An improved transformation-based kernel estimator for population abundance with shoulder condition

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Abstract. The estimation of parameter $f_X(0)$, which is the probability density function at the perpendicular distance $x = 0$, is a common target in the line transect sampling to estimate the population abundance, D. The key assumption of the density shape in the line transect sampling is known as the shoulder condition ($f'_X(0) = 0$). In this paper, we propose a log-transformation application as an adaptation of the classical kernel method to estimate the population abundance in the line transect sampling. The proposed transformation produces a simple and interpretable estimator as the usual kernel estimator while holding theoretical and practical advantages. The asymptotic properties of the proposed transformation are derived. A simulation study using the half-normal detection function is also investigated and applied using various sample

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sizes. Theoretical and practical results show the superior potential properties of the proposed transform estimator over the usual kernel estimator.

Keywords: line transect, log-transformation, kernel estimator, shoulder condition, abundance, bandwidth.

1. Introduction

Several common methods to estimate population density; the line transect method is highly recommended in recent years. In this sampling technique, the study area consists of at least one strip or several non-overlapping strips of total length L. Then, the transect line(s) are located at random in each strip. Furthermore, an observer moves in each line and detects an object from the line while recording the perpendicular distance, x. It is assumed that the observer detects a random sample of n objects $x_1, x_2, ..., x_n$.

Let $g_X(x)$ be the detection function, which is the conditional probability function of detecting an object given that its vertical distance x from the transect line. The random variable X has a probability density function $f_X(x)$ for $0 \leq x \leq w$, where the functions $g_X(x)$ and $f_X(x)$ have the same shape, such that $f_X(x) = \frac{g_X(x)}{\int_0^w g_X(t)dt}$. Then, the population abundance is $D = \frac{nf_X(0)}{2L}$ and its estimation is $\hat{D} = \frac{n\hat{f}_X(0)}{2L}$ [1], where $\hat{f}_X(0)$ is an estimate of $f_X(0)$ obtained based on detected perpendicular distances $x_1, x_2, ...x_n$ (a random sample of size n), while L is the length of the transect line. If the population area A is known, the population size N is easy to estimate using the formula $\hat{N} = A\hat{D}$.

In line transect sampling, several assumptions about the probability of detecting objects (function $g_X(x)$) are made. The first assumption is $g_X(0) = 1$, which means that all objects that are placed on the transect line itself cannot be missed. The second logical assumption is $g_X(x)$ is non-increasing, i.e., the highest probability of detecting an object is for the closest object to the transect line.

Many practical cases in the literature are assuming that the shape of the detection function $g_X(x)$ have a shoulder at distance $x = 0$ (see [1-3]). Mathematically, this condition is equivalent to $f'_X(0) = 0$, which is known in the line transect literature as the shoulder condition. Several test suggestions are stated in the literature to examine the shoulder condition [4]. Generally, the shoulder condition is commonly assumed for several population densities because the probability of detected objects closer to the transect line is still certain.

As shown above, the estimation of $f_X(0)$ is important to estimate density D. Many approaches are introduced in the line transect sampling literature to estimate $f_X(0)$. The two common estimation methods are parametric and nonparametric. For the parametric method, one fits the sample density shape with an appropriate parametric distribution. Then the parameter(s) are estimated using a robust method (e.g. maximum likelihood estimation). The negative exponential [5] and the half-normal [6] are prevalent models for fitting the density of the perpendicular distances and making an estimate of $f_X(0)$. However, recent researches have focused on using the non-parametric kernel method for estimation.

