A posteriori error analysis for discontinuous Galerkin methods for time discrete semilinear parabolic problems

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Abstract. *A posteriori* error estimates for time discretisations for semilinear parabolic (evolution) problems by the discontinuous Galerkin method DG(r) of arbitrary order $r \geq 0$ are derived and analysed. The semilinear evolutionary problems of the form $u' + Au = f(u)$ with A is either linear or monotone γ^2-angle bounded operator are considered. The main tool in this analysis is the time reconstruction function $\hat{U}$ of the approximate discrete solution U of the exact solution u. Two classes of nonlinearities are addressed: firstly, when the source term is globally Lipschitz continuous and secondly when the source term is locally Lipschitz continuous.

Keywords: a posteriori error analysis, finite element methods, semilinear parabolic problems, discontinuous Galerkin methods, reconstruction technique, time-stepping methods.

1. Introduction

The discontinuous Galerkin time-stepping scheme is a variational one-step implicit time marching scheme. The variational time marching schemes are dated back to the 1960s. The research paper of Argyris and Sharpf [2] in 1969 is considered the first work used the variational formulation of time integration in a systemic way. Hulme, in 1972 [23,24] addressed for the first time the variational continuous in time Galerkin (cG) finite element method for ordinary differential equations. In 1973, Reed and Hill [42] studied for the first time the discontinuous Galerkin finite element methods for first order hyperbolic problems. The first error analysis is done by Lesaint and Raviart in 1974 for the variational cG time–stepping schemes [32]. The first error analysis of linear parabolic problems using variational time-marching schemes is investigated by Jamet in 1978 [25]. Eriksson and coworkers investigated in detail the analysis of DG time-stepping schemes in [13]. Nonlinear parabolic problems are considered in [10,31,52,57] and the references therein for different numerical aspects of the finite element methods. The p- and hp-versions Galerkin time-stepping methods are considered and addressed from different numerical viewpoints in [4,5,18,28,51,53,54,55,56,62].

A posterior error analysis for transient problems recently has witnessed an increasing activity especially for the DG, Runge–Kutta and multistep methods [1,14,15,17,20,21,22,28,29,60]. The available error estimators either can be derived analytically by using stability features of the underlying problems (for this reason they are called analytical error estimators) [11,33,34,40,41,59], or computationally by using suitable computational methods (computational error estimators) [3,12,16]. The main techniques in deriving the error estimators are the energy, duality and reconstruction techniques.

We use in this research paper for deriving our *a posteriori* error estimates for higher order DG(r) methods the energy technique (analytical approach) since it is more flexible in dealing with the nonlinearities (the forcing functions) with the aid of the ideas of reconstruction and continuation argument. The basic idea behind the energy technique is deriving the error representation formula and then to test it by a quantity which hinges upon the error, this quantity depends on the underlying problem. The elliptic reconstruction technique was introduced by Makridakis and Nochetto [36] in 2003 for deriving optimal order *a posteriori* error estimates for semidiscrete linear parabolic problems. Then extended to the fully discrete setting for linear parabolic PDEs [30]. Also, Makridakis and Nochetto in 2006 [37] extended the idea of the elliptic reconstruction technique by presenting a novel time reconstruction for time semidiscrete linear and nonlinear parabolic problems. In [48] Y. Sabawi studied and investigated the *a posteriori* error bounds for DG methods for semilinear interface problems. Cangiani et al [8] examined and derived the *a posteriori* error estimates for the discontinuous Galerkin finite element methods for elliptic interface problems. Also, they considered and examined the convergence of the adaptive DG methods for elliptic interface problems in [7]. Georgoulis and coworkers [18] introduced a new space-time reconstruction for fully discrete *hp*-version Galerkin timestepping for linear evolutionary problems. Cangiani and coworkers [6] extended the work in [18] from linear to semilinear parabolic problems, and M. Sabawi [44] investigated the *a posteriori* and *a priori* error analyses for fully discrete *hp*-version Galerkin timestepping for semilinear parabolic problems. The *a posteriori* error analysis for DG time discretisation of semilinear evolutionary problems and fully discrete backward Euler-Galerkin finite element method for semilinear parabolic problems are derived in [45,46] respectively. Y. Sabawi derived *a posteriori* error bounds for DG methods in the $L_2(H^1) + L_\infty(L_2)$- and $L_2(H^1)$-norms in [47,49], respectively. The basic idea of the elliptic reconstruction technique is to construct an appropriate auxiliary continuous function satisfies a perturbed equation of the original problem. The counterpart of the time discretisation is the time reconstruction which simplifies the analysis and results in optimal order-regularity error estimators. For more details about the ideas such as the elliptic reconstruction and time reconstruction see the survey in [35]. In [37] the DG timestepping reconstruction uses the Gauss-Radua nodes, and these nodes have an appealing property that the DG method has superconvergence at these points. A review of superconvergence in DG methods and postprocessing aspects are considered in [43]. For more details about superconvergence and the related issues see [26,39,61].

In this work, we extend the work of Makridakis and Nochetto in [37] to the semilinear case for globally and locally Lipschitz continuous nonlinear source terms, and we mostly use the same notation. The remainder of this work is organised as follows: In Section 2, we introduce the preliminary and basic definitions, notation, the mathematical problem and the discontinuous Galerkin method. Section 3 is devoted for deriving the *a posteriori* error estimators for linear, monotone and γ^2-angle bounded operators. The conclusions are given in Section 4.

