Prime-valent one-regular graphs of order $12p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its full automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify connected one-regular graphs of prime valency and order $12p$ for each prime p. By analyzing the structure of the full automorphism group of such graphs and using the classification of arc-transitive graphs of order $2p$, we prove that there is only one such graph, that is, the cycle C_{12p} with valency two and order $12p$.
Keywords: one-regular graph, arc-transitive graph, covering graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [16, 19] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An s-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-

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transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* or s-*transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n, that is, the cycle C_n, which is 1-regular with full automorphism group D_{2n}. Let p be a prime. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [4] and [5], respectively. Wang [18] characterized the prime-valent s-transitive graphs of order $4p$. Feng and Kwak [10] classified cubic symmetric graphs of order $4p$ or $6p$. The classification of pentavalent and heptavalent s-transitive graphs of order $12p$ was given in [14] and [11], respectively.

For 2-valent case, s-transitivity always means 1-regularity, and for cubic case, s-transitivity always means s-regularity by Miller [9]. However, for the other prime-valent case, this is not true, see for example [12] for pentavalent case and [13] for heptavalent case. Thus, characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graph of order $12p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric-transitive graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [15, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and

$H(p,q)$ the unique subgroup of Z_p^* of order q. Define the graph $G(2p,q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x-y \in H(p,q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p,q primes. Then X is isomorphic to K_{2p} with $q = 2p-1$, $K_{p,p}$ or $G(2p,q)$ with $q \,\big|\, (p-1)$. Furthermore, if $(p,q) \neq (11,5)$ then* $\mathrm{Aut}(G(2p,q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$*; if $(p,q) = (11,5)$ then* $\mathrm{Aut}(G(2p,q)) = \mathrm{PGL}(2,11)$.

From [8, pp.12-14] and [17, Theorem 2], we can deduce the non-abelian simple groups whose orders have at most four different prime divisors.

Proposition 2.3. *Let p and q be two odd primes, and let G be a non-abelian simple group. If the order $|G|$ divides $2^2{\cdot}3{\cdot}p{\cdot}q$, then G is isomorphic to* A_5, $\mathrm{PSL}(2,11)$ *or* $\mathrm{PSL}(2,13)$. *If the order $|G|$ has at most three different prime divisors, then G is called K_3 simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2,3,p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(3,3)$	$2^4 \cdot 3^3 \cdot 13$
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSU}(3,3)$	$2^5 \cdot 3^3 \cdot 7$
$\mathrm{PSL}(2,8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSU}(4,2)$	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $12p$ for each prime p. Let q be a prime. In what follows, we always denote by X the connected q-valent one-regular graph of order $12p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 12pq$. Clearly, if $q = 2$, then $X \cong C_{12p}$ with $A \cong D_{24p}$. Now we deal with the case $q = 3$.

Lemma 3.1. *Suppose that $q = 3$. Then there is no cubic one-regular graph of order $12p$.*

Proof. Since $q = 3$ and X has order $12p$, by Checking the information in [7] we have that there is no such graph for $p \leq 17$. Thus, in what follows, we may assume that $p > 17$. Since $|A| = 2^2{\cdot}3^2{\cdot}p$, we have that A is solvable by Proposition 2.3. Let N be a minimal normal subgroup of A. Then N is solvable and hence $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_p$. By Proposition 2.1, the quotient graph X_N is also a cubic symmetric graph. Note that there is no regular graph of odd order and odd valency. Thus, $N \ncong \mathbb{Z}_2^2$.

Let $N \cong \mathbb{Z}_3$. Then $|X_N| = 4p$. Recall that $p > 17$. By [10, Theorem 6.2], there is no cubic symmetric graph of order $4p$ for $p > 17$, a contradiction.

Let $N \cong \mathbb{Z}_3^2$. Then X_N is a cubic symmetric graph of order $4p$. By Proposition 2.1, N is semiregular on $V(X)$. This is impossible because $N_v \cong \mathbb{Z}_3$.

Let $N \cong \mathbb{Z}_p$. Then X_N is a cubic symmetric graph of order 12. By [7] or [6], there is no cubic symmetric graph of order 12, a contradiction.