2. Kernel estimation method

Kernel method is a popular non-parametric estimation method to estimate $f(x)$ based on a random sample. The advantage of the kernel method compared to the parametric method is that it allows the data to be illustrated itself. Silverman [7] introduces an extended description of this method. The classical Rosenblatt–Parzen kernel estimator is given by

$$\hat{f}_X(x) = \frac{1}{nh}\sum_{i=1}^{n} K(\frac{x-x_i}{h}), \ -\infty < x < \infty, \tag{1}$$

where $K(u)$ is the kernel function, which is usually chosen as a symmetric density function. Silverman [7] denoted that there are very few differences in efficiency to adopt between the different kernel functions. Moreover, it is common to choose K as the standard normal while the smoothing parameter h is a positive real number. Several methods can be chosen to compute the value of h; a value of h that minimizing the estimator asymptotic mean squared error is common. It is worth noting that the kernel estimator performance is sensitive to the value of the smoothing parameter h, and thus, several techniques are recommended in the literature to find the 'leading' one. Therefore, several methods for the optimal bandwidth are suggested in the subsection 'Optimal Bandwidth'.

In line transect sampling, the perpendicular distances are positive. While kernel estimator for non-negative data at equation (1) effected by the boundary at $x = 0$ which it has high negative bias. A common adaptation method of it to apply a so-called "re-normalization" by applying a reflection technique; which change each sampled data value x_i with x_i and it's reflection $-x_i$ [7-8], if $K(u)$ is assumed to be an even and symmetric function around zero; the obtained reflected estimator is:

$$\hat{f}_X(x) = \frac{1}{nh}\sum_{i=1}^{n} K(\frac{x-x_i}{h}) + K(\frac{x+x_i}{h}), \ \ x \geq 0. \tag{2}$$

The reflection kernel estimator of $f_X(0)$ is gained by substituting $x = 0$ in (2), which gives [9]

$$\hat{f}_X(0) = \frac{2}{nh}\sum_{i=1}^{n} K(\frac{x_i}{h}). \tag{3}$$

Assuming the shoulder condition is valid ($f'_X(0) = 0$), the bias and variance of the estimator (3) are:

$$Bias[\hat{f}_X(0)] = \ h^2 f''_X(0)\int_0^{\infty} u^2 \ K(u)du + o(h^2), \tag{4}$$

$$Var[\hat{f}_X(0)] = \frac{4}{nh} f_X(0) \int_0^\infty K^2(u)du + o(\frac{1}{nh}). \tag{5}$$

The asymptotic mean squared error (AMSE) of $\hat{f}_X(0)$ is obtained after the small-expression $o(.)$ are negligent:

$$AMSE[\hat{f}_X(0)] = \frac{4}{nh} f_X(0) \int_0^\infty K^2(u)du + h^4 [f''_X(0)]^2 [\int_0^\infty u^2\, K(u)du]^2. \tag{6}$$

In practical cases of the line transect sampling, the kernel method is commonly used to locate a good estimate of $f_X(0)$. An extended description of this method can found in the literature. Jones [10] proposed different boundary correction methods for the kernel estimator while Marron and Ruppert [11] suggested different transformations that reduced boundary bias in the kernel estimator. On the other hand, Chen [9] derived mean, variance, and the asymptotic mean squared error of estimator (3). Zhang, Karunamuni, and Jones [12] introduced a new transformation for kernel density estimation to adaptive the boundary and Mack [13] proposed several bias reduction techniques for $\hat{f}_X(0)$. In addition, Eidous [14] introduced some adaptations of the kernel estimator of $f_X(0)$ while Karunamuni and Zhang [15] suggested new techniques to correction kernel estimator boundary effect. Eidous [16] suggested a new kernel estimator of $f_X(0)$ for densities that do not have a shoulder and Wen and Wu [17] proposed data transformation for the kernel density estimation. Charpentier and Flachaire [18] applied the logarithmic transformation of the kernel density estimator for income data distribution. Albadareen and Ismail [19] suggested the power transformation to improve kernel estimator for population abundance. Recently, Albadareen and Ismail [20] studied several parameters of Epanechnikov kernel function and recommended one for population abundance estimation, Al-bassam and Eidous [3] modified the common kernel estimator to be more appropriates for population density estimation, and Albadareen and Ismail [21-22] applied data transformation for the kernel estimator based on generalized form of power transformation.