2. Problem setting and the numerical method

We consider the discretisations in time by the use of the discontinuous Galerkin time-stepping method DG(r) of any order $r \geq 0$ of the dissipative evolutionary initial value problem

$$u'(t) + Au(t) = f(u(t)),\ u(0) = u_0, \tag{1}$$

A is a γ^2-angle bounded operator in a Hilbert space H with inner product $\langle ., . \rangle$ and norm $\|.\|_H$. For simplicity in notations we omit the dependence upon t and write u instead of $u(t)$. Although the analysis will be done in an abstract setting, the typical case will be when $H = L_2, V = H_0^1$ and $V^* = H^{-1}$. The operator A is called γ^2-angle bounded operator if

$$\langle A(v) - A(w), w - z \rangle \leq \gamma^2 \langle A(v) - A(z), v - z \rangle \ \forall v, w, z \in D(A), \tag{2}$$

where $\gamma \geq \frac{1}{2}$. We restrict our analysis to the case when $\gamma = 1$. The operator A is monotone if it satisfies the monotonicity condition

$$\langle A(v) - A(w), v - w \rangle \geq 0 \ \forall v, w \in D(A). \tag{3}$$

Also, the operator A satisfies the coercivity condition if

$$\langle A(v) - A(w), v - w \rangle \geq \| v - w \|^2 \ \ \forall v, w \in D(A). \tag{4}$$

The function f is globally Lipschitz continuous if

$$|f(v) - f(w)| \leq C|v - w| \ \forall v, w \in R, \tag{5}$$

and is said to be locally Lipschitz continuous if it satisfies the following growth condition

$$|f(v) - f(w)| \leq C|v - w|(1 + |v|^a + |w|^a),\ a \geq 0, \tag{6}$$

where $|.|$ denotes the Euclidean norm on R^N.

2.1 Reconstruction function

The reconstruction function $\hat{U}$ is a continuous interpolant of degree $r+1$ (higher degree) of the discontinuous approximate solution U which is of degree r, $\hat{U}$ satisfies the perturbed ODE

$$\hat{U}' + \Pi A(U) = \Pi f(U), \tag{7}$$

where $\Pi : H \longrightarrow V_k(r)$, is a suitable projection or interpolation operator and this depends upon the nature of the problem, $V_k(r)$ is the space of discontinuous piecewise polynomials of degree $\leq r$, and $\Pi f(U)$ is a discontinuous piecewise polynomial approximation of $f(U)$. Let $0 = t_0 < t_1 < ... < t_N = T$ be a partition Υ of the time interval $[0, T]$, $I_n = (t_n, t_{n+1}]$, and $k_n = t_{n+1} - t_n$ is the variable time step. We define $C(r)$ as the space of polynomials of degree $\leq r$, both spaces $C(r)$ and $V_k(r)$ are defined over the partition Υ with values in $D(A)$, if $\upsilon \in V_k(r)$, then

$$\upsilon|_{I_n(t)} = \sum_{i=0}^{r} \chi_i t^i,\ \chi_i \in D(A),\ 0 \leq i \leq r. \tag{8}$$

The discontinuous Galerkin discretisation of (1) is: find $U \in V_k(r)$ such that $U_0 = u_0$ and

$$\int_{I_n} (\langle U', v\rangle + \langle A(U), v\rangle)dt + \langle U_n^+ - U_n, v_n^+\rangle = \int_{I_n} \langle f(U), v\rangle dt,\ \forall v \in V_k(r), \tag{9}$$

where $0 \leq n \leq N-1,\ v_n = v(t_n),\ v_n^+ = \lim_{\epsilon\to 0^+} v(t_n + \epsilon)$.
The discontinuous Galerkin discretisation (9) for (1) is a finite element discretisation in time and when use with it a suitable Radau quadrature rules it turns out to the classical implicit Runge-Kutta-Radau methods which refer to it as IRK-R(r). Let $\{\tau_i\}_{i=1}^{r+1}$ be the Radau points in the interval $[0, 1]$, such that $0 < \tau_1 < ... < \tau_{r+1} = 1$ with the corresponding weights $\{w_i\}_{i=1}^{r+1}$. The Radau quadrature rule on $[0, 1]$

$$\int_0^1 \psi(\tau)d\tau \cong \sum_{i=1}^{r+1} w_i \psi(\tau_i), \tag{10}$$

is exact for polynomials of degree less than or equal to $2r$. Let $\{\ell_i\}_{i=1}^{r+1} \subset C(r)$ be the Lagrange polynomials of degree r associated with $\{\tau_i\}_{i=1}^{r+1}$ and $\{\hat{\ell}_i\}_{i=0}^{r+1} \subset C(r+1)$ be the Lagrange polynomials of degree $r+1$ associated with $\{\tau_i\}_{i=0}^{r+1}$ with $\tau_0 = 0$. The Radau points in the interval $\bar{I}_n = [t_n, t_{n+1}]$ are denoted by $t_{n,i}$ and the corresponding Lagrange polynomials by $\ell_{n,i},\ \hat{\ell}_{n,j}$, respectively, and they satisfy

$$\begin{aligned} t_{n,i} &= t_n + \tau_i k_n,\ i = 0, ..., r+1\ (t_{n,0} = t_n,\ t_{n,r+1} = t_{n+1}), \\ \ell_{n,i}(t) &= \ell_i(\tau),\ \hat{\ell}_{n,i}(t) = \hat{\ell}_i(\tau), t = t_n + \tau k_n. \end{aligned} \tag{11}$$

Let the interpolation operator $I : C(0,T;D(A)) \longrightarrow V_k(r)$ be defined by

$$Iv|_{I_n(t)} = \sum_{i=1}^{r+1} \ell_{n,i}(t)v(t_{n,i}), \tag{12}$$

hence, if $v(t) \in V_k(r)$, then $v(t) = Iv(t)$, $\forall t$. Now, we define the reconstruction operator $\hat{I} : V_k(r) \longrightarrow V_k(r+1)$ by $\hat{U} = \hat{I}U \in V_k(r)$ and satisfies on I_n

$$\begin{aligned} \hat{U}_n^+ &= U_n, \\ \int_{I_n} \langle \hat{U}', v\rangle dt &= \int_{I_n} \langle U', v\rangle dt + \langle U_n^+ - U_n, v_n^+\rangle, \ \forall v \in V_k(r). \end{aligned} \tag{13}$$

The basic idea of this research is to obtain *a posteriori* error estimates of the form

$$\| u - U \| \leq \eta(u_0, U, f(U)), \tag{14}$$

where u is the exact solution of (1), U is the approximate solution of u and $\| \,.\, \|$ is the energy norm. The quantity $\eta(u_0, U, f(U))$ is of optimal order-regularity and computable in terms of the known quantities u_0. U, and $f(U)$.

