Injectivity versus dense injectivity in topological spaces

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Abstract. An injective space is a topological space with a strong extension property for continuous functions with values on it. The topological characterizations by injectivity turns out to follow from the algebraic characterizations and general category theory (Escardo 1998). In this paper we study injectivity in the context of topological spaces. We show that, in the category $\mathcal{T}op_*$, dense injective spaces are precisely the injective spaces, and in Thychonoff spaces dense injective spaces are precisely compact spaces. This way we obtain enriched versions of known results about injective topological spaces and obtain new proofs and new characterizations of topological spaces by injectivity.

Keywords: injectivity, compactness, Tychonoff spaces.

1. Introduction and preliminaries

The injective objects are investigated in many branches of mathematics. The topological characterization by injectivity turns out to follow from the algebraic characterization and general category theory ([4]). In the category $\mathcal{T}op$ ($\mathcal{T}op_0$) of (T_0) topological spaces, injective objects with respect to the embeddings are characterized as retracts of products of the three-point space $S = \{0, 1, 2\}$(0 unique non-trivial open) (S the Sierpinski space for $\mathcal{T}op_0$), see [8, 6]. Dana Scott [8] characterized the continuous lattices endowed with the Scott topology precisely as the spaces that are injective over all subspace embeddings. The continuous Scott domains are the algebras of the proper filter monad [9]. For this monad, the associated embeddings are precisely the dense subspace embeddings, and hence the injective spaces over dense embeddings are characterized as the continuous Scott domains.

In this paper we study injectivity in the context of topological spaces. We show that, in the category $\mathcal{T}op_*$, of all topological spaces which one point sets are closed, dense injective spaces are precisely the injective spaces, and in Thychonoff spaces dense injective spaces are precisely compact spaces. In this way we obtain enriched versions of known results about injective topological spaces and obtain new proofs and new characterizations of topological spaces by injectivity. The main result of this article is Theorem 2.2, which says that the concepts of injectivity and dense injectivity are equivalent in the category of topological spaces.

Consider $\mathcal{T}op$ as a category of all topoplogical spaces and continuous functions between them. We denote the open sets of a topological space X by $\mathcal{T}_X$. For a subtopological space A of a topological space B, consider $\overline{A}$ to be the elements x in B such that the intersection of A and every open subset of B containing x is non-empty. A space A is a *dense* (*closed*) subspace of a topological space B, if $\overline{A} = B(\overline{A} = A)$. Also a continuous function $f : A \to B$ is dense (closed), if $f(A)$ is a dense (closed) subspace of B. A topological space X is said to be *completely regular space*, if for every closed subset C of X and every $x \in X \setminus C$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(C) = 1$.

In mathematics, an adjunction space (or attaching space) is a common construction in topology where one topological space is attached onto another. Specifically, let X and Y be topological spaces with A a subspace of Y. Let $f : A \to X$ be a continuous function, called the attaching function. One forms the adjunction space $X \cup_f Y$ by taking the disjoint union of X and Y and identifying x with $f(x)$ for all $x \in A$. Schematically, $X \cup_f Y = (X \sqcup Y)/\{f(A) \sim A\}$.

As a set, $X \cup_f Y$ consists of the disjoint union of X and $(Y \setminus A)$. The topology, however, is specified by the quotient construction.

The attaching construction is an example of a pushout in the category of topological spaces. That is to say, the adjunction space is universal with respect to the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\tau} & Y \\ f\downarrow & & \downarrow h_Y \\ X & \xrightarrow{h_X} & X \cup_f Y \end{array} \qquad (*)$$

where τ is the inclusion function and h_X, h_Y are the functions obtained by composing the quotient function with the canonical injections into the disjoint union of X and Y.

2. Injectivity

Injectivity is one of the most central notions in many branches of mathematics.

A topological space A is said to be *injective* if for any subtopological space $\tau : B \to C$, any continuous function $f : B \to A$ can be lifted to a continuous function $\bar{f} : C \to A$, one can state that the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\tau} & C \\ f\downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

A topological space X is said to be *dense injective*(*closed injective*) if it is injective with respect to all dense subspaces (closed subspaces), which means, for each dense subspaces (closed subspaces) A of B and each continuous function $f: A \to X$, there is a continuous function $g: B \to X$ such that $g\mid_A = f$.

The following theorem which has an easy proof, concludes the equivalent conditions for injectivity.

Theorem 2.1. *A topological space E is injective if and only if it is closed injective as well as dense injective.*

Proof. We show only the unclear direction. Let $\tau: A \to B$ be an inclusion function and $f: A \to E$ a continuous function. Since A is a dense subtopological space of $\overline{A}$, there is a continuous function $f_1: \overline{A} \to E$ which extends f and since $\overline{A}$ is closed in B, there exists $\bar{f}: B \to E$ such that $\bar{f}\mid_{\overline{A}} = f_1$. Hence we have $\bar{f}\tau = f$. $\square$

Lemma 2.1. *In the pushout diagram $(*)$, h_X is one to one.*

Proof. As we mentioned earlier, $X \cup_f Y$ consists of the disjoint union of X and $(Y \setminus A)$. Thus $h_X: X \to X \cup_f Y$ is an embedding continuous function. $\square$

A subtopological space A of B is a retract of B, if there is a continuous function $f: B \to A$ such that $f|_A = id_A$. A topological space A is said to be absolute (dense, closed) retract if it is a retract of each of its (dense, closed) extensions.

By Lemma 2.1, we have the following proposition.