Let $N \cong \mathbb{Z}_2$. Then X_N has order $6p$ and $|A/N| = 2{\cdot}3^2{\cdot}p$. Recall that $p > 17$. By Proposition 2.3, A/N is also solvable and has a normal Sylow p-subgroup $M/N \cong \mathbb{Z}_p$. Let P be a Sylow p-subgroup of M. Then $P \cong \mathbb{Z}_p$. Since $p > 17$, by Sylow Theorem P is characteristic in M and hence normal in A. This implies that A has a normal subgroup $P \cong \mathbb{Z}_p$. By the above argument, we also have a contradiction. $\square$

For $q = 5$ or 7, by [14, Theorem 4.1] and [11, Theorem 3.1], it is easy to see that there is no new graph. By [6], there is no prime-valent one-regular graph of order 24. Thus, we treat with the case $p \geq 3$ and $q > 7$ by proving the following lemma.

Lemma 3.2. *Let $p \geq 3$ and $q > 7$. Then there is no new graph.*

Proof. Recall that $|A| = 12pq$, $A_v \cong \mathbb{Z}_q$, $q > 7$ and $p \geq 3$. Let N be a minimal normal subgroup of A. We divide the proof into the following two cases: $p = q$ and $p \neq q$.

Case 1: Suppose that $p = q$. Then $|A| = 12p^2$ and $A_v \cong \mathbb{Z}_p$ with $p > 7$.

Since $p > 7$, we have that A is solvable by Proposition 2.3. Let P be a Sylow p-subgroup of A. Then $|P| = p^2$. Note that $p = q > 7$. Thus, by Sylow Theorem, the number of Sylow p-subgroups of A is $kp + 1 = |A : N_A(P)|$ for some integer k. Since $|A| = 12p^2$, we have that $(kp + 1) \,\big|\, 12$. Suppose that P is not normal in A. Then $kp + 1 > 1$ and hence $k \geq 1$. It is easy to see that the only possible is $p = 11$ and $k = 1$. Since P is not normal in A, we have that $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. By Proposition 2.1, X_N is a p-valent symmetric graph. Note that there is no regular graph of odd order and odd valency. Thus, $N \ncong \mathbb{Z}_2^2$. If $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$, then X_N has order $6p$ or $4p$ and $|A/N| = 6p^2$ or $4p^2$. However, $p = 11$ forces that A/N has a normal Sylow p-subgroup by Sylow Theorem. It then follows that P is normal in A, contrary to our hypothesis. If $N \cong \mathbb{Z}_p$, then X_N has order 12. Since $p = 11$, we have that $X_N \cong K_{12}$ and $A/N \lesssim \mathrm{S}_{12}$ with $|A/N| = 12{\cdot}11$. However, by Magma [3], S_{12} has no transitive subgroup of order $12{\cdot}11$, a contradiction.

Thus, P is normal in A. This means that P is the only Sylow p-subgroup of A. Since $A_v \cong \mathbb{Z}_p$, we have that $A_v \leq P$, that is, $A_v = P_v \neq 1$. By Proposition 2.1, P is transitive or has two orbits on $V(X)$. Clearly, both are impossible because $|P| = p^2$ and $|V(X)| = 12p$.

Case 2: Suppose that $p \neq q$. Then $|A| = 12pq$ and $A_v \cong \mathbb{Z}_q$ with $q > 7$.

Since $|A| = 12pq$ and $A_v \cong \mathbb{Z}_q$, we have that A_v is a Sylow q-subgroup of A. It forces that the Sylow q-subgroups of A cannot be normal in A. If A is non-solvable, then A must contain a non-solvable composition factor H, which is isomorphic to a non-abelian simple group. It forces that $|H| \,\big|\, 12pq$ and H is a K_3 or K_4 simple group. By Proposition 2.3, $H \cong \mathrm{A}_5$, $\mathrm{PSL}(2,11)$ or $\mathrm{PSL}(2,13)$.

Let $H \cong \mathrm{A}_5$. Then $p = 5$ and $A = HA_v$. Since $|V(X)| = 12{\cdot}5$, we have H is normal in A and regular on $V(X)$. Thus, X is a normal Cayley graph on the group H. If forces that $A_v \lesssim \mathrm{Aut}(H) \cong \mathrm{Aut}(\mathrm{A}_5) \cong \mathrm{S}_5$. However, this is impossible because $|A_v| = q \nmid |\mathrm{S}_5|$.