2.2 The discontinuous Galerkin method

By using (13), then (9) can be written as

$$\int_{I_n} (\langle \hat{U}', v\rangle + \langle A(U), v\rangle)dt = \int_{I_n} \langle f(U), v\rangle dt, \ \forall v \in V_k(r). \tag{15}$$

Eq. (15) can be written in the pointwise form as

$$\hat{U}' + \Pi A(U) = \Pi f(U), \tag{16}$$

where the operator $\Pi : H \longrightarrow V_k(r)$ is either $\Pi = P$ the L_2-projection operator if we use DG(r) or $\Pi = I$ where I is the Lagrange interpolation operator at the Radau points if we use IRK-R(r). We will restrict our analysis to the case when $\Pi = P$, and for the other case when $\Pi = I$, we can obtain the same results with simple and straightforward modifications. Note that these methods are connected and equivalent for more details, see [37]. We derive in the next section the representation formula for DG(r) and by applying the energy technique on this formula we derive later *a posteriori* error estimators. For more details about the method see [38,50,58,63] and for a general and brief introduction to the DG method for linear problems see [19].

3. *A Posteriori* error analysis in the case of linear operators

Now, we consider the case when the operator A is linear, namely $\Pi A(U) = A(U)$ for DG(r), hence the representation formula (16) simplifies to

$$\hat{U}' + A(U) = Pf(U). \tag{17}$$

Let $A : D(A) \longrightarrow H$ is a linear operator, the energy semi-norm related to the operator A over $D(A)$ is defined by

$$\| w \| = \langle A(w), w\rangle^{1/2}, \ \forall w \in D(A), \tag{18}$$

and we define $\Re = \{w \in H : \|w\| < \infty\} \subset H$. We can now prove the upper and lower *a posteriori* error estimates and before that we first define the error measure Λ

(19)
$$\Lambda = \left\{\max\left(\max_{0\leq t\leq T} \|(u-\hat{U})(t)\|_H^2, \frac{1}{2}\int_0^T \|u-\hat{U}\|^2 dt, \frac{1}{2}\int_0^T \|u-U\|^2 dt\right)\right\}^{1/2}.$$

We use in this paper the same error measure Λ as in [37].

Proposition 3.1 (Reconstruction [37]). *The function $\hat{U}$ is uniquely defined by* (13)*, is globally continuous, and satisfies*

$$\hat{U}(t_{n,i}) = U(t_{n,i}), \ i = 0, ..., r+1 \ (U(t_{n,0} = U_n). \tag{20}$$

Proposition 3.2 (Properties of $\hat{U}$ [37]). *The following representation of $\hat{U} - U$ is valid*

$$(\hat{U} - U)|_{I_n(t)} = \hat{\ell}_{n,0}(t)(U_n - U_n^+), \ \forall t \in I_n. \tag{21}$$

In addition, if

$$\alpha_p = (\int_0^1 |\hat{\ell}_0(\tau)|^p d\tau)^{1/p}, \ \forall 1 \leq p \leq \infty, \tag{22}$$

then for any semi-norm $\| . \|$ in H

$$(\int_{I_n} \| \hat{U} - U \|)^{1/p} = \alpha_p k_n^{1/p} \| U_n^+ - U_n \|, \ \forall \leq p \leq \infty. \tag{23}$$

Theorem 3.3 (Upper Bound of Globally Lipschitz Case). *Let A be a linear operator and $u_0 \in \Re$ and the function $f(u)$ is globally Lipschitz continuous, then the following upper bound estimate for DG(r) is valid for any $r \geq 0$:*

$$\max_{0\leq t\leq T} \|(u-\hat{U})(t)\|_H^2 \leq \xi(U)e^{C_1 T}, \tag{24}$$

where

$$\xi(U) = 2\alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt + C\alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|_H^2.$$

Proof. By subtracting (16) from (1) and using $\Pi = P$, we obtain the error representation formula

$$(u-\hat{U})' + A(u-U) = f(u) - Pf(U) \text{ (since A is linear operator).} \tag{25}$$

Multiplying the equation (25) by $(u-\hat{U})$ to have

$$\frac{1}{2}\frac{d}{dt}\|u-\hat{U}\|_H^2 - \langle A(U-u), u-\hat{U}\rangle = \langle f(u) - Pf(U), u-\hat{U}\rangle. \tag{26}$$

The nonlinear term in the righthand side of (26) is the more difficult term to deal with. To bound the righthand side term $\langle f(u) - Pf(U), u-\hat{U}\rangle$ we rewrite it in following form

$$\begin{aligned} \langle f(u) - Pf(U), u-\hat{U}\rangle &= \langle f(u) - f(U) + f(U) - Pf(U), u-\hat{U}\rangle = \\ \langle f(u) - f(U), u-\hat{U}\rangle &+ \langle f(U) - Pf(U), u-\hat{U}\rangle. \end{aligned} \tag{27}$$

Substituting (27) in (26), using (2), (4) and Young's inequality, yield

$$\begin{aligned} \frac{d}{dt}\|u-\hat{U}\|_H^2 - 2\gamma^2\ \|U-\hat{U}\|^2 &\le \|f(u) - f(U)\|_H^2 \\ &+\|f(U) - Pf(U)\|_H^2 + 2\|u-\hat{U}\|_H^2. \end{aligned} \tag{28}$$

Now, if f is globally Lipschitz continuous, namely,

$$\|f(u) - f(U)\|_H \le C\|u-U\|_H \le C\|u-\hat{U}\|_H + C\|\hat{U}-U\|_H, \tag{29}$$

inserting (29) in (28) and integrating in time to obtain

$$\begin{aligned} \|(u-\hat{U})(t)\|_H^2 &\le 2\gamma^2\int_0^T \|\hat{U}-U\|^2 dt + \int_0^T \|f(U) - Pf(U)\|_H^2 dt \\ &+ C_1\int_0^T \|u-\hat{U}\|_H^2 dt + C\int_0^T \|U-\hat{U}\|_H^2 dt, \end{aligned} \tag{30}$$

where $C_1 = C+2$. Using Proposition 3.1 the first and last terms on the righthand side of (30) can be expressed as follows

$$\int_0^T \|U-\hat{U}\|^2 dt = \alpha_2^2\sum_{n=0}^{N-1} k_n^2\|U_n^+ - U_N\|^2, \tag{31}$$

$$\int_0^T \|U-\hat{U}\|_H^2 dt = \alpha_2^2\sum_{n=0}^{N-1} k_n^2\|U_n^+ - U_N\|_H^2. \tag{32}$$

