *Logical entropy of dynamical systems-a general model*, Mathematics, 5 (2017), Article number 4.

[7] D. Ellerman, *An introduction to logical entropy and its relation to Shannon entropy*, International Journal of Semantic Computing, 7 (2013), 121-145.

[8] D. Markechová, B. Riečan, Logical entropy of fuzzy dynamical systems, Entropy, 18 (2016), Article number 157.

[9] D. Markechová, B. Riečan, *Logical entropy and logical mutual information of experiments in the intuitionistic fuzzy case*, Entropy, 19 (2017), Article number 429.

[10] D. Markechová, B. Riečan, *Logical entropy of dynamical systems in product MV-algebras and general scheme*, Advances in Difference Equations, (2019), Article number 9.

[11] D. Markechová, B. Mosapour, A. Ebrahimzadeh, *Logical divergence, logical entropy, and logical mutual information in product MV-algebras*, Entropy, 20 (2018), Article number 129.

[12] F. Marty, *Sur une generalization de la notion de groupe*, 8iem Congres Math. Scandinaves, Stockholm, 1934, 45-49.

[13] F. Montagna, *An algebraic approach to propositional fuzzy logic*, Journal of Logic, Language and Information, 9 (2000), 91-124.

[14] D. Mundici, *Interpretation of AF $C^*$-algebras in Łuksiewicz sentential calculus*, Journal of Functional Analysis, 65 (1986), 15-63.

[15] D. Mundici, *Advanced Łukasiewicz calculus and MV-algebras*, Springer: Dordrecht, the Netherlands, 2011.

[16] J. Petrovičová, *On the entropy of partitions in product MV-algebras*, Soft Computing, 4 (2000), 41-44.

[17] J. Petrovičová, *On the entropy of dynamical systems in product MV-algebras*, Fuzzy Sets and Systems, 121 (2001), 347-351.

[18] S. Rasouli, B. Davvaz, *Roughness in MV-algebras*, Information Sciences, 180 (2010), 737-747.

[19] B. Riečan, T. Neubrunn, *Integral, measure and ordering*, Springer: Dordrecht, the Netherlands, 1997.

[20] B. Riečan, *On the product MV-algebras*, Tatra Mountains Mathematical Publications, 16 (1999), 143-149.

[21] B. Riečan, *On the probability theory on MV-algebras*, Soft Computing, 4 (2000), 49-57.

[22] B. Riečan, *Kolmogorov-Sinaj entropy on MV-algebras*, International Journal of Theoretical Physics, 44 (2005), 1041-1052.

[23] B. Riečan, D. Mundici, *Probability on MV-algebras*, In: Pap, E. (ed.) Handbook of Measure Theory, Elsevier, Amsterdam, (2002), 869-910.

Accepted: June 24, 2020