In this article, the non-parametric kernel method is considered when the shoulder condition is valid. A logarithmic transformation is applied to the data with the kernel estimation method, and the original density is gained using back-transformation. The proposed estimator is mathematically proven to compare its efficiency with the reflection kernel estimator. A simulation study is also carried out to compare the efficiency of the proposed estimators and the kernel estimator.

3. Kernel estimator with log-transformation

The estimation of the kernel density based on transformed data is widely considered in many types of research. The logarithmic transformation $Y = \log(X)$

was suggested by Devroye and Györfi [23], briefly discussed by Silverman [7] and proposed by Marron and Ruppert [11]. Other studies can also be found in Charpentier and Flachaire [18] who showed that the log-transformation of $Y = \log(X)$ of heavy-tailed distribution such as income distribution fit much better density estimate. Geenens and Wang [24] proposed a kernel estimators that combining the transformations with local likelihood density estimation methods. Nguyen, Jones, and McLachlan [25] introduced the R package for positive data kernel density estimation using log-transformation. Bii et al. [26] proposed a kernel estimator based on combines the transformation and the reflection methods.

The outcomes of the line transect sampling are the perpendicular distances, which are non-negative data and the estimation of interest is $f_X(x)$ at $x = 0$. Since the logarithmic transformation $Y = \log(X)$ is unsuitable when $x = 0$, we propose shifted logarithm transformation $Y = \log(X + 1)$ such that Y is a strictly increasing function. Thus, the original and proposed density functions are $f_X(x)$ and $f_Y(y)$ respectively. By applying the transformation method of the density function $f_X(x)$, we obtain $f_Y(y)$,

$$f_Y(y) = f_X(x)|\frac{dx}{dy}| = f_X(e^y - 1)e^y,\ x \geq 0. \tag{7}$$

The propose transformation is $Y = \log(X + 1)$. $f_y(0) = f_x(0)$ at $x = 0$, $y = 0$. Moreover, the population abundance D is obtained using $D = \frac{nf_X(0)}{2L}$. Therefore, it is equivalent to $D = \frac{nf_Y(0)}{2L}$.

To make a good abundance estimation of D, the estimation of $f_Y(0)$ is critical. It requires $f_Y(0)$ to be substituted with $\hat{f}_Y(0)$. In this article, the kernel method is applied where the proposed kernel estimator $\hat{f}_Y(0)$ is:

$$\hat{f}_Y(0) = \frac{2}{nh}\sum_{i=1}^{n} K(\frac{Y_i}{h}) \ \ ,\ \ Y = \log(X + 1). \tag{8}$$

The bias and variance of $\hat{f}_Y(0)$ are:

$$Bias[\hat{f}_Y(0)] = 2hf'_Y(0)\int_0^\infty uK(u)du + O(h^2). \tag{9}$$

Assuming the shoulder condition is valid (i.e. $f'_X(0) = 0$),

$$Bias[\hat{f}_Y(0)] = 2h(f_X(0))\int_0^\infty uK(u)du + O(h^2), \tag{10}$$

$$Var[\hat{f}_Y(0)] = \frac{4}{nh}f_Y(0)\int_0^\infty K^2(u)du + o(\frac{1}{nh}) \tag{11}$$

$$= \frac{4}{nh}f_X(0)\int_0^\infty K^2(u)du + o(\frac{1}{nh}) \tag{12}$$

By ignoring the small terms $o(.)$, the asymptotic mean squared error of $\hat{f}_Y(0)$ is:

$$AMSE[\hat{f}_Y(0)] = \frac{4}{nh} f_X(0) \int_0^\infty K^2(u)du + [2h(f_X(0)) \int_0^\infty uK(u)du]^2. \tag{13}$$

By comparing estimator variances in terms (12) and (5), it was found that $Var[\hat{f}_Y(0)] = Var[\hat{f}_X(0)]$.