Consequently, by substituting (31) and (32) in (30), we obtain

$$\begin{aligned} &\|(u-\hat{U})(t)\|_H^2 \le 2\gamma^2\alpha_2^2\sum_{n=0}^{N-1} k_n^2\|U_n^+ - U_N\|^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt \\ &+ C_1\int_0^T \|u-\hat{U}\|_H^2 dt + C\alpha_2^2\sum_{n=0}^{N-1} k_n^2\|U_n^+ - U_N\|_H^2. \end{aligned} \tag{33}$$

Let

$$\xi(U)=2\gamma^2\alpha_2^2\sum_{n=0}^{N-1}k_n^2\|U_n^+-U_N\|^2+\int_0^T\|f(U)-Pf(U)\|_H^2dt+C\alpha_2^2\sum_{n=0}^{N-1}k_n^2\|U_n^+-U_N\|_H^2,$$

and by substituting that in (33), we have

$$\|(u-\hat{U})(t)\|_H^2 \leq \xi(U)+C_1\int_0^T\|u-\hat{U}\|_H^2dt, \tag{34}$$

and by Grönwall's inequality, we obtain the required result. □

Theorem 3.4 (Upper Bound of Locally Lipschitz Case)**.** *Let A be a linear operator and $u_0 \in \Re$ and the function $f(u)$ is locally Lipschitz continuous, then the following upper bound estimate for $DG(r)$ is valid for any $r \geq 0$:*

$$\max_{0\leq t\leq T}\|(u-\hat{U})(t)\|_H^2 \leq e^{\hat{C}_1\int_0^t G(U)dt}\big(\eta(U)\hat{C}_2\left(\eta(U)\right)^{\frac{a+1}{a}}\big), \tag{35}$$

where

$$\begin{aligned}\eta(U)=2\int_0^T\|U-\hat{U}\|^2dt+\int_0^T\|f(U)-Pf(U)\|_H^2dt\\ +C\int_0^T G(U)\|U-\hat{U}\|^2dt+C\int_0^T\|U-\hat{U}\|_H^a\|\nabla(U-\hat{U})\|_H^2dt.\end{aligned}$$

Proof. Now, we consider the case when the function f is locally Lipschitz continuous, namely, f satisfies the following growth condition. Now, we bound the nonlinear term by using the growth condition (6)

$$|\langle f(u)-f(U),u-\hat{U}\rangle| \leq C\int_\Omega|u-U|(1+|u|^a+|U|^a)|u-\hat{U}|dx, \tag{36}$$

by using the following inequality

$$|u|^a=|u-U+U|^a\leq C(|u-U|^a+|U|^a), \tag{37}$$

we can rewrite (36) as follows

$$\begin{aligned}|\langle f(u)-f(U),u-\hat{U}\rangle| \leq C\int_\Omega(1+|U|^a)|u-U||u-\hat{U}|dx\\ +C\int_\Omega|u-U|^{a+1}|u-\hat{U}|dx.\end{aligned} \tag{38}$$

The first term on the righthand side of (38), can be bounded as

$$\begin{aligned}&\int_\Omega(1+|U|^a)|u-U||u-\hat{U}|dx\\ &\leq\int_\Omega(1+|U|^a)\big(|u-\hat{U}|+|U-\hat{U}|\big)|U-\hat{U}|dx\\ &\leq CG(U)\big(\|u-\hat{U}\|^2+\|U-\hat{U}\|^2\big),\end{aligned} \tag{39}$$

where $G(U) = \big(1 + \|U\|^a_{L_\infty}\big)$. The second term, we bound it by using the following inequality [9],

$$\int \rho^{\sigma+1}\delta dx \le \frac{\sigma+1}{\sigma+2}\int \rho^{\sigma+2}dx + \frac{1}{\sigma+2}\int \delta^{\sigma+2}dx,\ \rho,\delta,\sigma > 0, \tag{40}$$

the second term can be written as

$$\int |u-U|^{a+1}|u-\hat{U}|dx \le \int |u-\hat{U}|^{a+2}dx + \int |U-\hat{U}|^{a+1}|u-\hat{U}|dx. \tag{41}$$

Now by applying the inequality (40) on the second term in (41) and after some maths, we obtain

$$\int |u-U|^{a+1}|u-\hat{U}|dx \le C\left(\|U-\hat{U}\|^{a+2}_{L_{a+2}} + \|u-\hat{U}\|^{a+2}_{L_{a+2}}\right). \tag{42}$$

Substituting (39) and (42) in (36), and after some mathematical manipulations, we find

$$\begin{aligned} &|\langle f(u)-f(U), u-\hat{U}\rangle| \le CG(U)\big(\|u-\hat{U}\|^2 + \|U-\hat{U}\|^2\big)\\ &+C\|U-\hat{U}\|^a_H\|\nabla(U-\hat{U})\|^2_H + C\|u-\hat{U}\|^a_H\|\nabla(u-\hat{U})\|^2_H, \end{aligned} \tag{43}$$

and by putting (43) in (28), we have

$$\begin{aligned} &\frac{d}{dt}\|u-\hat{U}\|^2_H - 2\gamma^2\ \|U-\hat{U}\|^2 \le \|f(U)-Pf(U)\|^2_H\\ &+ 2\|u-\hat{U}\|^2_H + CG(U)\big(\|u-\hat{U}\|^2 + \|U-\hat{U}\|^2\big)\\ &+ C\|U-\hat{U}\|^a_H\|\nabla(U-\hat{U})\|^2_H + C\|u-\hat{U}\|^a_H\|\nabla(u-\hat{U})\|^2_H, \end{aligned} \tag{44}$$

and by grouping the same and similar terms together, and integrating in time, we obtain

$$\begin{aligned} &\|(u-\hat{U})(t)\|^2_H \le 2\gamma^2\int_0^T \|U-\hat{U}\|^2dt + \int_0^T \|f(U)-Pf(U)\|^2_H dt\\ &+ \int_0^T (CG(U)+2)\|u-\hat{U}\|^2dt + C\int_0^T G(U)\|U-\hat{U}\|^2dt\\ &+C\int_0^T \|U-\hat{U}\|^a_H\|\nabla(U-\hat{U})\|^2_H dt + C\int_0^T \|u-\hat{U}\|^a_H\|\nabla(u-\hat{U})\|^2_H dt. \end{aligned} \tag{45}$$