Proposition 2.1. *A topological space A is an injective space if and only if it is an absolute retract.*

Consider $\mathcal{T}op_*$, the sub category of $\mathcal{T}op$, of all topological spaces which one point sets are closed. It is clear that every injective space is dense injective. For the converse, we have,

Theorem 2.2. *In the category $\mathcal{T}op_*$, each dense injective space X is injective.*

Proof. Let A be a closed subspace of B and $f: A \to X$ be a continuous function. Consider Σ the set of all pairs (C, f_C) of closed subspaces of B containing A and a continuous function $f_C: C \to X$ which extends f. The set Σ is a partially ordered set by the order relation $(C_1, f_{C_1}) \leq (C_2, f_{C_2})$ if and only if $C_1 \subseteq C_2$ and $f_{C_2}\mid_{C_1} = f_{C_1}$. Let $\{(C_i, f_{C_i})\}_{i\in I}$ be a chain in Σ. The function $f_C: C = \cup C_i \to X$ defined by $f_C(t) = f_{C_i}(t) (t \in C_i)$ is a continuous function. Indeed, let $V \in \mathcal{T}_X$. We have $(f_C^{-1}(V))^c = (\cup f_{C_i}^{-1}(V))^c = \cap(f_{C_i}^{-1}(V))^c$, in which $(f_C^{-1}(V))^c = C \setminus f_C^{-1}(V)$. Since for each $i \in I$, $(f_{C_i}^{-1}(V))^c$ is closed in C_i, hence $(f_{C_i}^{-1}(V))^c$ is a closed subset of B. So $(f_C^{-1}(V))^c = \cap(f_{C_i}^{-1}(V))^c$ is closed in B and hence it is closed in C. Thus $f_C^{-1}(V)$ is an open subset of C, which implies that $f_C: C \to X$ is a continuous function. Since X is dense injective, f_C can be extended to $f_{\bar{C}}: \bar{C} \to X$ which is an upper bound for the chain. By Zorn's lemma Σ has a maximal element (M, f_M). Let $M \neq B$. So there exists

$b_0 \in B \setminus M$. Consider $N = M \cup \{b_0\}$ as a subtopological space of B. Choose and fixed $x_0 \in X$ and define a continuous function $g : \{b_0\} \to X$ by $g(b_0) = x_0$. By Pasting theorem there exists a continuous function $h : N \to X$ extending f_M, which contradicts the maximality of M. Thus $B = M$ that shows X is closed injective. Now Theorem 2.1 complets the proof. □

Here, we have a relationship between injectivity and compactness. We show that every completely regular dense injective topological space is compact, but the converse is not generally true.

Theorem 2.3. *In the category of Tychonoff spaces, a topological space X is compact if and only if it is dense injective.*

Proof. Let Y be a dense extension of X. Since X is a compact space, $X = \overline{X} = Y$. Thus X is an absolute dense retract. Now the proof is complete by using Proposition 2.1.

For the converse, let X be a dense injective space and Y be a Stone-Cech compactification of X. Hence with $\overline{X} = Y$ and Proposition 2.1, there exists a continuous function $g : Y \to X$ such that $g\,|_X = id_X$. Since the image of any compact set under a continuous function is compact, $X = g(Y)$ is a compact space. □

Corollary 2.1. *Let $X \in \mathcal{T}op_*$ be a Tychonoff space. Then X is an injective space if and only if it is a compact space.*

Corollary 2.2. *Every completely regular dense injective topological space is compact.*

In the following we show that the converse of Corollary 2.2 is not generally true.

Suppose that each compact topological space is completely regular dense injective. The two-point discrete space $2 = \{0, 1\}$ and the Euclidean interval $[0, 1]$ are both completely regular, and $A = 2$ is a subspace of $B = [0, 1]$. Now let $X = 2$, and $f : A \to X$ be the identity function. Since X, being finite, is compact, we conclude that $f : A \to X$ extends to a continuous function $g : B \to X$. That is a continuous function $g : [0, 1] \to 2$ with $g(0) = 0$ and $g(1) = 1$. But this is impossible because $[0, 1]$ is connected, whereas the existence of such a $g : [0, 1] \to 2$ gives non-empty disjoint open sets $g^{-1}(0)$ and $g^{-1}(1)$, which means that $[0, 1]$ is disconnected.

Analogously to the Tychonoff's theorem, we have the following corollary.

Corollary 2.3. *If every compact topological space is dense injective, then the product of each family of compact spaces is compact in the product topology.*

Proof. Let $\{A_i\}_{i \in I}$ be a family of compact spaces. So it is a family of dense injective spaces. By [7, Theorem 33.2], $\prod A_i$ is Tychonoff and by [1, Proposition

10.40], $\prod A_i$ is a dense injective space, and then a compact space by applying Corollary 2.2. $\square$

The following corollary is a more generality of Tietze extension theorem.

Corollary 2.4. *If every compact topological space is dense injective, then every retract of* $[a,b]^I$ *for the set indexed* I *is an injective object in the category of normal topological spaces with continuous functions.*

Proof. By Tietze extension theorem (see, [7, Theorem 35.1]), the interval $[a,b]$ is closed injective and since each closed subset of $\mathbb{R}$ is a compact space, $[a,b]$ is a dense injective space by using Corollary 2.2. From Theorem 2.1, one can deduces that $[a,b]$ is an injective space. By [1, Proposition 10.40], $[a,b]^I$ is injective where [1, Proposition 9.5] shows that every retract of $[a,b]^I$ is an injective space. $\square$

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