4. Simulation study

The theoretical characteristics of the proposed transform estimator in equations (10) and (12) are asymptotic, which implies that the sample size is assumed to grow infinitely (i.e. $n \to \infty$). In this simulation study, different sample sizes are applied ($n = 50,\ 100,\ and\ 500$). A comparison between the performance of the proposed estimator and the reflection kernel estimator is introduced. The relative bias (RB) and the relative mean error (RME) are computed using $RB = E[\hat{f}(0) - \ f(0)]/f(0)$ and $RME = \sqrt{MSE[\ \hat{f}(0)]}/\ f(0)$.

In this article, the case of the shoulder condition is assumed. A common density in the line transect sampling that has a shoulder at ($x = 0$) is the half-normal model which will have applied in this study, where $f_X(x) = \frac{2e^{-x^2/2\beta^2}}{\beta\sqrt{2\pi}}$, $\beta > 0,\ 0 \le x \le w$. Five detection functions are chosen under the validity of the shoulder condition. The parameter values are $\beta = 0.1,\ 0.2,\ 0.3,\ 0.4$ and 0.5, and the truncation point is $\omega = 1.5$. The detection functions cover several possible model cases for the perpendicular distances in the case of the validity of the shoulder condition, as demonstrated in Figure 1.

Optimal bandwidth

The estimator in equation (3) yields convergent AMSE when the symmetric kernel functions $K(.)$ are chosen, such as Gaussian, biweight, and Epanechnikov [27]. Therefore, the Gaussian kernel function is considered in this study.

The performance key in the kernel method is the smoothing parameter h [28]. From past studies, different approaches were suggested to choose an efficient bandwidth. Generally, a sequence of bandwidth values can be applied and their results are compared to understand the structure of the unknown probability density function because there is no 'best' method in choosing the bandwidth parameter [27]. The same approach is also suggested by Silverman [29]. Thus, different bandwidth methods are used in this simulation study, along with some comparisons between classical and proposed estimators.

For this simulation, we consider the following bandwidth methods, each with underlying estimators:

a) Silverman [7] rule of thumb: $h = 1.06\hat{\sigma}n^{(-1/5)}$, this bandwidth is derived by minimizing the approximate mean squared error (AMSE) of $\hat{f}_X(x)$.

In the line transect method, the half-normal distribution is assumed as the reference density of $f_X(x)$ to estimate σ via the maximum likelihood estimator $\hat{\sigma} = \sqrt{\sum_{i=1}^{n} x_i^2/n}$. This bandwidth is applied to the original data, which results in the estimator Est1.

b) Silverman [7] modified the rule of thumb: $h = 0.9 \min(\frac{IQR(X)}{1.349}, \hat{\sigma})n^{(-1/5)}$, this bandwidth is similar to the previous bandwidth except that it is modified for the long-tailed and skewed distribution. $IQR(X)$ is the interquartile range of the perpendicular with distance X. This bandwidth is applied on the original data, which results in the estimator Est2.

c) Similar to (a), this bandwidth is applied to the transformed data, which results in the estimator Est3.

d) Similar to (b), this bandwidth is applied to the transformed data, which results in the estimator Est4.

Another efficient bandwidth can be obtained by minimizing the $AMSE[\hat{f}_Y(0)]$ in equation (13). The bandwidth is $h = (\frac{\int_0^\infty [K(t)]^2 dt}{2n f_X(0)(\int_0^\infty t^2 K(t) dt)^2})^{(1/3)}$, by assuming that the shoulder condition is valid and the unknown value $f_X(0)$ are estimated using a suitable reference distribution model (see [2-3,14,30,31]). This bandwidth value will be applied in parts (e) and (f).