Now, let $\eta(U)^2 = 2\gamma^2 \int_0^T \|U-\hat{U}\|^2dt + \int_0^T \|f(U)-Pf(U)\|_H^2dt + C\int_0^T G(U)\|U-\hat{U}\|^2dt + C\int_0^T \|U-\hat{U}\|_H^a\|\nabla(U-\hat{U})\|_H^2dt$ and by substituting that in (45), implies

$$\begin{aligned}\|(u-\hat{U})(t)\|_H^2 &\le \eta(U)^2 + \int_0^T (CG(U)+2)\|u-\hat{U}\|^2dt\\ &+ C\sup_{t\in[0,T]}\|(u-\hat{U})(t)\|_H^a\int_0^T \|\nabla(u-\hat{U})\|_H^2dt\\ &\le \eta(U)^2 + \hat{C}_1\int_0^T G(U)\|u-\hat{U}\|^2dt\\ &+ \hat{C}_2\left(\sup_{t\in[0,T]}\|(u-\hat{U})(t)\|_H^2 + \int_0^T \|\nabla(u-\hat{U})\|_H^2dt\right)^{\frac{a+2}{2}},\end{aligned} \tag{46}$$

where, $\hat{C}_1, \hat{C}_2 > 0$. Now, consider the set

$$A = \{t\in[0,T] : \sup_{t\in[0,T]}\|(u-\hat{U})(t)\|_H^2 + \int_0^T \|\nabla(u-\hat{U})\|_H^2dt \le 4\delta E(t,U)\}.$$

The set A is nonempty since $u(0)-\hat{U}(0)=0$ and by continuity of left hand side it should be closed. Let $t^* = \max_{0\le t\le T} A$, be such that $t^* < t^\#$, implies that $t^\# \notin A$, then, we have $\eta(t^\#,U) \ge \eta(t^*,U)$. Also, for simplicity of notation let $E(t,U) = e^{\hat{C}_1\left(\int_0^t G(U)dt\right)}$. When $t=t^\#$, from (46), we conclude that

$$\|(u-\hat{U})(t^*)\|_H^2 \le \eta(U)^2 + \hat{C}_1\int_0^{t^*} G(U)\|u-\hat{U}\|^2dt + \hat{C}_2\left(4\delta E(t^\#,U)\right)^{\frac{a+2}{2}}. \tag{47}$$

Using Grönwall's inequality, we have

$$\|(u-\hat{U})(t^*)\|_H^2 \le E(t^\#,U)\left(\delta + \hat{C}_2\left(4\delta E(t^*,U)\right)^{\frac{a+2}{2}}\right), \tag{48}$$

since $\delta(t^\#) \ge \delta(t^*)$, upon assuming δ is such that

$$\delta \le \hat{C}_2^{\frac{-2}{a}}\left(4E(t^\#,U)\right)^{\frac{a+2}{2}},$$

which contradicts the assumption that t^* is the maximum element of A, hence we conclude that $t^* = t^\#$ and $A=[0,T]$. Consequently, we get the required estimate in (35). □

Theorem 3.5 (Lower Bound)**.** *If A is a linear operator and $u_0 \in \Re$, then the following error bound is valid for $DG(r)$, $r \ge 0$:*

$$\frac{1}{8}\alpha_2^2\sum_{n=0}^{N-1} k_n\|U_n^+ - U_n\|^2 \le \Lambda^2. \tag{49}$$

Proof.

$$\left(\int_0^T \|\hat{U}-U\|_H^2 dt\right)^{1/2} = \left(\sum_{n=0}^{N-1}\int_{I_n} \|\hat{U}-U\|_H^2 dt\right)^{1/2}, \tag{50}$$

which implies that

$$\int_0^T \|\hat{U}-U\|^2 dt = \sum_{n=0}^{N-1}\int_{I_n} \|\hat{U}-U\|^2 dt, \tag{51}$$

by using (23) in Proposition 3.2, we have

$$\int_{I_n} \|\hat{U}-U\|^2 dt = \alpha_2^2 k_n \|U_n^+ - U_n\|^2, \tag{52}$$

by substituting (52) in (51), we obtain

$$\int_0^T \|\hat{U}-U\|^2 dt = \sum_{n=0}^{N-1}\int_{I_n} \|\hat{U}-U\|^2 dt = \alpha_2^2 \sum_{n=0}^{N-1} k_n \|U_n^+ - U_n\|^2. \tag{53}$$

Consequently, using (19), we have

$$\begin{aligned}
&\alpha_2^2 \sum_{n=0}^{N-1} k_n \|U_n^+ - U_n\|^2 \le 2\int_0^T \|u-\hat{U}\|^2 dt + 2\int_0^T \|u-U\|^2 dt \\
&\le 4\max\left(\frac{1}{2}\int_0^T \|u-\hat{U}\|^2 dt\right) + 4\max\left(\frac{1}{2}\int_0^T \|u-U\|^2 dt\right) \\
&\le 4\Lambda^2 + 4\Lambda^2 = 8\Lambda^2.
\end{aligned} \tag{54}$$

Finally, from (54), we obtain the estimate in (49). □

Theorem 3.6 (Error Estimate of Globally Lipschitz Case)**.** *Let A be a linear monotone operator and $u_0 \in D(A)$ and f is globally Lipschitz continuous, then*

$$\max_{0\le t\le T} \|(u-\hat{U})(t)\|_H \le \beta(U) e^{CT}, \tag{55}$$

where

$$\beta(U) = \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \left(\|A(U_n^+ - U_n)\|_H^2 + C\|U_n^+ - U_n\|_H^2\right) + \int_0^T \|f(U) - Pf(U)\|_H^2 dt.$$

Proof. By repeating the same first two steps in the previous theorem, we arrive at (26) which can be rewritten in the following way