Let $H \cong \mathrm{PSL}(2,11)$ or $\mathrm{PSL}(2,13)$. By the order of H, we can deduce that $A = H$. An easy calculation implies that $p = 5$ and $q = 11$ for $A \cong \mathrm{PSL}(2,11)$, $p = 7$ and $q = 13$ for $A \cong \mathrm{PSL}(2,13)$. By Magma [3], there is only one symmetric graph admitting $\mathrm{PSL}(2,11)$ and $\mathrm{PSL}(2,13)$ as an arc-transitive automorphism group, respectively. However, its full automorphism group is $\mathrm{PGL}(2,11)$ for the former, and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$ for the latter. Both cases are not one-regular, a contradiction.

Thus, we may assume A is solvable and hence N is also solvable. It follows that $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. Clearly, $N \ncong \mathbb{Z}_2^2$ because there is no regular graph of odd order and odd valency.

Suppose that $N \cong \mathbb{Z}_p$. Then X_N is a q-valent symmetric graph of order 12 and $|A/N| = 12{\cdot}q$. Recall that $q > 7$. By [6], $X_N \cong K_{12}$, $q = 11$ and $A/N \lesssim \mathrm{S}_{12}$. However, by Magma [3], S_{12} has no transitive subgroup of order $12{\cdot}11$, a contradiction.

Suppose that $N \cong \mathbb{Z}_3$. Then X_N is a q-valent symmetric graph of order $4p$ and $|A/N| = 4pq$. Let M/N be a minimal normal subgroup of A/N. Then the solvability of A implies that M/N is also solvable. Clearly, $M/N \ncong \mathbb{Z}_2^2$. Thus, $M/N \cong \mathbb{Z}_2$ or $\mathbb{Z}_p$.

Let $M/N \cong \mathbb{Z}_p$. Then X_M is a q-valent symmetric graph of order 4 with $q > 7$, a contradiction.

Let $M/N \cong \mathbb{Z}_2$. Then X_M is a q-valent symmetric graph of order $2p$. by Proposition 2.2, $X_M \cong K_{2p}$ with $q = (2p-1)$ or $G(2p,q)$ with $q \,\big|\, (p-1)$. For the former, $A/M \lesssim \mathrm{S}_{2p}$ and $|A/M| = 2pq$. By Burnside's Theorem, any 2-transitive permutation group is almost simple or affine. Since A/M is solvable, we have that A/M is affine. It forces that A/M must have a normal subgroup isomorphic to $\mathbb{Z}_p$. A similar argument as above, we also have a contradiction. For the later, $A/M \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. It is easy to see that A/M has a normal Sylow p-subgroup K/M. Note that $q > 7$ and $q \,\big|\, (p-1)$. It follows that $p > 11$. By Sylow Theorem, K has a normal Sylow p-subgroup P and hence P is characteristic in K. This forces that P is normal in A. With the similar argument as above, we deduce a contradiction.

Suppose that $N \cong \mathbb{Z}_2$. Then X_N is a q-valent symmetric graph of order $6p$ and $|A/N| = 6pq$. Let M/N be a minimal normal subgroup of A/N. Then solvability of A implies that M/N is also solvable. Similarly, $M/N \ncong \mathbb{Z}_2$ and $M/N \cong \mathbb{Z}_3$ or $\mathbb{Z}_p$.

Let $M/N \cong \mathbb{Z}_3$. Then $M \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and X_M is a q-valent symmetric graph of order $2p$. By Proposition 2.2, $X_M \cong K_{2p}$ with $q = (2p-1)$ or $G(2p, q)$ with $q \mid (p-1)$. The same argument as above, these two cases are impossible.

Let $M/N \cong \mathbb{Z}_p$. Then $M \cong \mathbb{Z}_2 \times \mathbb{Z}_p$. Since $p \geq 5$, we have that M has a normal Sylow p-subgroup P. It then forces that P is normal in A and X_P is a q-valent symmetric graph of order 12. Again by [6], $X_P \cong K_{12}$ and $|A/P| = 12 \cdot 11$ with $q = 11$. However, by Magma [3], S_{12} has no transitive subgroup of order $12 \cdot 11$, a contradiction. □

Combining the above arguments with the cases $q = 2, 3, 5, 7$, and by Lemmas 3.1-3.2, we have the following result.

Theorem 3.1. *Let p, q be two primes and X a connected q-valent one-regular graph of order $12p$. Then $X \cong C_{12p}$ with valency 2 and $\mathrm{Aut}(X) \cong D_{24p}$.*

4. Acknowledgements

This work was supported by the National Natural Science Foundation of China (11301154).

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Accepted: March 04, 2020