e) Assuming that $f_X(x)$ is the negative exponential model ($f_X(x) = \frac{1}{\beta} e^{-x/\beta}$, $0 \le x \le w$) and the MLE of β, $\hat{\beta} = \bar{X}$, is used, this bandwidth is applied to the transformed data which results in the estimator Est5.

f) Assuming that $f_X(x)$ is a half-normal model ($f_X(x) = \frac{2e^{-x^2/2\beta^2}}{\beta\sqrt{2\pi}}$, $\beta > 0, 0 \le x \le w$), and the MLE of β, $\hat{\beta} = \sqrt{\sum_{i=1}^{n} x_i^2/n}$, is used, this bandwidth is applied to the transformed data which results in the estimator Est6.

Our simulation study focuses on six estimation values of $f_Y(0)$, where Est1 and Est2 are applied to the original data as stated in the literature, while Est3, Est4, Est5, and Est6 are the proposed estimators applied to the transformed data. Table 1 shows the relative bias (RB) and the relative mean error (RME) for the estimators based on the simulation study.

The simulation results show that the proposed transformed estimator (Est3) is superior, in which it has smaller absolute relative bias and relative mean error compared to the classical kernel estimators (Est1 and Est2) for each density family and all cases. Moreover, the RME of (Est3) decreases as the sample size increases, hence it is concluded that (Est3) is asymptotically consistent. Figure 2 depicted these results.

The proposed transformed estimator (Est5) also has efficient performance compared to estimators (Est1 and Est2) for all sample sizes. The proposed transformed estimator (Est6) also works well for a small value of β (i.e. $\beta = 0.1, 0.2, and\ 0.3$) with various sample sizes.

Looking at the modified bandwidth [7], ($h = 0.9\ \min(\frac{IQR(X)}{1.349}, \hat{\sigma}) n^{(-1/5)}$), the classical kernel estimator (Est2), and the proposed transformed estimator (Est4) have unsatisfactory performance for the line transect data when the shoulder condition is valid. Therefore, the bandwidth suggested by Silverman [7] ($h = 1.06\hat{\sigma} n^{(-1/5)}$) is more appropriate for the log-transformed data in this study.

5. Conclusion

In this study, an efficient and consistent estimator for wildlife population density using the line transect sampling is proposed. The application of log-transformation to the kernel estimation is proven to perform well compared to the reflection kernel estimator in the case of the shoulder condition. The asymptotic mathematical properties (bias, variance, and mean squared error (MSE)) of the proposed transform estimator were obtained. The proposed transformed estimator was shown to have smaller AMSE if the detection function has a shoulder. A simulation study using the half-normal model was employed to compare the efficiency of the proposed and the reflection kernel estimators. The simulation results showed that the proposed transformed estimators have smaller relative mean error than the classical reflection kernel estimators. Generally, the proposed transformed estimators have better performance than the classical kernel estimators in terms of absolute relative error with fixed variance, especially for small samples.

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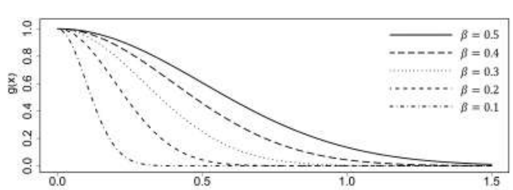

Figure 1: The detection functions of the half-normal models.

Table 1: Relative bias and relative mean error of the half-normal models.