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-\hat{U}\|_H^2 + \langle A(u-\hat{U}), u-\hat{U}\rangle = \langle A(U-\hat{U}), u-\hat{U}\rangle \\
&+ \langle f(u)-f(U), u-\hat{U}\rangle + \langle f(U)-Pf(U), u-\hat{U}\rangle.
\end{aligned} \tag{56}$$

Consequently, after some mathematical manipulations, we get

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \int_0^T \|A(U-\hat{U})\|_H^2 + \int_0^T \|f(U)-Pf(U)\|_H^2 + \int_0^T \|u-\hat{U}\|_H^2 \\ &+ 2\int_0^T |\langle f(u)-f(U), u-\hat{U}\rangle|. \end{aligned} \tag{57}$$

Now, we bound the first term in the righthand side of (57), we have

$$\int_0^T \|A(U-\hat{U})\|_H^2 dt = \sum_{n=0}^{N-1} \int_{I_n} \|A(U-\hat{U})\|_H^2 dt, \tag{58}$$

and by Proposition 3.2, we have

$$\int_{I_n} \|A(U-\hat{U})\|_H^2 dt = \alpha_1^2 k_n^2 \|A(U_n^+ - U_n)\|_H^2, \text{ (since } \hat{U}_n = U_n\text{)}, \tag{59}$$

by substituting (58) and (59) in (57), we get

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U)-Pf(U)\|_H^2 \\ &+ \int_0^T \|u-\hat{U}\|_H^2 + 2\int_0^T |\langle f(u)-f(U), u-\hat{U}\rangle|. \end{aligned} \tag{60}$$

Now, we bound the last term in the righthand side of (60) $|\langle f(u)-f(U), u-\hat{U}\rangle|$ by using Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned}&|\langle f(u)-f(U), u-\hat{U}\rangle| \\ &\leq \|f(u)-f(U)\|_H \|u-\hat{U}\|_H \leq \frac{1}{2}\|f(u)-f(U)\|_H^2 + \frac{1}{2}\|u-\hat{U}\|_H^2, \end{aligned} \tag{61}$$

and by substituting (61) in (60), we obtain

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U)-Pf(U)\|_H^2 + \\ &+ \int_0^T \|f(u)-f(U)\|_H^2 dt + 2\int_0^T \|u-\hat{U}\|_H^2 dt, \end{aligned} \tag{62}$$

and using Lipschitz continuity of f and Proposition 3.2, to find that

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + C\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 \\ &+ \int_0^T \|f(U)-Pf(U)\|_H^2 dt + C\int_0^T \|u-\hat{U}\|_H^2 dt. \end{aligned} \tag{63}$$

Let

$$\beta(U) = \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \left(\|A(U_n^+ - U_n)\|_H^2 + C\|U_n^+ - U_n\|_H^2\right) + \int_0^T \|f(U) - Pf(U)\|_H dt,$$

and by substituting that in (63), we get

$$\|u - \hat{U}\|_H^2 \leq \beta(U) + C\int_0^T \|u - \hat{U}\|_H^2 dt. \tag{64}$$

Finally, by using Grönwall's proposition, we have the estimate in (55). □

Theorem 3.7 (Error Estimate of Locally Lipschitz Case)**.** *Let A be a linear monotone operator and $u_0 \in D(A)$ and f is locally Lipschitz continuous, then*

$$\max_{0\leq t\leq T} \|(u - \hat{U})(t)\|_H^2 \leq 2\lambda(U)^2 e^{\bar{C}_1 T}, \tag{65}$$

where

$$\begin{aligned}\lambda(U)^2 = {} & \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt \\ & + K\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 dt + \bar{C}\int_0^T \|U - \hat{U}\|_H^a \|\nabla(U - \hat{U})\|_H^2 dt.\end{aligned}$$

Proof. In the case of the locally Lipschitz continuous function f, we return to (60)

$$\begin{aligned}\|u - \hat{U}\|_H^2 \quad \leq \quad & \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 + \\ + \quad & 2\int_0^T \|u - \hat{U}\|_H^2 + 2\int_0^T |\langle f(u) - f(U), u - \hat{U}\rangle|.\end{aligned} \tag{66}$$

Now, we bound the nonlinear term $\int_0^T |\langle f(u) - f(U), u - \hat{U}\rangle|$ by using the growth condition (6) and following the same steps as in the Theorem 2, we arrive at

$$\begin{aligned}\|u - \hat{U}\|_H^2 \leq {} & \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt + \\ & + \bar{C}_1 \int_0^T \|u - \hat{U}\|_H^2 dt + K\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 dt \\ & + \bar{C}_2 \int_0^T \|u - \hat{U}\|_H^a \|\nabla(u - \hat{U})\|_H^2 dt \\ & + \bar{C}\int_0^T \|U - \hat{U}\|_H^a \|\nabla(U - \hat{U})\|_H^2 dt,\end{aligned} \tag{67}$$

where $\bar{C} = 2C\tilde{C}$, $\bar{C}_1 = 3K + 2$ and $\bar{C}_2 = 2C + \bar{C}$. Now, let

$$\lambda(U)^2 = \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt$$
$$+ K\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 dt + \bar{C} \int_0^T \|U - \hat{U}\|_H^a \|\nabla(U - \hat{U})\|_H^2 dt,$$

and by substituting that in (67), we have

$$\|u - \hat{U}\|_H^2 \leq \lambda(U)^2 + \bar{C}_1 \int_0^T \|u - \hat{U}\|_H^2 dt$$
$$+ \bar{C}_2 \left(\sup_{t \in [0,T]} \|u - \hat{U}\|_H^2 + \int_0^T \|\nabla(u - \hat{U})\|_H^2 dt \right)^{\frac{a+2}{2}}. \tag{68}$$

To bound the last term, we follow the same techniques as in Theorem 2, we end up with the estimate in (65). □

4. Conclusions

Optimal order *a posteriori* error estimators of discontinuous Galerkin method for semidiscretised in time semilinear parabolic problems have been derived and presented. These error bounds are derived for the linear, monotone and γ^2-angle bounded operators. The main tool in deriving these error estimates is the time reconstruction operator introduced in [37]. The nonlinear evolutionary problems when the nonlinearity is both globally and locally Lipschitz continuous are investigated.

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References

[1] G. Akrivis and C. Makridakis, *Galerkin time-stepping methods for nonlinear parabolic equations*, M2AN Math. Model. Numer. Anal., 38 (2004), 261-289.

[2] J. Argyris and D. Scharpf, *Finite elements in time and space*, Nuclear Engineering and Design, 10 (1969), 456-464.

[3] W. Bangerth and R. Rannacher, *Adaptive finite element methods for differential equations*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2003.

[4] A. Cangiani, Z. Dong, and E. H. Georgoulis, *hp-version space-time discontinuous Galerkin methods for parabolic problems on prismatic meshes*, SIAM Journal on Scientific Computing, 39 (2017), A1251-A1279.

[5] A. Cangiani, Z. Dong, E. H. Georgoulis, and P. Houston, *hp-version discontinuous Galerkin methods for advection-diffusion-reaction problems on polytopic meshes*, ESAIM Math. Model. Numer. Anal., 50 (2016), 699-725.

[6] A. Cangiani, E. Georgoulis, and M. Sabawi, *A posteriori error analysis for implicit-explicit hp- discontinuous Galerkin timestepping methods for semilinear parabolic problems*, J. Sci. Comput., 82 (2020).

[7] A. Cangiani, E. Georgoulis, and Y. Sabawi, *Convergence of an adaptive discontinuous Galerkin method for elliptic interface problems*, J. Comput. Appl. Math., 367 (2020), 112397.

[8] A. Cangiani, E. Georgoulis, and Y. Sabawi, *Adaptive discontinuous Galerkin methods for elliptic interface problems*, Math. Comput., 87 (2018), 2675-2707.

[9] A. Cangiani, E. H. Georgoulis, and M. Jensen, *Discontinuous Galerkin methods for mass transfer through semipermeable membranes*, SIAM Journal on Numerical Analysis, 51 (2013), 2911-2934.

[10] A. Cangiani, E. H. Georgoulis, I. Kyza, and S. Metcalfe, *Adaptivity and blow-up detection for nonlinear evolution problems*, SIAM Journal on Scientific Computing, 38 (2016), A3833-A3856.

[11] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems IV: Nonlinear problems*, SIAM, 32 (1995), 1729-1749.

[12] K. Eriksson, C. Johnson, and A. Logg, *Adaptive computational methods for parabolic problems. in: Encyclopedia of computational mechanics*, Stein, E., de Borst, R., Houghes, T.J.R. (eds), RAIRWiley New York, (2004), 1-44.

[13] K. Eriksson, C. Johnson, and V. Thomée, *Time discretization of parabolic problems by the discontinuous Galerkin method*, RAIRO, 19 (1985), 611-643.

[14] A. Ern, I. Smears, and M. Vohralík, *Guaranteed, locally space-time efficient, and polynomial-degree robust a posteriori error estimates for high-order discretizations of parabolic problems*, Tech. Rep., 2016.

[15] A. Ern, I. Smears, and M. Vohralík, *Equilibrated flux a posteriori error estimates in $L^2(H^1)$-norms for high-order discretizations of parabolic problems*, tech. rep., 2017.

[16] D. Estep, M. Larson, and R. Williams, *Estimating the error of numerical solutions of systems of reaction-diffusion equations*, Mem. Amer. Math. Soc., 146 (2000), viii+109.

[17] F. Gaspoz, C. Kreuzer, K. Siebert, and D. Ziegler, *A convergent time-space adaptive DG(s) finite element method for parabolic problems motivated by equal error distribution*, https://arxiv.org/abs/1610.06814, (2017).

[18] E. Georgoulis, O. Lakkis, and T. Wihler, *A posteriori error bounds for fully-discrete hp- discontinuous Galerkin timestepping methods for parabolic problems*, accepted for publishing.

[19] E. H. Georgoulis, *Discontinuous Galerkin methods for linear problems: an introduction*, research reports in mathematics, University of Leicester, May 2010.

[20] E. H. Georgoulis, E. Hall, and C. G. Makridakis, *A posteriori error estimates for Runge-Kutta discontinuous Galerkin methods for hyperbolic problems*, in preparation.

[21] E. H. Georgoulis and C. Makridakis, *On a posteriori error control for the Allen-Cahn problem*, Mathematical Methods in the Applied Sciences, 173-179.

[22] B. Holm and T. P. Wihler, *Continuous and discontinuous Galerkin time stepping methods for nonlinear initial value problems with application to finite time blow-up*, Numer. Math., 138 (2018), 767-799.

[23] B. Hulme, *Discrete Galerkin and related one-step methods for ordinary differential equations*, Math. Comput., 26 (1972), 881-891.

[24] B. Hulme, *One-step piecewise polynomial Galerkin methods for initial value problems*, Math. Comput., 26 (1972), 415-426.

[25] P. Jamet, *Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain*, SIAM, 15 (1978), 912-928.

[26] L. Ji, P. Slingerland, J. Ryan, and C. Vuik, *Superconvergent error estimates for a position-dependent smoothness-increasing accuracy-conserving filter for DG solutions*, Math. Comput., 83 (2014), 2239-2262.

[27] S. Karaa, A. Pani, and S. Yadav, *A priori hp-estimates for discontinuous Galerkin approximations to linear hyperbolic integro-differential equations*, Appl. Numer. Math., 96 (2015), 1-23.

[28] I. Kyza, S. Metcalfe, and T. P. Wihler, *hp-adaptive Galerkin time stepping methods for nonlinear initial value problems*, J. Sci. Comput., 75 (2018), 111-127.

[29] O. Lakkis, A. Madazvamuse, and C. Venkataraman, *Implicit-explicit timestepping with finite element approximation of reaction-diffusion systems on evolving domains*, SIAM J. Numer. Anal., 51 (2013), 2309-2330.

[30] O. Lakkis and C. Makridakis, *Elliptic reconstruction and a posteriori error estimates for fully discrete linear parabolic problems*, Math. Comp., 75 (2006), 1627-1658.

[31] A. Lasis, *Discontinuous Galerkin finite approximation of nonlinear parabolic problems*, PhD thesis, University of Oxford, Oxford, UK, 2005.