		$n = 50$		$n = 100$		$n = 500$	
	Estimator	RB	RME	RB	RME	RB	RME
$\beta = 0.1$	Est1	-0.104	0.161	-0.079	0.122	-0.042	0.068
	Est2	-0.053	0.203	-0.036	0.154	-0.01	0.083
	Est3	-0.078	0.144	-0.055	0.106	-0.022	0.057
	Est4	-0.036	0.2	-0.021	0.151	0.002	0.083
	Est5	-0.232	0.24	-0.158	0.167	-0.051	0.065
	Est6	-0.226	0.233	-0.154	0.162	-0.049	0.063
$\beta = 0.2$	Est1	-0.106	0.156	-0.075	0.12	-0.043	0.068
	Est2	-0.053	0.189	-0.025	0.151	-0.013	0.084
	Est3	-0.054	0.124	-0.025	0.095	-0.001	0.052
	Est4	-0.019	0.183	0.007	0.15	0.012	0.084
	Est5	-0.08	0.121	-0.033	0.089	0.006	0.059
	Est6	-0.079	0.119	-0.033	0.088	0.006	0.06
$\beta = 0.3$	Est1	-0.096	0.157	-0.077	0.12	-0.042	0.068
	Est2	-0.041	0.203	-0.032	0.151	-0.01	0.083
	Est3	-0.019	0.12	-0.002	0.089	0.022	0.057
	Est4	0.011	0.201	0.016	0.15	0.027	0.088
	Est5	-0.003	0.125	0.015	0.105	0.029	0.078
	Est6	-0.005	0.124	0.014	0.106	0.028	0.079
$\beta = 0.4$	Est1	-0.092	0.151	-0.077	0.123	-0.041	0.067
	Est2	-0.03	0.194	-0.028	0.157	-0.009	0.081
	Est3	0.012	0.115	0.024	0.095	0.045	0.068
	Est4	0.04	0.197	0.036	0.16	0.041	0.091
	Est5	0.044	0.149	0.045	0.134	0.042	0.09
	Est6	0.041	0.149	0.043	0.136	0.042	0.091
$\beta = 0.5$	Est1	-0.092	0.152	-0.072	0.12	-0.042	0.068
	Est2	-0.041	0.198	-0.023	0.155	-0.012	0.082
	Est3	0.039	0.121	0.054	0.105	0.065	0.083
	Est4	0.048	0.201	0.059	0.166	0.051	0.096
	Est5	0.067	0.174	0.068	0.158	0.048	0.101
	Est6	0.063	0.175	0.066	0.159	0.047	0.102

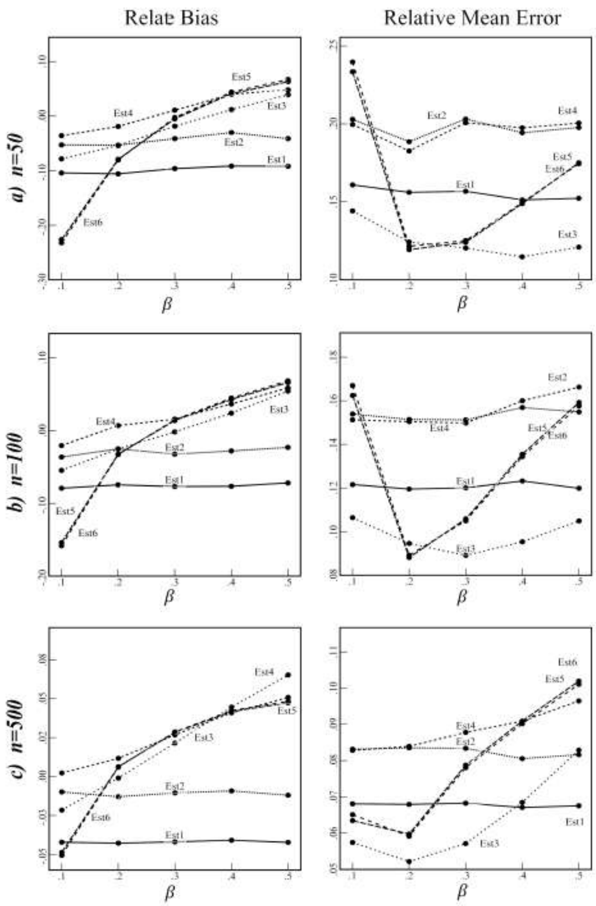

Figure 2: The detection functions of the half-normal models.

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