[32] P. Lesaint and P. Raviart, *On a finite element method for solving the neutron transport equation*, Publication, (1974), 89-123.

[33] G. Lippold, *Error estimates for the implicit Euler approximation of an evolution inequality*, Nonlinear Anal., 15 (1990), 1077-1089.

[34] G. Lippold, *Error estimates and step-size control for the approximate solution of a first order evolution equation*, RAIRO Modél. Math. Anal. Numer, 25 (1991), 111-128.

[35] C. Makridakis, *Space and time reconstructions in a posteriori analysis of evolution problems*, ESIAM: Procedings, 21 (2007), 31-44.

[36] C. Makridakis and R. Nochetto, *Elliptic reconstruction and a posteriori error estimates for parabolic problems*, SIAM J. Numer. Anal., 41 (2003), 1585-1594.

[37] C. Makridakis and R. Nochetto, *A posteriori error analysis for higher order dissipative methods for evolution problems*, Numer. Math., 104 (2006), 489-514.

[38] G. Matthies and F. Schieweck, *Higher order variational time discretizations for nonlinear systems of ordinary differential equations*, Preprint Otto von Guericke Universitat Magdeburg, 23 (2011), 130.

[39] K. Mustapha and J. Ryan, *Post-processing discontinuous Galerkin solutions to Volterra integro-differential equations: analysis and simulations*, J. Comput. Appl. Math., 253 (2013), 89-103.

[40] R. Nochetto, G. Savaré, and C. Verdi, *Error control for nonlinear evolution equations*, Acad. Sci. Paris, Ser. I (1998), 1437-1442.

[41] M. Picasso, *Adaptive finite elements for a linear parabolic problem*, Comput. Methods Appl. Mech. Eng., 167 (1998), 223-237.

[42] W. Reed and T. Hill, *Triangular mesh methods for the neutron transport equation*, tech. report la-ur- 73-479, Los Alamos Scientific Laboratory, 1973.

[43] J. Ryan, *Exploiting superconvergence through smoothness-increasing accuracy-conserving (SIAC) filtering, in Spectral and high order methods for partial differential equations ICOSA-HOM 2014*, vol. 106 of Lect. Notes Comput. Sci. Eng., Springer, Cham, 2015, 87-102.

[44] M. Sabawi, *Discontinuous Galerkin timestepping for nonlinear parabolic problems*, PhD thesis, University of Leicester, Leicester, UK, 2018.

[45] M. Sabawi, *A posteriori error analysis for fully discrete semilinear parabolic problems*, 2nd International Conference for Engineering, Technology and Science (2nd ICETS), Al-Kitab University, Kirkuk, Iraq, 4-5 Dec., 2018.

[46] M. Sabawi, *A posteriori error analysis for semidiscrete semilinear parabolic problems*, Al Mansour International Conference on New Trends in Computing, Communications, and Information Technology (NTCCIT), Al- Mansour University College, Baghdad, Iraq, 14-15 Nov., 2018.

[47] Y. Sabawi, *A posteriori $L_\infty(L_2) + L_2(H^1)$-error bounds in discontinuous Galerkin methods for semidiscrete semilinear parabolic interface problems*, Baghdad Sci. J., To appear.

[48] Y. Sabawi, *Adaptive discontinuous Galerkin methods for interface problems*, PhD thesis, University of Leicester, Leicester, UK, 2017.

[49] Y. Sabawi, *A posteriori $L_\infty(H^1)$-error bounds for the finite element approximation of semilinear parabolic problems*, First International Conference of Computer and Applied Sciences (1st CAS), Mustansiriyah University, Baghdad, Iraq, 2019.

[50] F. Schieweck, *A stable discontinuous Galerkin-Petrov time discretization of higher order*, J. Numer. Math., 18 (2010), 429-456.

[51] D. Schötazu and T. Wihler, *A posteriori error estimation for hp-version time-stepping methods for parabolic partial differential equations*, Numer. Math., p. 475-509.

[52] D. Schötzau, *Discontinuous Galerkin finite approximation of nonlinear parabolic problems*, PhD thesis, ETH, Zürich, 1999.

[53] D. Schötzau and C. Schwab, *Time discretization of parabolic problems by the hp-version of the discontinuous Galerkin finite element method*, Research Reports in Mathematics 99-04, Research Report, Seminar für Angewandte Mathematik, Eidgenössische Technische Hochschule, Zürich, Switzerland, 1999.

[54] D. Schötzau and C. Schwab, *An hp a priori error analysis of the DG time-stepping method for initial value problems*, CALCOLO, 37 (2000), 207-232.

[55] D. Schötzau and C. Schwab, *Time discretization of parabolic problems by the hp-version of the discontinuous Galerkin finite element method*, SIAM J. Numer. Anal., 38 (2000), 837-875.

[56] D. Schötzau and C. Schwab, *hp-discontinuous Galerkin time-stepping for parabolic problems*, C. R. Acad. Sci. Paris, t. 333, (2001), 1121-1126.

[57] K. Segeth, *A posteriori error estimation with the finite element method of lines for a nonlinear parabolic equation in one space dimension*, Numer. Math., 38 (1999), 455-475.

[58] Thomée, *Galerkin finite element methods for parabolic problems*, Second Edition, Springer, 2006.

[59] R. Verfürth, *A posteriori error estimates for finite element discretizations of the heat equations*, Calcolo, 40 (2003), 195-212.

[60] R. Verfürth, *A posteriori error estimates for non-stationary non-linear convection-diffusion equations*, Calcolo, 55 (2018), Art. 20, 18.

[61] M. Vuik and J. Ryan, *Multiwavelet troubled-cell indicator for discontinuity detection of discontinuous Galerkin schemes*, J. Comput. Phys., 270 (2014), 138-160.

[62] T. Werder, K. Gerdes, D. Schötzau, and C. Schwab, *hp-discontinuous Galerkin time stepping for parabolic problems*, Comput. Methods Appl. Mech. Engrg., 190 (2001), 6685-6708.

[63] S. Zhao and G. Wei, *A unified discontinuous Galerkin framework for time integration*, Math. Methods Appl. Sci., 37 (2014), 1042-1071.

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